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CONNECTEDNESS IN TRANSFINITE GRAPHS AND THE EXISTENCE AND UNIQUENESS OF NODE VOLTAGES

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Abstract — Unlike connectedness in ordinary graphs, transfinite connectedness need not be transitive. As a result, sections of a transfinite graph that are maximal with respect to transfinite connectedness may overlap while being different, as is shown by an example. A sufficient condition is established under which transitivity holds, in which case the said sections partition the transfinite graph. A related phenomenon is that it may not be possible to assign a unique voltage to a node of a transfinite electrical network because the sum of the branch voltages along a path between that node and a chosen ground node may depend upon the choice of the path. This too is shown by example. Sufficient conditions are established that insure that all nodes have unique node voltages, being independent of the choices of the paths to ground. The proofs are based on a characterization of the totally ordered set of nodes along any transfinite path, the characterization being a certain hierarchical structure of nested sequences.

1 Introduction

The idea of a transfinite graph arises quite naturally from reflections about infinite electrical networks [2, Section 8], [3, Examples 1.6-4 and 1.6-5]. The key difference between transfinite graphs and the usual infinite graphs discussed heretofore is that, in the latter, two nodes are either connected through a finite path or not connected at all whereas in the former two nodes may also be connected through a transfinite path, that is, through a sequential connection of many — possibly infinitely many — infinite paths. In fact, for transfinite graphs there is a hierarchy of connectedness concepts, that hierarchy being indexed by the countable ordinals. Thus, we may speak of two nodes being k-connected but not l-connected, where k and l are countable ordinals with $l < k$. (This idea of "k-connectedness" is different from the usual concept, in which k is a cardinal number [1, Section 3].) 0-connectedness is the same as ordinary connectedness for graphs, but k-connectedness, where $k \geq 1$, is a weaker and more general concept.

Moreover, k-connectedness ($k \geq 1$) is peculiar in that it may not be transitive as a binary relation between branches. We show this by example. A consequence of this possible nontransitivity relates to the k-sections ($k \geq 1$) of a transfinite graph; these are the reduced graphs induced by maximal sets of k-connected branches. Different k-sections may overlap. We establish a sufficient condition for the transitivity of k-connectedness between branches, in which case the k-sections comprise a partition of the transfinite graph. The sufficient condition is that, if two perceptible infinite or transfinite paths meet infinitely often in a certain way, then their infinite extremities are either required to be shorted together or at least one of them is open (i.e., not shorted to any other node or infinite extremity).

Another related pathology that can arise concerns transfinite electrical networks, that is, electrical networks whose graphs are transfinite and whose branches contain resistors and voltage sources. A node n_0 may be assigned a node voltage with respect to a chosen ground node n_g if all the branch voltages along some (possibly transfinite) path between n_0 and n_g sum to a finite amount. It can happen that the node voltage may depend upon the choice of the path, in contrast to the situation for ordinary (i.e., 0-connected), finite or infinite, electrical networks. This too we show by example. We then establish sufficient conditions that insure that all node voltages are unique, whatever be the choices of the paths connecting nodes to n_g. One condition is that node voltages be assigned only along paths that are perceptible (i.e., their resistances sum to a finite amount). Another condition is similar to — but not exactly the same as — the prior condition requiring the shorting together of infinite extremities of paths that meet infinitely often.

A substantial part of this paper is devoted to a characterization of the totally ordered set of all the embraced nodes along a transfinite path, that characterization being a certain hierarchical structure of nested sequences, called "k-sequences." k-sequences generalize ordinary sequences in much the same way as transfinite paths generalize ordinary paths.

2 Some Definitions

Transfinite graphs $\mathcal{G}^k$ were introduced in [2]. To define them once again would be repetitious. Please refer to [2] for any definitions not specified below. (Another exposition is given in [3].) For the sake of definiteness, we shall establish our results for the case where either the rank k is a natural number p or k is the first transfinite ordinal ω. The former case extends directly to higher ranks that are successor ordinals and the latter to higher ranks that are limit ordinals. Also, for the latter case we have to consider the $\vec{\omega}$-graphs $\mathcal{G}^{\vec{\omega}}$ of rank $\vec{\omega}$ used in constructing the ω-graphs $\mathcal{G}^{\omega}$. By definition $\mathcal{G}^k$ contains no more than countably many branches. We allow $\mathcal{G}^k$ to have infinite 0-nodes, self-loops, parallel branches, and nodes that embrace nodes of lower ranks.

Henceforth, p and q will always denote natural numbers. Recall that a node n of rank p (or of rank ω) is defined [2, Sections 4 and 5] as a set whose elements are $(p-1)$-tips (respectively, $\vec{\omega}$-tips) except possibly for one element; that exceptional element, if it exists, is a node n_0 of rank q, where $q \leq p-1$ (respectively, where q is some natural number). Also, every node n is required to have at least one such tip. The node n is called a *nonsingleton* if it contains at least two elements. Furthermore, n is said to *embrace* itself, all its elements, all elements of its embraced node n_0 if n_0 exists, all elements of the node that n_0 embraces if that too exists, and so forth through a finite sequence of embraced nodes of decreasing ranks. As an immediate consequence of these definitions, we have

Lemma 2.1. *If a node m is a nonsingleton, then any node n that embraces m is also a nonsingleton.*

It is a fact, that if two nodes a and c embrace a third node x, then either a embraces c or c embraces a [2, Proposition 4.1]. It follows that all the nodes of a k-graph can be partitioned into subsets, with two nodes being in the same subset if one node embraces the other. Moreover, each such subset can be identified by any one of its nodes. Its node of maximal rank will be called a *maximal* node. All the nodes in any such subset are said to be *shorted* together.

We shall say that a p-tip t^p and a node n are *shorted* together if the $(p+1)$-node that contains t^p either embraces n or is embraced by n. Similarly, we say that an $\vec{\omega}$-tip $t^{\vec{\omega}}$ and a node n are *shorted* together if the ω-node that contains $t^{\vec{\omega}}$ embraces n (or is embraced by n — a case that will not arise in this work because we are restricting ourselves to graphs with ranks no larger than ω). Also, two tips of possibly differing ranks are said to be *shorted* together if the node that contains one of those tips embraces or is embraced by the node that contains the other tip.

A p-path P^p is said to *meet* a node n, whose rank need not be p, if P^p embraces n or embraces a node that is shorted to n or if P^p has a p-tip that is shorted to n. In the former case, we say that P^p meets n *with a node* or *nodally* meets n; in the latter case, we say that P^p meets n *with a p-tip*. The nodes m and n, again of possibly differing ranks, are said to be *p-connected* if there exists a finite q-path P^q (i.e., P^q contains only finitely many q-nodes) such that $q \leq p$ and P^q nodally meets m and n. It follows from the last definition that, if m and n are p-connected, then they are r-connected for all $r \geq p$. Two branches are called *p-connected* if their incident 0-nodes are p-connected.

The corresponding definitions for $\vec{\omega}$-paths and ω-paths are not much different. An $\vec{\omega}$-path $P^{\vec{\omega}}$ (or an ω-path P^{ω}) is said to *meet* a node n if $P^{\vec{\omega}}$ (or, respectively, P^{ω}) embraces n or embraces a node that is shorted to n or if $P^{\vec{\omega}}$ (or P^{ω}) has an $\vec{\omega}$-tip (or ω-tip) that is shorted to n. Again we say that the path meets n *with a node* or correspondingly *with an $\vec{\omega}$-tip* (or *ω-tip*). The nodes m and n are said to be *$\vec{\omega}$-connected* (or *ω-connected*) if there is a p-path (respectively, a p-path or an $\vec{\omega}$-path or a finite ω-path) that meets m and n. Two branches are *$\vec{\omega}$-connected* (or *ω-connected*) if their 0-nodes are.

A *p-section* of $\mathcal{G}^k$, where $p \leq k$, is a reduction [2, Sections 3 and 4] of $\mathcal{G}^k$ induced by a maximal set of branches that are pairwise p-connected, and similarly for an *$\vec{\omega}$-section* and an *ω-section*. A node of any rank is said to be *incident* to a p-section S^p if it is shorted to a node of S^p or to a p-tip of S^p; by replacing p by $\vec{\omega}$ or ω we get analogous definitions.

A *partition* of $\mathcal{G}^k$ is a collection of reduced graphs [2, Sections 3 to 5] whose branch sets comprise a partition of the branch set of $\mathcal{G}^k$. On the other hand, two reduced graphs of $\mathcal{G}^k$ are said to *overlap* if they share branches.

We now wish to extend the definition of nondisconnectable 0-tips [2, Section 14] to tips of higher ranks. First some preparatory ideas: Recall that a representative of a p-tip is a one-ended p-path which in turn is a one-way infinite alternating sequence of p-nodes n_i^p and $(p-1)$-paths P_i^{p-1} of the form:

$$P^p = \{n_0^q, P_0^{p-1}, n_1^p, P_1^{p-1}, n_2^p, P_2^{p-1}, \cdots\} \tag{1}$$

where the first node n_0^q has a rank $q \leq p$ and certain conditions are satisfied [2, Section 4]. Similarly, a representative of an $\vec{\omega}$-tip is a one-ended $\vec{\omega}$-path which in turn is an alternating sequence of the form:

$$P^{\vec{\omega}} = \{n_0^q, P_0^{p_0-1}, n_1^{p_1}, P_1^{p_1-1}, n_2^{p_2}, P_2^{p_2-1}, \cdots\} \tag{2}$$

where $q \leq p_0 < p_1 < p_2 < \cdots$ and again certain conditions are satisfied [2, Section 5].

Now consider an infinite sequence of nodes $\{m_1, m_2, m_3, \ldots\}$ of possibly differing ranks. We shall say that the m_l *approach* a p-tip t^p (alternatively, an $\vec{\omega}$-tip $t^{\vec{\omega}}$) if there is a representative (1) for t^p (respectively, (2) for $t^{\vec{\omega}}$) such that, for each natural number i, all but finitely many of the m_l are shorted to nodes embraced by the members of (1) (respectively, (2)) lying to the right of n_i^p (respectively, $n_i^{p_i}$). Later on, we shall also say that those nodes lie *beyond* n_i^p. We shall also say the m_l *approach* any node that embraces t^p (respectively, $t^{\vec{\omega}}$).

Let t_a and t_b be two tips, not necessarily of the same rank. We say that t_a and t_b are *nondisconnectable* or *not disconnectable* if there is an infinite sequence of nodes that approach both t_a and t_b.

3 An Example

Consider the 1-graph shown in Figure 1. It contains the 0-nodes n_j^0 where $j = 1, 2, 3 \ldots$, the parallel branches a_j and b_j incident to n_j^0 and n_{j+1}^0, the nonsingleton 1-node $n_a^1 = \{t_a^0, n_a^0\}$ where n_a^0 is an embraced 0-node and t_a^0 is the 0-tip having as a representative the 0-path induced by the a_j, the nonsingleton 1-node $n_b^1 = \{t_b^0, n_b^0\}$ where n_b^0 is another embraced 0-node and t_b^0 is the 0-tip having as a representative the 0-path induced by the b_j, and finally two more branches β_a and β_b — the first one incident to n_a^0 and the 0-node n_c^0 and the second one incident to n_b^0 and the 0-node n_d^0.

In this 1-graph, the 0-node n_1^0 is 1-connected to both n_c^0 and n_d^0. However, n_c^0 and n_d^0 are not 1-connected because there is no 0-path meeting n_a^1 and n_b^1; indeed, any tracing from n_a^1 to n_b^1 would perforce meet at least one of the n_j^0 at least twice — thereby preventing that tracing from being a 1-path. Thus, 1-connectedness is not transitive as a binary relation between the nodes or between the branches.

Moreover, this 1-graph contains exactly three 0-sections: S_1^0 induced by all the branches a_j and b_j; S_2^0 induced by β_a alone; finally, S_3^0 induced by β_b alone. These 0-section do not overlap because 0-connectedness (i.e., ordinary connectedness) is transitive, whatever be the transfinite graph. On the other hand, there are exactly two 1-sections: S_1^1 induced by β_a and all the a_j and b_j; S_2^1 induced by β_b and all the a_j and b_j. Because of the nontransitivity of 1-connectedness for branches in this case (i.e., a_1 is 1-connected to β_a and to β_b, but β_a and β_b are not 1-connected), these two 1-sections overlap but are not the same.

Furthermore, if every branch in Figure 1 is replaced by an endless p-path, where p is a natural number, and if the rank of each node shown in Figure 1 is increased by $p+1$, then in the resulting $(p+2)$-graph, two $(p+2)$-sections will overlap but will not be the same. However, that graph will have three $(p+1)$-sections, which partition the graph.

Similarly, let us replace the a_j and b_j by one-ended j-paths, the n_j^0 by j-nodes, and n_a^1 and n_b^1 by ω-nodes, but let us leave n_a^0, n_b^0, n_c^0, n_d^0, β_a, and β_b as they are. The result is an ω-graph having two different ω-sections which overlap. On the other hand, its three $\vec{\omega}$-sections partition the ω-graph; two of those $\vec{\omega}$-sections are also 0-sections.

The overlapping of the two 1-sections in the 1-graph of Figure 1 is the result of the nontransitivity of 1-connectedness for branches. However, if another branch were to be appended incident to n_c^0 and n_d^0, the 1-nodes n_a^1 and n_b^1 would become 1-connected and 1-connectedness would become transitive for all nodes and branches in the resulting 1-graph. It is tempting therefore to conjecture that the transitivity of 1-connectedness for branches may be obtained for any 1-graph from the following condition: *If two nonsingleton 1-nodes are incident to the same 0-section, then they are 1-connected.* However, this conjecture is not true.

As a counterexample, consider the 1-graph of Figure 2. Each heavy dot therein represents a nonsingleton 1-node. Each S_j^0 is a 0-section like the 0-section of Figure 1 induced by the a_j and b_j branches. The 0-node of S_j^0 corresponding to n_1^0 in Figure 1 is embraced by a 1-node of Figure 2 — except for that 0-node of S_1^0. (Note that, there is no 0-path in S_j^0 that meets the 1-nodes corresponding to n_a^1 and n_b^1 in Figure 1.) Furthermore, the R_j^0 and T_j^0 are 0-sections, each consisting of a single endless 0-path. Finally, this 1-graph extends infinitely to the right.

It can be seen that every two nonsingleton 1-nodes in Figure 2 that are incident to the same 0-section are 1-connected, and therefore the above condition is fulfilled. For example, consider the two nonsingleton 1-nodes incident to S_1^0; they are connected by a 1-path that passes along R_1^0, then through S_3^0, and finally through S_2^0. Moreover, the branches of all the S_j^0 and all the R_j^0 induce a 1-section W^1, and the branches of all the S_j^0 and all the T_j^0 induce another 1-section Z^1. However, there is no 1-path connecting any branch of R_j^0 to any branch of any T_j^0; the "forked ends" of the S_j^0 block such 1-paths. Thus, 1-connectedness is not transitive for the branches in this 1-graph; moreover, W^1 and Z^1 overlap but are not the same. A necessary and sufficient condition for the transitivity of branch 1-connectedness remains to be found.

4 About Tips and Nodes

A p-path can be represented as a $(p+1)$-path, and also as a $(p+2)$-path, and so forth. For example, consider the one-ended 0-path

$$P^0 = \{n_0^0, b_0, n_1^0, b_1, \ldots\}$$

embedded in a k-graph $\mathcal{G}^k$. Let n^1 be a 1-node that embraces the 0-tip t^0 for which P^0 is a representative. Then, we have the 1-path

$$P^1 = \{n_0^0, P^0, n^1\}.$$

Moreover, P^1 can be rewritten as paths of higher ranks:

$$P^2 = \{n_0^0, P^1, n^1\},$$

$$P^3 \; = \; \{n_0^0, P^2, n^1\},$$

and so forth. Since P^1 contains n^1, it embraces more than P^0 embraces. However, P^2 and P^3 are the same as P^1 — just written differently. We wish to identify 1 as the minimum rank one can associate with P^1, P^2 and P^3 and will call 1 the "essential rank" of those paths.

To this end, note that every path embraces all the tips embraced by all the nodes embraced by that path. For example, P^0 embraces all the elementary tips [3, Section 1.3] of all its branches plus the elementary tips of all the other branches in $\mathcal{G}^k$ that are incident to the n_i^0. Also, P^1 embraces all those elementary tips plus the 0-tip for which P^0 is a representative plus all the other tips that n^1 embraces. On the other hand, P^2 and P^3 do not embrace any tips other than those embraced by P^1.

We also need the idea of a "traversed tip" for a path. A tip of rank 0 or higher is said to be *traversed* by a path if the path embraces a representative of the tip, that is, the path embraces all the members of that representative. Also, an elementary tip is said to be *traversed* by a path if the path embraces the branch having that elementary tip. A tip may be embraced but not traversed by a path; for example, P^1 embraces all the 0-tips of n^1 but traverses only that 0-tip t^0 for which P^0 is a representative. Also, a tip may be traversed but not embraced by a path; indeed, P^0 traverses t^0 but does not embrace it.

Let us denote the rank of an elementary tip by $\vec{0}$. Let $\mathcal{R}$ be the totally ordered set of all ranks. Thus, $\mathcal{R}$ is obtained from the set of all countable ordinals by inserting the symbol $\vec{\nu}$ just before the countable limit ordinal ν and $\vec{0}$ before 0. Thus,

$$\mathcal{R} \; = \; \{\vec{0}, 0, 1, 2, \ldots, \vec{\omega}, \omega, \omega+1, \ldots, \overrightarrow{\omega \cdot 2}, \omega \cdot 2, \omega \cdot 2 + 1, \ldots\}$$

(All ranks of the form $\vec{\nu}$ will be called *arrowed ranks.*) $\mathcal{R}$ is a well-ordered set; that is, it is totally ordered and each nonvoid subset has a least member.

Given any path P, let $\mathcal{R}(P)$ denote the set of all the ranks for all the tips that are both embraced and traversed by P. If $\mathcal{R}(P)$ contains a rank μ, it will also contain all ranks less than μ [2, Section 7]. Let ρ be the smallest rank that is larger than every member of $\mathcal{R}(P)$. Then, ρ is defined to be the *essential rank* of P and is denoted by $\rho \, = \, \mathrm{essrank}(P^0)$.

Thus, for the examples of paths given above, essrank$(P^0) = 0$, whereas essrank$(P^\mu) = 1$ for $\mu = 1, 2, 3$. Note also that for no path can the essential rank be $\vec{0}$.

As another example, consider the one-ended path

$$P = \{n_0^0, P_0^0, n_1^1, P_1^1, n_2^2, P_2^2, \ldots\}$$

where, for each natural number m, P_m^m is a one-ended path of essential rank m that starts at n_m^m and meets n_{m+1}^{m+1} with an m-tip. P embraces and traverses tips of all the natural-number ranks. On the other hand, P traverses an $\vec{\omega}$-tip but does not embrace any $\vec{\omega}$-tip. In fact, $\mathcal{R}(P)$ is exactly the set of all natural numbers and $\vec{0}$ as well. Thus, essrank$(P) = \vec{\omega}$.

Lemma 4.1. *A finite path embraces every tip that it traverses.*

Proof. Assume that the path P traverses a tip t without embracing t. This means that P embraces a representative of t without embracing any node that embraces t. Since that representative is a one-ended path, P cannot terminate; that is, P is not finite. ♣

As always, p denotes a natural number.

Lemma 4.2. *Let*

$$m_1,\ m_2,\ m_3,\ \ldots \tag{3}$$

be an infinite sequence of nodes in a p-graph $\mathcal{G}^p$ and let P be a finite p-path in $\mathcal{G}^p$ that meets those nodes in the order given. (P may meet other nodes as well.) Then, P embraces a one-ended ρ-path R, where $\rho < p$, such that R meets all of the m_l except possibly finitely many of them, R is a representative of a ρ-tip t^ρ traversed and embraced by P, and the m_l approach t^ρ.

Proof. For any two consecutive nodes m_l and m_{l+1}, P embraces a finite path Q_l that terminates at m_l and m_{l+1}. Let $\rho_l =$ essrankQ_l. Let ρ be the largest of the values ρ_l for which there are an infinity of Q_l with that essential rank ρ. We must have that $\rho < p$; indeed, since P is a finite p-path, it can embrace only finitely many nodes of rank p, and therefore only finitely many of the Q_l can be of essential rank p. Furthermore, there will be only finitely many paths Q_l whose essential ranks are larger than ρ. So, by choosing l_0 large enough, we can ensure that all Q_l with $l \geq l_0$ have $\rho_l =$ essrank$Q_l \leq \rho$. Infinitely many of the Q_l with $l \geq l_0$ will have $\rho_l = \rho$.

The path R induced by all the branches embraced by all the Q_l with $l \geq l_0$ is the one-ended ρ-path that we seek. Indeed, R clearly meets all except perhaps finitely many of the m_l. Also, its essential rank is ρ. Since P embraces R, it traverses the ρ-tip t^ρ that has R as a representative. Also, since P is a finite path, we can invoke Lemma 4.1 to conclude that P embraces t^ρ. Finally, the m_l obviously approach t^ρ. ♣

Lemma 4.3. *Let (3) be an infinite sequence of nodes in an ω-graph $\mathcal{G}^\omega$ and let P be a finite μ-path in $\mathcal{G}^\omega$ that meets those nodes in the order given, where either μ is a natural number or $\mu = \omega$. Then, P embraces a one-ended path R such that R meets all of the m_l except possibly finitely many of them, R is a representative of a ρ-tip t^ρ traversed and embraced by P where $\rho < \mu$, and the m_l approach t^ρ.*

Proof. Let Q_l and ρ_l be as in the preceding proof. All except perhaps finitely many of the Q_l will have natural numbers as their essential ranks ρ_l, for otherwise P would traverse an infinity of tips of rank $\vec{\omega}$ and therefore would embrace an infinity of ω-nodes according to Lemma 4.1, in which case P would not be a finite μ-path with $\mu \leq \omega$.

Now, if all but finitely many of the ρ_l are bounded by some fixed natural number, we can proceed as in the preceding proof to find a representative R of a ρ-tip, as asserted in the conclusion, where now ρ is a natural number less than μ.

So, assume that the ρ_l that are natural numbers are not bounded by any fixed natural number. We can now choose l_0 so large that all Q_l with $l \geq l_0$ will have natural numbers as their essential ranks ρ_l. We can find a representative R of an $\vec{\omega}$-tip embraced by P as follows: Let R_{l_1} be a path embraced by P, starting at m_{l_0}, proceeding toward the m_l of higher indices $l > l_0$, and embracing a Q_{l_1} such that $\rho_{l_0} < \rho_{l_1}$. R_{l_1} exists because of the unboundedness of the ρ_l. Inductively, for $i = 2, 3, 4, \ldots$, let R_{l_i} be a path embraced by P, starting at $m_{l_{i-1}}$, proceeding toward the m_l of higher indices $l > l_{i-1}$, and embracing a Q_{l_i} such that $\rho_{l_{i-1}} < \rho_{l_i}$. R_{l_i} exists for the same reason. $R = \bigcup_{i=1}^{\infty} R_{l_i}$ is a one-ended $\vec{\omega}$-path , which uniquely determines an $\vec{\omega}$-tip $t^{\vec{\omega}}$. R meets all of the m_l except possibly finitely many of them. P traverses $t^{\vec{\omega}}$ and also embraces it because P is finite (Lemma 4.1). Finally, the m_l obviously approach t^ρ, as before. ♣

Lemma 4.4. *A finite path cannot have $\vec{\omega}$ as its essential rank.*

Proof. Let the path P have $\vec{\omega}$ as its essential rank. Consequently, P embraces and traverses tips of all ranks that are natural numbers, but no tips of rank $\vec{\omega}$ or higher. It follows that P embraces finite paths whose essential ranks comprise all the natural numbers. We now proceed as in the last paragraph of the proof of Lemma 4.3 to construct a one-ended $\vec{\omega}$-path R traversed by P. Were P to embrace a node that embraces the $\vec{\omega}$-tip for which R is a representative, P's essential rank would be ω or higher. Hence, the latter does not happen, which implies that P does not terminate and therefore is not a finite path. ♣

(A similar argument shows that no finite path can have any arrowed rank $\vec{\rho}$.)

5 k-sequences

The nodes of a path of rank 1 or higher in a k-graph have a particular structure — a hierarchy of sequences of sequences, which we need to explicate. We will refer to the elements at the lowest level of this hierarchy as "nodes" and interpret them as maximal nodes in some k-graph, but this interpretation is not at all essential.

0-sequences:

A 0-*sequence*

$$s^0 = \{\cdots, n_m, n_{m+1}, \cdots\} \tag{4}$$

is an ordinary sequence, that is, a nonvoid set whose elements are indexed by some or all of the integers m and are ordered according to those integers. A 0-sequence may be finite, one-ended, or endless. The *trivial* 0-sequence is a singleton $\{n\}$. For every nontrivial 0-sequence s^0, one can construct a nontrivial 0-path P^0 [2, Section 2] by inserting a branch between every pair of adjacent nodes in s^0. Conversely, the nodes of P^0 comprise a 0-sequence.

A 0-sequence is said to *terminate on the left (right)* when there is a leftmost (respectively, rightmost) node in (4), and it is said to *extend infinitely leftward (rightward)* when there is no such leftmost (respectively, rightmost) node. We say that s^0 *embraces* itself and all its nodes.

$\mathcal{X}\backslash\mathcal{Y}$ will denote the set of elements in the set $\mathcal{X}$ that are not in the set $\mathcal{Y}$. Let $\mathcal{S}$ be a totally ordered set of nodes and let s^0 be a 0-sequence of some of the nodes of $\mathcal{S}$ with a compatible ordering. s^0 is called *maximal with respect to* $\mathcal{S}$ (or simply *maximal* when $\mathcal{S}$

is understood) if there does not exist any node n_0 in $\mathcal{S}\backslash s^0$ such that $\{n_0\} \cup s^0$ with the ordering induced by $\mathcal{S}$ is a 0-sequence.

1-sequences:

A 1-*sequence*

$$s^1 = \{\cdots, s^0_m, s^0_{m+1}, \cdots\} \tag{5}$$

is a 0-sequence of 0-sequences s^0_m such that the following holds: For every two adjacent members s^0_m and s^0_{m+1} in s^1, either s^0_m extends infinitely rightward and s^0_{m+1} terminates on the left, or s^0_m terminates on the right and s^0_{m+1} extends infinitely leftward.

Thus, s^0_m and s^0_{m+1} do not extend infinitely toward each other, nor do they terminate next to each other. To save words, we shall say that *infinite extensions are separated by nodes*, that the terminal node n_0 between s^0_m and s^0_{m+1} *abuts an infinite extension*, and that n_0 *separates* s^0_m and s^0_{m+1}. We also say that s^1 *embraces* itself, all its 0-sequences, and all the nodes of its 0-sequences.

Any 0-sequence s^0 can be treated as a singleton 1-sequence $s^1 = \{s^0\}$; in this case, we say that the *minimum rank* of $s^1 = \{s^0\}$ is 0. Furthermore, if s^0 is also a singleton $\{n\}$, we have the *trivial* 1-sequence $s^1 = \{\{n\}\}$.

Let $\mathcal{E}(s^1)$ denote the set of all nodes in all the 0-sequences in s^1, and endow $\mathcal{E}(s^1)$ with the total ordering induced by the orderings of s^1 and its 0-sequences. $\mathcal{E}(s^1)$ will be called the *elementary set* of s^1. Note also that each 0-sequence s^0_m in (5) is maximal with respect to $\mathcal{E}(s^1)$. If s^0_m is not a first or last maximal 0-sequence in s^1 and is not a singleton, then there are exactly two nodes that separate s^0_m from all other maximal 0-sequences in s^1.

Example. A finite 1-sequence having four members is

$$\begin{aligned} s^1 &= \{s^0_1, s^0_2, s^0_3, s^0_4\} \\ &= \{\{n_1, n_2, n_3, \cdots\}, \{n_a, n_b, n_c, \cdots\}, \{n_x\}, \{\cdots, n_\alpha, n_\beta, n_\gamma, \cdots\}\} \end{aligned}$$

Here, $s^0_3 = \{n_x\}$ is a trivial 0-sequence. Note that n_a and n_x are the nodes that abut infinite extensions and separate s^0_2 from the other s^0_m. The elementary set for s^1 is

$$\mathcal{E}(s^1) = \{n_1, n_2, n_3, \ldots, n_a, n_b, n_c, \cdots, n_x, \cdots, n_\alpha, n_\beta, n_\gamma, \cdots\}.$$

Also note that each s_i^0 is maximal with respect to $\mathcal{E}(s^1)$; that is, we cannot contiguously extend any s_i^0 within $\mathcal{E}(s^1)$ as a 0-sequence. ♣

For any given 1-sequence s^1, let us imagine that a branch has been inserted between every two adjacent nodes embraced by $\mathcal{E}(s^1)$. This yields a 1-path P^1 [2, Section 3]. Indeed, it is a routine matter to check that all the conditions in the definition of a 1-path are fulfilled. The nodes of $\mathcal{E}(s^1)$ that abut infinite extensions take the roles of the 1-nodes in P^1, and all other nodes of $\mathcal{E}(s^1)$ become the 0-nodes embraced by P^1. Also, distinct nodes in $\mathcal{E}(s^1)$ are taken to be totally disjoint nodes in P^1.

Conversely, given any 1-path P^1 in a k-graph $\mathcal{G}^k$, the maximal nodes in $\mathcal{G}^k$ that P^1 nodally meets comprise the elementary set $\mathcal{E}(s^1)$ of a 1-sequence s^1. This fact follows from [2, Proposition 4.2]. Moreover, we can uniquely specify P^1 by specifying the said maximal nodes in $\mathcal{G}^k$ — so long as a 1-path is truly obtained thereby.

Lemma 5.1. *Let $\mathcal{A}$ be an infinite subset of the elementary set $\mathcal{E}(s^1)$ of a given 1-sequence s^1. With $\mathcal{A}$ endowed with the ordering induced by $\mathcal{E}(s^1)$, assume that, for every strictly increasing (strictly decreasing), ordinary, infinite sequence $\{a_i\} \subset \mathcal{A}$, the set $\{s \in \mathcal{A} : s > a_i \ \forall i\}$ has a minimum member a (respectively, $\{s \in \mathcal{A} : s < a_i \ \forall i\}$ has a maximum member a). Then, $\mathcal{A}$ is the elementary set of a 1-sequence (whose minimum rank may be 0).*

Note. The hypothesis concerning a can be restated as follows: Given $\{a_i\}$ as stated, there exists an $a \in \mathcal{A}$ with $a_i < a \leq s$ for all i and for all $s \in \mathcal{A}$ such that $s > a_i$ for all i (respectively, there exists an $a \in \mathcal{A}$ with $s \leq a < a_i$ for all i and for all $s \in \mathcal{A}$ such that $s < a_i$ for all i). Note also that $\{a_i\}$ need not be one of the members of s^1.

Proof. $\mathcal{A}$ has the structure of a 0-sequence of 0-sequences (perhaps just a single 0-sequence alone) because $\mathcal{E}(s^1)$ has that structure. We have to show that in $\mathcal{A}$ infinite extensions are separated by nodes. Let A_m^0 and A_{m+1}^0 be any two adjacent maximal 0-sequences in $\mathcal{A}$. Assume that A_m^0 extends infinitely rightward. Let $\{a_i\}$ be a strictly increasing, infinite subsequence of A_m^0. By hypothesis, there exists an $a \in \mathcal{A}$ such that $a_i < a \leq s$ for all i and for all $s \in A_{m+1}^0$. It follows that $a \notin A_m^0$ and that a is a member of A_{m+1}^0 lying to the left of all other members of A_{m+1}^0. Hence, A_{m+1}^0 does not extend infinitely leftward. Its leftmost node a is the node we seek. A similar argument works when A^0_{m+1} extends infinitely leftward. ♣

As before, let $\mathcal{S}$ be a totally ordered set of nodes and let s^1 be a 1-sequence such that $\mathcal{E}(s^1) \subset \mathcal{S}$. s^1 is called *maximal with respect to* $\mathcal{S}$ (or simply *maximal* when it is clear what $\mathcal{S}$ is) if there does not exist any node n_0 in $\mathcal{S}\backslash\mathcal{E}(s^1)$ such that $\{n_0\} \cup \mathcal{E}(s^1)$ with the ordering induced by $\mathcal{S}$ is the elementary set of a 1-sequence.

p-sequences:

A "2-sequence" can be defined as a 0-sequence of 1-sequences such that infinite extensions are separated by nodes. In fact, our definitions can be extended recursively to obtain a "p-sequence" for any natural number p. To this end, let us now assume that q-sequences have been defined for $q = 0, 1, \cdots, p-1$, where $p \geq 2$. Consider a 0-sequence of $(p-1)$-sequences s^{p-1}_m.

$$s^p \;=\; \{\cdots, s^{p-1}_m, s^{p-1}_{m+1}, \cdots\} \tag{6}$$

(We allow s^p to be a singleton.) By recursion each s^{p-1}_m is a 0-sequence of $(p-2)$-sequences, which in turn are 0-sequences of $(p-3)$-sequences, and so forth down to 0-sequences of nodes. We shall say that s^p *embraces* itself, all its members, all members of its members, and so on down to the said nodes. Let $\mathcal{E}(s^p)$ be the set of all nodes embraced by s^p. We call $\mathcal{E}(s^p)$ the *elementary set* of s^p. $\mathcal{E}(s^p)$ has the total ordering endowed by this recursive sequences-of-sequences structure.

Let $\mathcal{S}$ be a superset of s^{p-1}_m, where $\mathcal{S}$ has a total ordering that is compatible with that of s^{p-1}_m. For example, $\mathcal{S}$ may be $\mathcal{E}(s^p)$. s^{p-1}_m is called *maximal with respect to* $\mathcal{S}$ if there does not exist any node n_0 in $\mathcal{S}\backslash\mathcal{E}(s^{p-1})$ such that $\{n_0\} \cup \mathcal{E}(s^{p-1})$ with the ordering induced by $\mathcal{S}$ is the elementary set of a $(p-1)$-sequence. (So far, this definition has been explicated for $p-1 = 0, 1$, and it will become explicitly defined for $p-1 > 1$ when we complete our recursive definitions.)

Furthermore, we shall say that s^{p-1}_m *extends infinitely leftward (rightward)* if s^{p-1}_m extends in that direction through an infinity of $(p-2)$-sequences that are maximal with respect to $\mathcal{E}(s^{p-1}_m)$. On the other hand, we shall say that s^{p-1}_m *terminates on the left (right) at a node* n_0 if there exists a node $n_0 \in \mathcal{E}(s^p)$ such that n_0 is embraced by s^{p-1}_m and no other node embraced by s_m^{p-1} lies to the left (right) of n_0. This occurs when and only when s_m^{p-1} contains a leftmost (rightmost) maximal $(p-2)$-sequence, which in turn contains a leftmost (rightmost) maximal $(p-3)$-sequence, and so on down to a leftmost (rightmost) maximal 0-sequence, which terminates on the left (right) at a node n_0. n_0 is called a *terminal node* of s_m^{p-1}. As a particular case, all these leftmost (rightmost) sequences may be trivial sequences of the form $\{\cdots\{n_0\}\cdots\}$.

Consider again s^p as given by (6), where $p \geq 2$. A *p-sequence* s^p is a 0-sequence of $(p-1)$-sequences such that the following conditions hold for every two adjacent members s_m^{p-1} and s_{m+1}^{p-1} of s^p:

Conditions 5.2. *Either s_m^{p-1} extends infinitely rightward and s_{m+1}^{p-1} terminates on the left at a node, or s_m^{p-1} terminates on the right at a node and s_m^{p-1} extends infinitely leftward.*

This definition insures that each s_m^{p-1} is truly maximal as a $(p-1)$-sequence with respect to $\mathcal{E}(s^p)$, which in turn insures that the representation (6) of s^p is unique under Conditions 5.2.

The statement that *infinite extensions embraced by s^p are separated by nodes* will mean that Conditions 5.2 hold not only for s^p but also for all maximal q-sequences embraced by s^p, where $q = 1, \cdots, p-1$. Moreover, any node that separates an infinite extension from its adjacent sequence of whatever rank will be said to *abut an infinite extension* and to *separate* adjacent maximal sequences.

Example. An illustration of this structure for a 3-sequence is indicted in Figure 3. So as not to clutter the diagram too much, we have deleted many of the subscripts. In that diagram, s_1^2 is a singleton 2-sequence $\{s_1^1\}$. Both s_1^2 and s_1^1 terminate on the right at the node n_0, which is the sole member of the singleton 0-sequence s_1^0; n_0 separates s_1^2 and s_2^2, as well as other maximal sequences of lower rank. For instance, according to our terminology, n_0 separates $\{n_0\}$ from all the other maximal 0-sequences. On the other hand, s_2^2 extends infinitely leftward. $\mathcal{E}(s^3)$ consists of the nodes at the lowest level of this diagram. Note that every infinite extension at that level has an abutting node. ♣

When a p-sequence s^p is a singleton $\{s^{p-1}\}$, its single member s^{p-1} may in fact represent a single q-sequence where $q < p-1$; for example,

$$s^p = \{s^{p-1}\} = \{\{s^{p-2}\}\} = \cdots = \{\cdots\{s^q\}\cdots\}$$

$$= \{\cdots\{\cdots, s_m^{q-1}, s_{m+1}^{q-1}, \cdots\}\cdots\}.$$

The minimum natural number q, for which either s^q has two or more members or $q = 0$, will be called the *minimum rank* of s^p. When $q = 0$ and in addition s^0 is a singleton (i.e., $s^p = \{\cdots\{n_0\}\cdots\}$), we have the *trivial* p-sequence.

Given any p-sequence s^p, let us imagine again that a branch has been inserted between every two adjacent nodes embraced by s^p. It is easy to check that the result is a p-path P^p [2, Section 4]. The embraced nodes of s^p that abut infinite extensions take the role of the embraced q-nodes ($0 < q \leq p$) of P^p. Conversely, given any p-path P^p in a k-graph $\mathcal{G}^k$, the maximal nodes in $\mathcal{G}^k$ that P^p meets nodally comprise the elementary set $\mathcal{E}(s^p)$ of a p-sequence s^p; this too is a consequence of [2, Proposition 4.2]. Moreover, a p-path in $\mathcal{G}^k$ can be specified by identifying the p-sequence of maximal nodes that the p-path nodally meets so long as a p-path is in fact obtained that way.

Example. In some k-graph let the following be a finite 3-path.

$$P^3 = \{n_0^2, P_0^2, n_1^3, P_1^2, n_2^3, P_2^2, n_3^3, P_3^2, n_4^3\}$$

where

$$P_0^2 = \{n_0^2, P_0^1, n_1^2, P_1^1, \cdots\}$$

is a one-ended 2-path with n_0^2 embracing a 1-tip of P_0^1 and n_1^3 embracing the 2-tip of P_0^2,

$$P_1^2 = \{n_a^2, P_a^1, n_b^2\}$$

is a finite 2-path with n_1^3 embracing n_a^2 and n_2^3 embracing n_b^2,

$$P_2^2 = \{\cdots, n_\alpha^2, P_\alpha^1, n_\beta^2, P_\beta^1, \cdots\}$$

is an endless 2-path with n_2^3 embracing a 2-tip of P_2^2 and n_3^3 embracing the other 2-tip of P_2^2, and finally

$$P_3^2 = \{\cdots, n_x^2, P_x^1, n_y^2, P_y^2, \cdots\}$$

is an endless 2-path whose 2-tips are embraced by n_3^3 and n_4^3. The set of maximal nodes that P^3 meets with embrace nodes is the elementary set of a 3-sequence

$$s^3 = \{s_0^2, s_1^2, s_2^2, s_3^2, s_4^2, s_5^2\}$$

where each $\mathcal{E}(s_i^2)$ consists of the maximal nodes met nodally by the following paths: For s_0^2 we have P_0^2. For s_1^2 we have P_1^2; note that n_1^3 and n_a^2 are embraced by the same maximal node, and similarly for n_2^3 and n_b^2. For s_2^2 we have P_2^2. For s_3^2 we have the trivial sequence consisting only of the maximal node that embraces n_3^3. For s_4^2 we have P_3^2. Finally, for s_5^2 we have a trivial sequence again embracing n_4^3 alone.

Note also that we can reverse this discussion. Starting with s^3 we can insert branches between adjacent nodes in $\mathcal{E}(s^3)$ to obtain P^3. ♣

Lemma 5.3. *Let $\mathcal{A}$ be an infinite subset of the elementary set $\mathcal{E}(s^p)$ of a given p-sequence s^p. With $\mathcal{A}$ endowed with the ordering induced by $\mathcal{E}(s^p)$, assume that, for every strictly increasing (strictly decreasing), ordinary, infinite sequence $\{a_i\} \subset \mathcal{A}$, the set $\{s \in \mathcal{A} : s > a_i \ \forall i\}$ has a minimum member a (respectively, $\{s \in \mathcal{A} : s < a_i \ \forall i\}$ has a maximum member a). Then, $\mathcal{A}$ is the elementary set of a p-sequence (whose minimum rank may be less than p).*

Proof. $\mathcal{A}$ will have the structure of a hierarchy of embraced sequences because $\mathcal{E}(s^p)$ has that structure. The rank of that hierarchy (that is, the number of levels within it minus one — see Figure 3) cannot be larger than p. We have to show that in $\mathcal{A}$ infinite extensions are separated by nodes. We can do so inductively. Arguing as in the proof of Lemma 5.1, we show that this is true for the maximal 0-sequences in $\mathcal{A}$. Next, for $q \leq p$, assume that this is true for all maximal ν-sequences where $\nu = 0, \cdots, q-1$. Let A_m^{q-1} and A_{m+1}^{q-1} be two adjacent maximal $(q-1)$-sequences in $\mathcal{A}$. For definiteness, assume that A_m^{q-1} extends infinitely rightward. Thus,

$$A_m^{q-1} = \{\cdots, A_i^{q-2}, A_{i+1}^{q-2}, \cdots\},$$

where the integers $i, i+1, \cdots$ extend infinitely rightward. Choose $a_i \in \mathcal{E}(A_i^{q-2})$ for all i. Thus, $\{a_i\}$ is a strictly increasing 0-sequence in $\mathcal{A}$. Let a be the node specified in the hypothesis. We can conclude that a is not embraced by A_m^{q-1}. Moreover, since $a \leq s$ for all

$s \in A^{q-1}_{m+1}$, A^{q-1}_{m+1} terminates on the left at a. This shows that the infinite extensions of the $(q-1)$-sequences in $\mathcal{A}$ are separated by nodes. By induction this is so for all $q = 0, \cdots, p$. (It may happen that, for some $q < p$, there will be only one q-sequence, in which case there will be nothing to prove for ranks higher than q.) ♣

$\vec{\omega}$-sequences:

Consider the following one-ended 0-sequence

$$s^{\vec{\omega}} = \{s_0^{p_0}, s_1^{p_1}, s_2^{p_2}, \cdots\} \tag{7}$$

of p_m-sequences $s_m^{p_m}$ of varying ranks p_m. The words and notations: "embraces", "elementary set $\mathcal{E}(s^{\vec{\omega}})$", "$s_m^{p_m}$ extends infinitely leftward (rightward)", "$s_m^{p_m}$ terminates on the left (right)", and "a node abuts and infinite extension and separates maximal sequences" are defined exactly as they were for p-sequences except that now p is replaced by $\vec{\omega}$ and s_m^{p-1} by $s_m^{p_m}$. In the same way, we speak of $s_m^{p_m}$ being "maximal with respect to some superset $\mathcal{S}$ of $\mathcal{E}(s_m^{p_m})$", it being understood that $\mathcal{S}$ has a compatible ordering; for example, $\mathcal{S}$ may be $\mathcal{E}(s^{\vec{\omega}})$.

A *rightward $\vec{\omega}$-sequence* is an infinite 0-sequence of the form (7), wherein $\max(p_0, \cdots, p_m) \to \infty$ as $m \to \infty$ and every two adjacent members $s_m^{p_m}$ and $s_{m+1}^{p_{m+1}}$ $(m \geq 0)$ satisfy Conditions 5.2 with s_m^{p-1} replaced by $s_m^{p_m}$ and s_{m+1}^{p-1} by $s_{m+1}^{p_{m+1}}$.

A *leftward $\vec{\omega}$-sequence* is an infinite 0-sequence:

$$s^{\vec{\omega}} = \{\cdots, s_{-3}^{p_{-3}}, s_{-2}^{p_{-2}}, s_{-1}^{p_{-1}}\} \tag{8}$$

of p_{-m}-sequences, where now $\max(p_{-1}, \cdots, p_{-m}) \to \infty$ as $m \to \infty$ and every two adjacent members $s_{-m}^{p_{-m}}$ and $s_{-m+1}^{p_{-m+1}}$ $(m \geq 2)$ satisfy Conditions 5.2 with s_m^{p-1} replaced by $s_{-m}^{p_{-m}}$ and s_{m+1}^{p-1} by $s_{-m+1}^{p_{-m+1}}$.

Finally, an *endless $\vec{\omega}$-sequence* is the conjunction of a leftward $\vec{\omega}$-sequence and a rightward $\vec{\omega}$-sequence:

$$s^{\vec{\omega}} = \{\cdots, s_{-1}^{p_{-1}}, s_0^{p_0}, s_1^{p_1}, \cdots\}$$

Here, the leftward part (7) and rightward part (8) of this 0-sequence satisfy the corresponding conditions given above. Moreover, $s_{-1}^{p_{-1}}$ and $s_0^{p_0}$ satisfy Conditions 5.2 with s_m^{p-1} replaced by $s_{-1}^{p_{-1}}$ and s_{m+1}^{p-1} by $s_0^{p_0}$.

Altogether then, an $\vec{\omega}$*-sequence* is one of these three kinds of sequences. Note that an $\vec{\omega}$-sequence is always an infinite 0-sequence of sequences — never a finite one. As a result, no p-sequence can be represented as a singleton $\vec{\omega}$-sequence; there are no singleton $\vec{\omega}$-sequences.

Lemma 5.4. *Let $\mathcal{A}$ be an infinite subset of the elementary set $\mathcal{E}(s^{\vec{\omega}})$ of a given $\vec{\omega}$-sequence $s^{\vec{\omega}}$. With $\mathcal{A}$ endowed with the ordering induced by $\mathcal{E}(s^{\vec{\omega}})$, assume that, for every strictly increasing (strictly decreasing), ordinary, infinite sequence $\{a_i\} \subset \mathcal{A}$, the set $\{s \in \mathcal{A} : s > a_i \ \forall i\}$ has a minimum member a (respectively, $\{s \in \mathcal{A} : s < a_i \ \forall i\}$ has a maximum member a). Then, $\mathcal{A}$ is the elementary set of a ρ-sequence, where ρ is either a natural number or $\vec{\omega}$.*

The hypothesis of this lemma reads exactly like that of Lemma 5.3 except that p is replaced by $\vec{\omega}$. Its proof is also the same as that of Lemma 5.3 except for some obvious modifications.

Here too, we can relate $\vec{\omega}$-sequences to the maximal nodes in a k-graph that an $\vec{\omega}$-path [2, Section 5] meets nodally. For instance, an $\vec{\omega}$-sequence becomes an $\vec{\omega}$-path when branches are connected between adjacent nodes in the $\vec{\omega}$-sequence. (There is an unimportant variation between the definitions of $\vec{\omega}$-sequences and $\vec{\omega}$-paths: For $\vec{\omega}$-paths, the ranks p_m are required to be strictly monotone for $m \geq 0$ and also for $m < 0$. However, by combining contiguous sequences in $s^{\vec{\omega}}$ appropriately, we get the needed monotonicities in the ranks.)

ω-sequences:

Finally, consider a (finite, one-ended, or endless) 0-sequence of the form

$$s^{\omega} \;=\; \{\cdots, s_m^{\rho_m}, s_{m+1}^{\rho_{m+1}}, \cdots\} \tag{9}$$

where each $s_m^{\rho_m}$ is a ρ_m-sequence whose rank ρ_m is either a natural number or $\vec{\omega}$. Again the definitions of "embrace", "elementary set $\mathcal{E}(s^{\omega})$", "maximal member $s_m^{\rho_m}$ with respect to $\mathcal{E}(s^{\omega})$", "$s_m^{\rho_m}$ terminates on the left (right) at a node", and "a node abuts an infinite extension and separates maximal sequences" read exactly as they do for p-sequences except for changes in notation. For instance, p is replaced by ω and $p-1$ by ρ_m. On the other hand, when $\rho_m = \vec{\omega}$, "$s_m^{\rho_m}$ extends infinitely leftward (rightward)" will now mean that $s_m^{\rho_m}$ is either a leftward (rightward) $\vec{\omega}$-sequence or an endless $\vec{\omega}$-sequence.

An ω*-sequence* is a 0-sequence of the form (9) such that every two adjacent members

$s_m^{\rho_m}$ and $s_{m+1}^{\rho_{m+1}}$ satisfy Conditions 5.2 with s_m^{p-1} replaced by $s_m^{\rho_m}$ and s_{m+1}^{p-1} by $s_{m+1}^{\rho_{m+1}}$. As a special case, an ω-sequence may be a singleton whose minimum rank is a natural number.

As before, an ω-sequence can be related to the set of maximal nodes in a k-graph that an ω-path [2, Section 5] meets nodally.

The same proof as that for Lemmas 5.3 and 5.4 yields

Lemma 5.5. *Invoke the hypothesis of Lemma 5.4 with $\vec{\omega}$ replaced by ω. Then, $\mathcal{A}$ is the elementary set of a ρ-sequence, where ρ is either a natural number or $\vec{\omega}$ or ω.*

Lemma 5.6. *Let P^ρ be a finite ρ-path and let Q^μ be a finite μ-path in a k-graph $\mathcal{G}^k$. Let t^{ρ_1} denote any arbitrary tip embraced and traversed by P^ρ (thus, $\rho_1 < \rho$) and let t^{μ_1} denote any arbitrary tip embraced and traversed by Q^μ (thus, $\mu_1 < \mu$). Assume that t^{ρ_1} and t^{μ_1} are shorted together whenever they are nondisconnectable. Let $\{n_i\}_{i\in I}$ be the set of maximal nodes that P^ρ and Q^μ both meet nodally and let $\{n_i\}$ have the total ordering induced by P^ρ. Then, $\{n_i\}$ is the elementary set of a ν-sequence, where $\nu \le \rho$.*

Proof. If $\{n_i\}$ is a finite set, it is the elementary set of a 0-sequence. So, assume that $\{n_i\}$ is an infinite set. Choose any ordinary, strictly increasing sequence $\{n_{i_j}\}_{j=1}^{\infty}$ in $\{n_i\}$. Set $m_1 = n_{i_1}$. Now, starting at n_{i_1}, trace along Q^μ. In at least one of the two possible directions of tracing Q^μ from n_{i_1}, Q^μ will meet an infinity of the n_{i_j}. Choose such a direction. In accordance with that direction of tracing, let m_2 be the first node in $\{n_{i_j}\}_{j=1}^{\infty}$ after m_1 that Q^μ meets. More generally, for each integer $l > 1$, let m_l be the first node in $\{n_{i_j}\}_{j=1}^{\infty}$ after m_{l-1} that Q^μ meets when tracing along Q^μ Then, $\{m_l\}_{l=1}^{\infty}$ is a strictly increasing sequence in $\{n_i\}_{i\in I}$ such that both P^ρ and Q^μ meet the m_l in the order given. Also, since P^ρ and Q^μ are finite (as a ρ-path and as a μ-path), neither ρ nor μ can be $\vec{\omega}$ (Lemma 4.4). By Lemmas 4.2 and 4.3, P^ρ traverses and embraces a ρ_1-tip t^{ρ_1} ($\rho_1 < \rho$) with a representative that meets all but possibly finitely many of the m_l; moreover, the m_l approach t^{ρ_1}. By the same lemmas, Q^μ traverses and embraces a μ_1-tip τ^{μ_1} ($\mu_1 < \mu$) with a representative that meets all but possibly finitely many of the m_l; also, the m_l approach τ^{μ_1}. Thus, t^{ρ_1} and τ^{μ_1} are nondisconnectable. Hence, they are shorted together, and the maximal node n_x that shorts them is met nodally by both P^ρ and Q^μ according to Lemma 4.1. Thus, we have that $n_{i_j} < n_x$ for all j and $n_x \le s$ for all s in $\{n_i\}$ such that $s > n_{i_j}$ for all j. That is, the set $\{s \in \{n_i\} : s > n_{i_j}\ \forall j\}$ has a minimum member n_x. (The analogous conclusion would hold had $\{n_{i_j}\}_{j=1}^{\infty}$ been chosen strictly decreasing.) Finally, recall that the set of all maximal nodes that P^ρ meets nodally is the elementary set of a ρ-sequence ($\rho = p$ or ω). Thus, the hypothesis of Lemma 5.1 or 5.3 or 5.4 or 5.5 holds with respectively $\mathcal{E}(s^1)$ or $\mathcal{E}(s^p)$ or $\mathcal{E}(s^{\vec{\omega}})$ or $\mathcal{E}(s^\omega)$ being the set of all maximal nodes that P^ρ meets nodally, with $\mathcal{A}$ being the set $\{n_i\}$, with $\{a_i\}$ being $\{n_{i_j}\}$, and with n_x being a. (The m_l were only used to find n_x.) By those lemmas, we can conclude that $\{n_i\}$ is a ν-sequence with $\nu \leq \rho$. ♣

6 Transitivity of k-Connectedness and Partitioning of Transfinite Graphs by Sections

In this section we shall show that, under the following Condition 6.1, the ρ-sections (for a given rank $\rho \leq k$) partition a transfinite graph $\mathcal{G}^k$ ($k \leq \omega$) because ρ-connectedness is then transitive.

We shall say that a tip is *open* if it is not shorted to any other tip — including any elementary tip of a branch; in other words, a tip is open if and only if it is embraced by only one node and that node is a singleton.

Condition 6.1. *If two tips (of possibly differing ranks) are nondisconnectable, then either the two tips are shorted together (i.e., they are both embraced by some node) or at least one of them is open.*

Theorem 6.2. *Let $\mathcal{G}^k$ be a k-graph that satisfies Condition 6.1. Let n_a, n_b, and n_c be distinct 0-nodes of $\mathcal{G}^k$ such that n_a and n_b are p-connected, and n_b and n_c are p-connected. Then, n_a and n_c are p-connected.*

Proof. There is a finite p-path P^p that terminates at n_a and n_b. Also, there is a finite p-path Q^p that terminates at n_b and n_c. Let $\{n_i\}_{i \in I}$ be the maximal nodes met nodally by both P^p and Q^p, but let $\{n_i\}$ be totally ordered in accordance with a tracing of P^p from n_b to n_a. The 0-node n_b is embraced by a maximal node in $\{n_i\}$, which we shall also denote by n_b.

If $\{n_i\}$ is a finite set, there will be a last node n_x in it. Then, a tracing of P^p from n_a

to n_x followed by a tracing of Q^p from n_x to n_c will be a finite p-path terminating at n_a and n_c. Hence, n_a and n_c are p-connected.

Now, assume that $\{n_i\}_{i\in I}$ is an infinite set. It is no loss of generality to assume that n_a, n_b, and n_c are all nonsingletons, for we can always append a self-loop to any one of those nodes to make it a nonsingleton. That self-loop will not affect the connectedness between n_a, n_b, and n_c. Now, no tip traversed by P^p (or Q^p) can be open because that tip will also be embraced by P^p (or respectively Q^p) according to Lemma 4.1 and every node embraced by P^p (or Q^p) will be a nonsingleton. Hence, by Condition 6.1, any tip traversed by P^p and any tip traversed by Q^p that are nondisconnectable will be shorted together. Therefore, by Lemma 5.6, $\{n_i\}$ is the elementary set of a q-sequence ($q \leq p$).

Thus, either $\{n_i\}$ has a last node n_x or it (that is, the said q-sequence) extends infinitely rightward through an infinite sequence $\{s_l^r\}_{l=1}^{\infty}$ of maximal r-sequences s_l^r, where $r < q$. (Here, s_l^r is a singleton 0-node if $q = 0$. Also, we have an illustration in Figure 3 for the case where $q = 3$ and $r = 0$.) Suppose $\{n_i\}$ does not have a last member, that is, it extends infinitely rightward as stated. For each natural number l, choose a node m_l that is embraced by s_l^r. Thus, $\{m_l\}_{l=1}^{\infty}$ is a sequence in $\{n_i\}_{i\in I}$ such that no node of $\{n_i\}$ lies to the right of all the m_l. As in the proof of Lemma 5.6, we can choose a subsequence $\{m_{l_\lambda}\}_{\lambda=1}^{\infty}$ of $\{m_l\}$ such that Q^p meets the m_{l_λ} in sequence, that is, in the same order that P^p meets the m_{l_λ}. So, by Lemma 4.2, P^p traverses a tip t^ρ and Q^p traverses a tip τ^μ such that the m_{l_λ} approach both tips. Hence, t^ρ and τ^μ are nondisconnectable. We have already noted that neither of them are open. By Condition 6.1, there is a node n_x that embraces both of them. Moreover, $n_x \in \{n_i\}$. Indeed, P^p being a finite path, embraces every tip it traverses (Lemma 4.1) and thus embraces a node that embraces such a tip; similarly for Q^p.

Finally, n_x lies to the right of all the m_{l_λ}, therefore to the right of all the m_l, and therefore to the right of all the nodes in $\{n_i\}$ according to our supposition that $\{n_i\}$ extends infinitely rightward. This is a contradiction. It follows that $\{n_i\}$ does have a last member n_x. We can now conclude as before that n_a and n_c are p-connected. ♣

Corollary 6.3. *Theorem 6.2 remains true when p is replaced either by $\vec{\omega}$ or ω.*

Proof. First replace p by $\vec{\omega}$. By the definition of $\vec{\omega}$-connectedness, there is a p such that n_a and n_b are p-connected and so too are n_b and n_c. By Theorem 6.2, n_a and n_c are p-connected and therefore $\vec{\omega}$-connected.

When p is replaced by ω, the proof that n_a and n_c are ω-connected is the same as that of Theorem 6.2 except for some obvious modifications in wording and notations, the use of Lemma 4.3 in place of Lemma 4.2, and the following additional alteration: When $\{n_i\}$ is taken to extend infinitely rightward, it may do so either through an infinite sequence $\{s_l^r\}_{l=1}^\infty$ of r-sequences s_l^r as before or through an $\vec{\omega}$-sequence $s^{\vec{\omega}}$ such as (7). In the latter case, we choose each m_l to be a node embraced by $s_l^{p_l}$. ♣

Corollary 6.4. *Let $\mathcal{G}^k$ be a k-graph ($k \leq \omega$) that satisfies Condition 6.1. Let ρ denote either p or $\vec{\omega}$ or ω. Then, the ρ-sections of $\mathcal{G}^k$ comprise a partition of $\mathcal{G}^k$.*

Proof. We need merely show that ρ-connectedness is an equivalence relation between the branches of $\mathcal{G}^k$. Reflexivity and symmetry are obvious. Since two branches are ρ-connected if and only if their 0-nodes are ρ-connected, Theorem 6.2 or Corollary 6.3 asserts the transitivity for ρ-connectedness. ♣

7 Node Voltages for Transfinite Electrical Networks

We turn now to transfinite electrical networks. In particular, let $\mathbf{N}^k$ be a k-network, that is, an electrical network whose graph is a k-graph $\mathcal{G}^k$, as above. The jth branch of $\mathbf{N}^k$ consists of a positive resistance r_j in series with a pure voltage source of real value e_j, which may be 0. The branch conductance is $g_j = 1/r_j$. The branch voltage v_j and branch current i_j are related by $v_j = i_j r_j - e_j$ in accordance with the polarity conventions shown in Figure 4. The branch's orientation is the direction in which current is measured or voltage is measured, that is, from left to right in Figure 4. Also, $-v_j$ is called the *voltage rise* for the branch. The graph-theoretic definitions given above for $\mathcal{G}^k$ are transferred directly to $\mathbf{N}^k$.

We assume henceforth that the voltage-current regime in $\mathbf{N}^k$ is the one specified by the fundamental theorem [2, Theorem 10.2], which invokes the hypothesis that the maximum total power available from all the sources is finite: $\sum e_j^2 g_j < \infty$. In that regime, $\sum i_j^2 r_j < \infty$; that is, the total power dissipation is finite as well. Our first task is to define what we mean by a "node voltage".

Let m and n be two nonembracing nodes of $\mathbf{N}^k$; their ranks need not be the same. Also, let P be a path that meets both m and n terminally; that is, P meets m either with an embraced terminal node or with a tip, and similarly for n. P is called *perceptible* if $\sum_{j\in\Pi} r_j < \infty$, where Π is the index set for all the branches embraced by P; in this case, n is said to be *perceptible from m along P*. Moreover, if P is a representative of a tip t, then t itself is called *perceptible.*

Now, the *algebraic sum of the voltage rises along P from m to n* is

$$\sum_{j\in\Pi} \mp v_j \tag{10}$$

where the minus (plus) sign is used if the branch's orientation agrees (respectively, disagrees) with a tracing of P from m to n. If P is perceptible, (10) converges absolutely. Indeed, for j restricted to Π,

$$\begin{aligned}\sum |v_j| \;=\; \sum |r_j i_j - e_j| \;&\le\; \sum \sqrt{r_j}|i_j|\sqrt{r_j} + \sum \sqrt{g_j}|e_j|\sqrt{r_j} \\ &\le \left[\sum r_j i_j^2 \sum r_j\right]^{1/2} + \left[\sum e_j^2 g_j \sum r_j\right]^{1/2} \;<\; \infty.\end{aligned}$$

If n is perceptible from m along P, then (10) is defined to be the *node voltage of n with respect to m along P*. We also say that *n obtains the node voltage (10) with respect to m along P*. (On the other hand, if m embraces n or conversely, we have a trivial path between m and n, and these definitions hold with (10) equal to 0.) Let us emphasize that by our definition node voltages are assigned only along perceptible paths. For instance, (10) may converge even when P is not perceptible, but in this case we do not use (10) with that P to define a node voltage at n with respect to m.

It can happen that (10) may be different for different perceptible paths between m and n. This will not occur if the ranks of those paths are both zero, but it may occur if at least one of them has a rank of 1 or higher. For example, consider a network whose graph is that of Figure 1 and assume that all branches are purely resistive (i.e., have zero voltage sources). Let the resistances of branches a_j and b_j be $1/2^j$ ohms, where $j = 1, 2, 3, \ldots$, and let the resistances for β_a and β_b be equal to 1 ohm. Finally, append one more branch β_c incident to n_c^0 and n_d^0, oriented from n_d^0 to n_c^0 and having a 1 ohm resistor in series with a 1 volt source. Let $\mathbf{N}^1$ be the resulting 1-network with the appended branch β_c. Every branch of $\mathbf{N}^1$ will have zero current. Indeed, no current can flow in any a_j or b_j because every such branch resides in only one loop and that loop is purely resistive. Similarly, there is no loop that passes through the three β branches because there is no path connecting n_a^1 and n_b^1 through the 0-section of the a_j and b_j branches. Thus, the source in β_c produces a 1 volt rise in voltage from n_d^0 to n_c^0. Now there is a perceptible 0-path P^0 connecting n_1^0 to n_a^1 and passing through the a_j branches only; the node voltage with respect to n_1^0 that n_a^1 obtains along P^0 is 0. Also, there is a perceptible 1-path P^1 that connects n_1^0 to n_a^1, which passes along the b_j branches, through n_b^1, and then along the β branches; the node voltage with respect to n_1^0 that n_a^1 obtains along P^1 is 1.

Thus, it is pertinent to ask when a node n has a unique node voltage with respect to some other node m, that is, when that node voltage does not depend upon the choice of perceptible path between m and n. An answer is given in the next section.

8 Existence and Uniqueness of Node Voltages

We will impose the following Condition 8.1 on all tips in the transfinite electrical network $\mathbf{N}^k$, where $k \leq \omega$. It is not required that the two tips mentioned therein be of the same rank, but it is understood that their ranks are either natural numbers or $\vec{\omega}$.

Throughout this section, we identify each node with the maximal node that embraces it, and any reference to a node will mean that maximal node.

Condition 8.1. *If two tips are perceptible and nondisconnectable, then those tips are shorted together.*

Theorem 8.2. *Assume that the tips of ranks no larger than $\vec{\omega}$ in the k-network $\mathbf{N}^k$ ($k \leq \omega$) satisfy Condition 8.1. Let n_g and n_0 be two nodes (of possibly different ranks), and let there be at least one perceptible path connecting n_g and n_0. Then, n_0 has a unique node voltage with respect to n_g; that is, n_0 obtains the same node voltage with respect to n_g along all perceptible paths between n_g and n_0.*

Proof. Assume that there are at least two perceptible paths P^ρ and Q^μ terminating at n_g and n_0. Orient P^ρ and Q^μ from n_g to n_0. We want to show that n_0 obtains the same node voltage along P^ρ as it does along Q^μ.

We can assume that P^ρ and Q^μ are finite paths, for, if either of them meets n_g and n_0 with tips, we can append n_g and n_0 to that path to obtain a finite path of higher rank.

Let $\{n_i\}_{i\in I}$ be the set of (maximal) nodes met by both P^ρ and Q^μ and let $\{n_i\}$ have the total ordering induced by P^ρ. Since P^ρ and Q^μ are both perceptible, every tip embraced and traversed by either path is also perceptible. Hence, if two such tips are nondisconnectable, they are shorted together according to Condition 8.1. By Lemma 5.6, $\{n_i\}$ is the elementary set of a ν-sequence s^ν, where $\nu \leq \rho$.

Consider any maximal 0-sequence s^0 embraced by s^ν (maximal with respect to $\{n_i\}$), and let n_1 and n_2 be two adjacent nodes in s^0. Then, a tracing from n_1 to n_2 along P^ρ followed by a tracing from n_2 to n_1 along Q^μ will follow a perceptible η-loop L^η, where $\eta \leq \max(\rho, \mu)$. By Kirchhoff's voltage law applied to L^η [2, Theorem 11.2], n_2 obtains the same node voltage with respect to n_1 along P^ρ as it does along Q^μ. Since this is true for all pairs of adjacent nodes in s^0, it is also true when n_1 and n_2 are any two nonadjacent nodes in s^0.

Now, let s^1 be a maximal 1-sequence embraced by s^ν and let s^0 be one of the maximal 0-sequences embraced by s^1. There will be a node n_a (alternatively, n_b) that separates s^0 from all the maximal 0-sequences to the right (left) of s^0 if there are such sequences; otherwise, n_a is identical to n_0 (respectively, n_g). Since P^ρ is perceptible, the voltages along the nodes of s^0 with respect to any fixed node n_1 of s^0 converge to a node voltage for n_a (respectively, n_b). Thus, n_a (respectively, n_b) obtains the same voltage with respect to n_1 along P^ρ as it does along Q^μ. Hence, the same is true for node voltages at n_a and n_b with respect to each other if both n_a and n_b exist.

We can continue this argument inductively. Let us assume that for some natural number p, where $p < \nu$, the following is true for every maximal p-sequence s^p embraced by s^ν.

Inductive assumption: Let n_a (alternatively, n_b) be the node that separates s^p from all the maximal q-sequences ($q \leq p$) to the right (left) of s^p if such sequences exist and let n_1 be any node embraced by s^p; then, n_a (or n_b) obtains the same node voltage with respect to n_1 along P^ρ as it does along Q^μ.

It follows that, if n_a and n_b both exist, each of them obtains the same node voltage with respect to the other along P^ρ as it does along Q^μ. Now, if both n_a and n_b exist, they are adjacent in the sense that they separate a single maximal p-sequence s^p from all other q-sequences ($q \leq p$) not embraced by s^p. On the other hand, if n_a and n_b are not adjacent in this sense, more particularly, if there are finitely many maximal p-sequences between n_a and n_b and if n_a and n_b separate those sequences from all other q-sequences ($q \leq p$) not embraced by any of those p-sequences, then the same conclusion regarding node voltages for n_a and n_b with respect to each other can be drawn. Because P^p is perceptible, we can once again take limits to obtain the above inductive assumption with p replaced by $p+1$. Moreover, if there are two nodes n_a and n_b that separate a maximal $(p+1)$-sequence s^{p+1} from all q-sequences ($q \leq p+1$) not embraced by s^{p+1}, then each node obtains the same node voltage with respect to the other along P^ρ as it does along Q^μ.

This inductive argument can be extended still further to include the cases where n_a and n_b separate a maximal $\vec{\omega}$-sequence $s^{\vec{\omega}}$ from all q-sequences ($q \leq \vec{\omega}$) not embraced by $s^{\vec{\omega}}$, and then finally for ω-sequences s^ω. The conjunction of all these results implies our theorem. ♣

Corollary 8.3. *Assume that the tips of all ranks no larger than $\vec{\omega}$ in the k-network $\mathbf{N}^k$ ($k \leq \omega$) satisfy Condition 8.1. Also, assume tht every two nodes of $\mathbf{N}^k$ are connected through at least one perceptible path. Choose a ground node n_g in $\mathbf{N}^k$ arbitrarily. Then, every node of $\mathbf{N}^k$ has a unique node voltage with respect to n_g.*

References

[1] C.Thomassen, Infinite graphs, *Graph Theory*, **2** (1983), 129-160.

[2] A.H.Zemanian, Transfinite graphs and electrical networks, *Trans. Amer. Math. Soc.*, in press.

[3] A.H.Zemanian *Infinite Electrical Networks*, Cambridge University Press, Cambridge, England, 1991.

FOOTNOTES

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Figure Numbers

(There are no legends for the figures.)

Figure 1.

Figure 2.

Figure 3.

Figure 4.

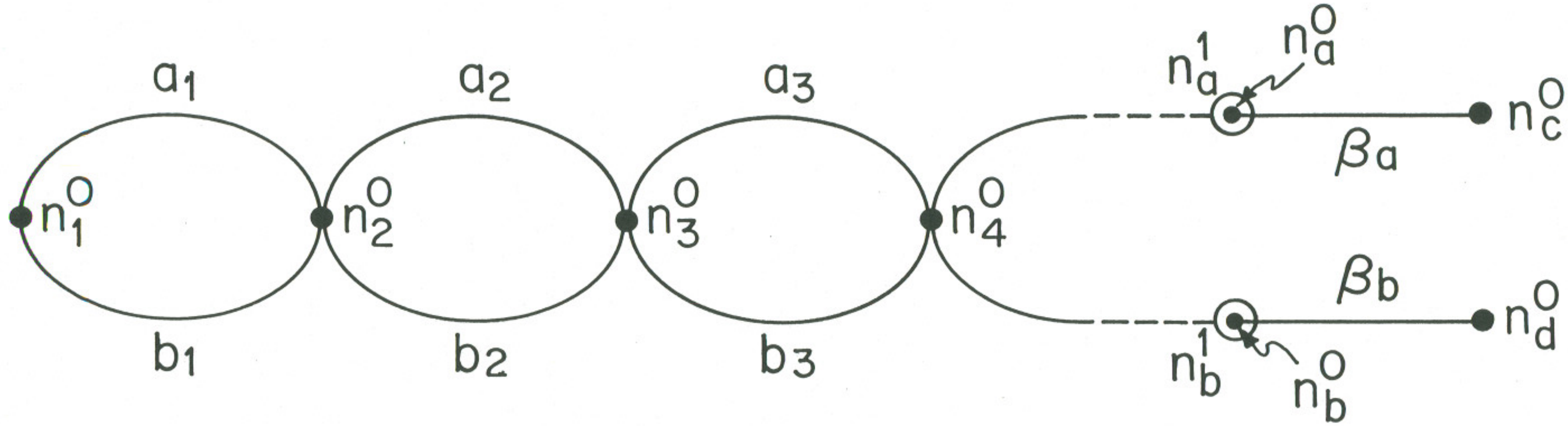

FIG. 1

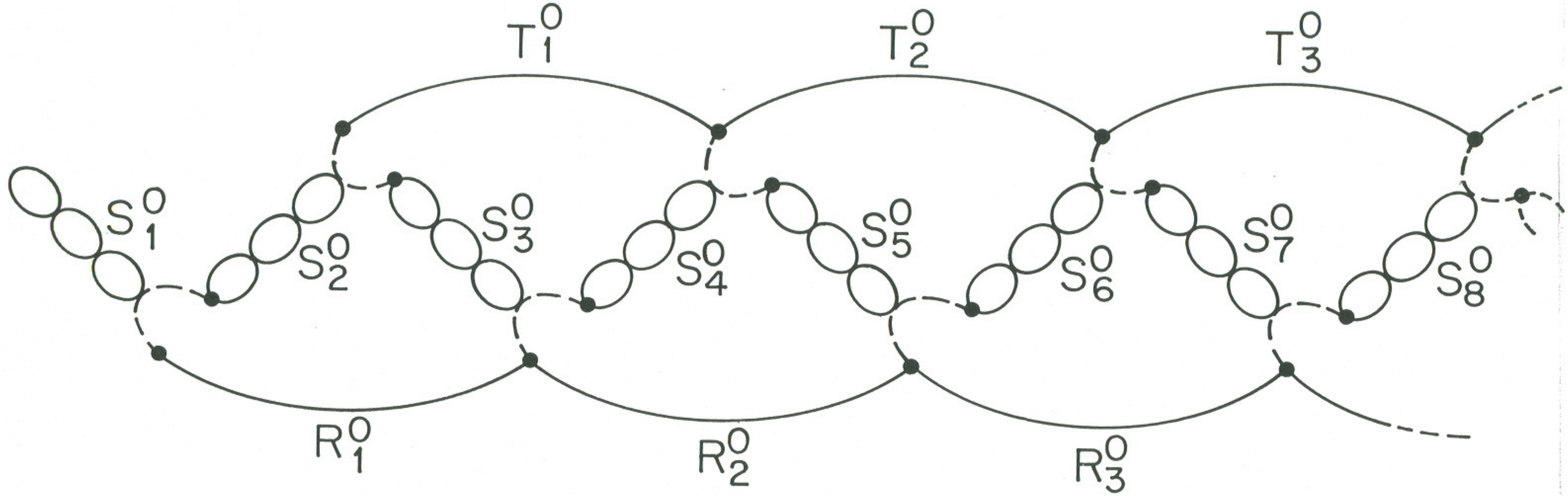

FIG. 2

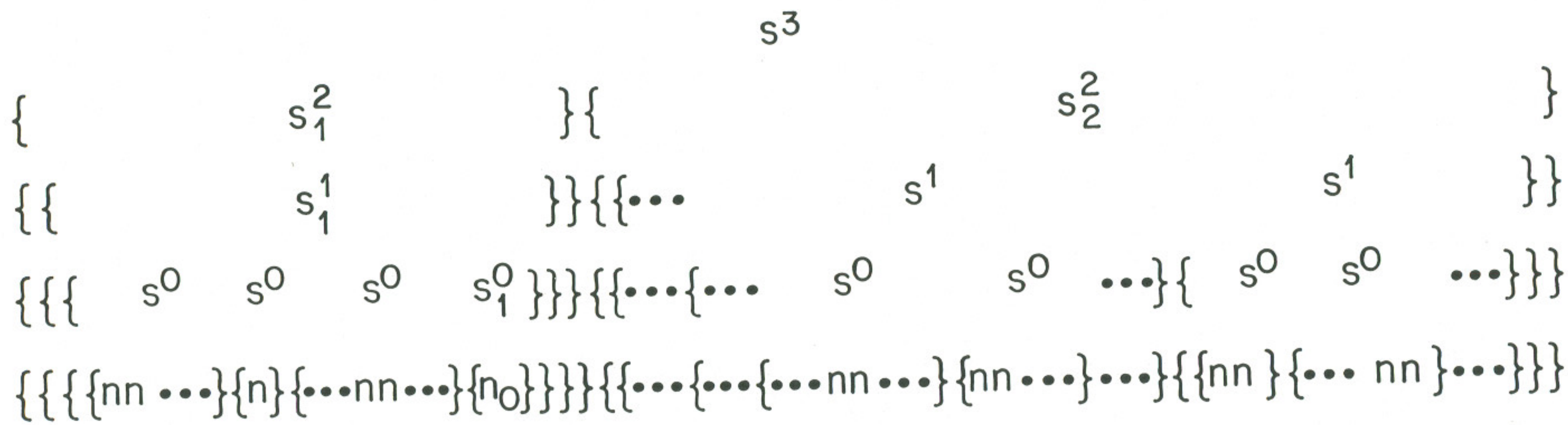

FIG. 3

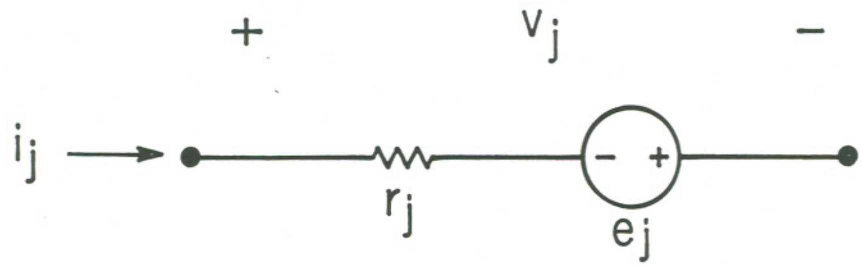

FIG. 4