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Abstract — In a prior work the galaxies of the nonstandard enlargements of conventionally infinite graphs and also of transfinite graphs of the first rank of transfiniteness were defined, examined, and illustrated by some examples. In this work it is shown how the results of the prior work extend to transfinite graphs of higher ranks. Among those results are following principal ones: Any such enlargement either has exactly one galaxy, its principal one, or it has infinitely many such galaxies. In the latter case, the galaxies are partially ordered by there "closeness" to the principal galaxy. Also, certain sets of galaxies that are totally ordered by that "closeness" criterion are identified.

Key Words: Nonstandard graphs, enlargements of graphs, transfinite graphs, galaxies in nonstandard graphs, graphical galaxies.

1 Introduction

In some prior works, the ideas of "nonstandard graphs" [2, Chapter 8] and the "galaxies of nonstandard enlargements of graphs" [3] were defined and examined. However, all this was done only for conventionally infinite graphs and transfinite graphs of the first rank of transfiniteness. The purpose of this work is to define and examine nonstandard transfinite graphs of higher ranks of transfiniteness.

This paper is written as a sequel to [3] and uses a symbolism and terminology consistent with that prior work. We also use a variety of results concerning transfinite graphs, and these may all be found in [2]. We refer the reader to those sources for such information. For instance, the "hyperordinals" are constructed in much the same way as are the hypernaturals, and their definitions are given in [3, Section 5]. Furthermore, we use herein the idea of "wgraphs," which are transfinite graphs based upon walks rather than paths. This avoids some of the difficulties associated with path-based transfinite graphs and is in fact both simpler and more general than the path-based theory of transfinite graphs. Wgraphs and their transfinite extremities are defined and examined in [2, Sections 5.1 to 5.6]. Lengths of walks on wgraphs and "wdistances" based on such lengths are discussed in [2, Sections 5.7 and 5.8]. All this is assumed herein as being known.

Our arguments will be based on ultrapower constructions, and, to this end, we assume throughout that a free ultrafilter $\mathcal{F}$ has been chosen and fixed. Finally, when adding ordinals, we always take it that ordinals are in normal form and that the natural summation of ordinals is being used [1, pages 354-355].

It is a fact about a transfinite graph G^ν of rank ν that it contains subgraphs of all ranks ρ with $0 \leq \rho \leq \nu$, called ρ-sections, that at each rank ρ the ρ-sections are ρ-graphs by themselves and induce a partitioning of the branch set of G^ν, and that the one and only ν-section is G^ν itself. We define the "enlargements" ${}^*G^\nu$ of a transfinite graph G^ν and of its ρ-sections in the next section. The galaxies of all ranks in ${}^*G^\nu$ are defined in Section 3. A galaxy of rank ρ $(0 \leq \rho \leq \nu)$ is called a "ρ-galaxy."

Within the enlargement ${}^*S^\rho$ of an ρ-section S^ρ of G^ν, there is either exactly one ρ-galaxy, the "principal ρ-galaxy," or infinitely many ρ-galaxies in addition to the principal ρ-galaxy. The latter case arises when G^ν is locally finite in a certain way (Section 4), but it may arise in other ways as well. Moreover, the enlargements of all the ρ-sections within a $(\rho+1)$-section lie within the principal $(\rho+1)$-galaxy of the enlargement of that $(\rho+1)$-section, and so on through the sections of higher ranks. In that latter case still, there will be a two-way infinite sequence of ρ-galaxies that are totally ordered according to their "closeness to the principal ρ-galaxy," and there may be many such totally ordered sequences of ρ-galaxies. When there are many ρ-galaxies in ${}^*S^\rho$, they are partially ordered, again according to their closeness to the principal ρ-galaxy (Section 5 again).

2 Enlargements of ν-Graphs and Hyperdistances in the Enlargements

First of all, the enlargement ${}^*G^0 = \{ {}^*X^0, {}^*B \}$ of a conventionally infinite 0-graph and the enlargement ${}^*G^1 = \{ {}^*X^0, {}^*B, {}^*X^1 \}$ of a transfinite 1-graph are discussed in [3, Sections 2 and 8]. These prior constructs will be encompassed by the more general development we now undertake. We shall assume that the rank ν is no larger than ω. The extensions to higher ranks of transfiniteness proceeds in much the same way.

Consider a wconnected transfinite wgraph of rank ν $(0 \leq \nu \leq \omega)$:

$$G^\nu = \{X^0, B, X^1, \ldots, X^\nu\}$$

where X^0 is a set of 0-nodes, B is a set of branches (i.e., two-element sets of 0-nodes), and X^ρ $(\rho = 1, \ldots, \nu)$ is a set of ρ-wnodes. It is assumed that each X^ρ $(\rho = 0, \ldots, \nu)$ is nonempty except possibly for $\rho = \vec{\omega}$. In general, $X^{\vec{\omega}}$ may be empty.

The "enlargement" ${}^*G^\nu$ of G^ν is defined as follows: Two sequences $\langle x_n^\rho \rangle$ and $\langle y_n^\rho \rangle$ of ρ-wnodes in G^ν (i.e., $x_n^\rho, y_n^\rho \in X^\rho$) are taken to be equivalent if $\{n \colon x_n^\rho = y_n^\rho\} \in \mathcal{F}$. This is truly an equivalence relation on the set of sequences of ρ-nodes, as is easily shown. Each equivalence class $\mathbf{x}^\rho$ will be called a *ρ-hypernode* and will be represented by $\mathbf{x}^\rho = [x_n^\rho]$ where the x_n^ρ are elements of any one of the sequences in that equivalence class. We let ${}^*X^\rho$ denote the set of all such equivalence classes (i.e., the set of all ρ-hypernodes). Then, the *enlargement* ${}^*G^\nu$ of G^ν is the set

$${}^*G^\nu = \{ {}^*X^0, {}^*B, {}^*X^1 \ldots, {}^*X^\nu \}.$$

The elements of ${}^*B$ are called *hyperbranches* and have been defined in [2, Section 8.1]. Here, too, ${}^*X^\rho$ is nonempty if $\rho \neq \vec{\omega}$; ${}^*X^{\vec{\omega}}$ may be empty.

Next, we wish to define the "hyperdistances" between the hypernodes of ${}^*G^\nu$. The "length" $|W_{xy}|$ of any two-ended walk W_{xy} terminating at two wnodes x and y of any ranks in G^ν is defined in [2, Section 5.7]. Also, the wdistance $d(x, y)$ between those two wnodes is

$$d(x, y) = \min\{|W_{x,y}|\}$$

where the minimum is taken over the lengths $|W_{xy}|$ of all the walks W_{xy} in G^ν that terminate at x and y. The minimum exists because those lengths comprise a well-ordered set of ordinals. Furthermore, a wnode is said to be *maximal* if it is not embraced by a wnode of higher rank. In these definitions, x and y may be either maximal or nonmaximal wnodes. Note that the wdistance measured from any nonmaximal wnode z is the same as that measured from the maximal wnode x that embraces z. We also set $d(x,x) = 0$. Thus, d is an ordinal-valued metric defined on the maximal wnodes in $\cup_{\rho=0}^{\nu} X^\rho$. (The axioms of a metric are readily verified.)

Given two hypernodes $\mathbf{x} = [x_n]$ and $\mathbf{y} = [y_n]$ of any ranks in ${}^*G^\nu$, we defined the *hyperdistance* $\mathbf{d}$ between them as the internal function

$$\mathbf{d}(\mathbf{x},\mathbf{y}) \;=\; [d(x_n, y_n)].$$

We say that a hypernode $\mathbf{x}^\rho = [x_n^\rho]$ is *maximal* if it is not embraced by a hypernode $\mathbf{y}^\gamma = [y_n^\gamma]$ of higher rank ($\gamma > \rho$) (i.e., x_n^ρ is not embraced by y_n^γ for almost all n). Upon restricting $\mathbf{d}$ to the maximal hypernodes in ${}^*G^\nu$, we have that this restricted $\mathbf{d}$ satisfies the metric axioms except that it is hyperordinal-valued. In particular, we have by the transfer principle that the triangle inequality holds for any three maximal hypernodes $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, namely,

$$\mathbf{d}(\mathbf{x},\mathbf{z}) \;\le\; \mathbf{d}(\mathbf{x},\mathbf{y}) \;+\; \mathbf{d}(\mathbf{y},\mathbf{z}). \tag{1}$$

3 The Galaxies of ${}^*G^\nu$

We continue to assume that the rank ν is no larger than ω. Also, we assume at first that the rank ρ is a natural number no larger than ν. Consider the ν-graph G^ν and its enlargement ${}^*G^\nu$. Two hypernodes $\mathbf{x} = [x_n]$ and $\mathbf{y} = [y_n]$ of any ranks in ${}^*G^\nu$ will be said to be in the same *nodal ρ-galaxy* $\dot{\Gamma}^\rho$ if there exists a natural number μ_{xy} depending on $\mathbf{x}$ and $\mathbf{y}$ such that $\{n\colon d(x_n, y_n) \le \omega^\rho \cdot \mu_{xy}\} \in \mathcal{F}$. In this case, we say that $\mathbf{x}$ and $\mathbf{y}$ are *ρ-limitedly distant.* This defines an equivalence relation on the set $\cup_{\gamma=0}^{\nu} {}^*X^\gamma$ of all the hypernodes in ${}^*G^\nu$, and thus $\cup_{\gamma=0}^{\nu} {}^*X^\gamma$ is partitioned into nodal ρ-galaxies. The proof of this is the same as that given in [3, Section 9] except that the rank 1 therein is now replaced by ρ. By the same arguments, we have that, for $\alpha \le \rho$, the nodal α-galaxies provide a finer partitioning of

$\cup_{\gamma=0}^{\nu} {}^*X^\gamma$ than do the nodal γ-galaxies. Moreover, any nodal ρ-galaxy is partitioned by the nodal α-galaxies ($\alpha < \rho$) in that nodal ρ-galaxy.

Corresponding to each nodal ρ-galaxy $\dot{\Gamma}^\rho$, we define a *ρ-galaxy* Γ^ρ of ${}^*G^\nu$ as the nonstandard subgraph of ${}^*G^\nu$ induced by all the 0-hypernodes in $\dot{\Gamma}^\rho$; that is, along with the hypernodes of $\dot{\Gamma}^\rho$, we have hyperbranches whose incident hypernodes are in $\dot{\Gamma}^\rho$. Note that every hyperbranch must lie in a single ρ-galaxy for every ρ because their incident 0-hypernodes are at a hyperdistance of 1.

Let us now turn to the case where $\rho = \vec{\omega}$. Now, ν is either $\vec{\omega}$ or ω. The definition of the $\vec{\omega}$-galaxies is rather different. Two hypernodes $\mathbf{x} = [x_n]$ and $\mathbf{y} = [y_n]$ in ${}^*G^\nu$ will be said to be in the same *$\vec{\omega}$-galaxy* $\dot{\Gamma}^{\vec{\omega}}$ if there exists a natural number μ_{xy} depending on $\mathbf{x}$ and $\mathbf{y}$ such that $\{n : d(x_n, y_n \leq \omega^{\mu_{xy}}\} \in \mathcal{F}$. Thus, $\mathbf{x}$ and $\mathbf{y}$ are in $\dot{\Gamma}^{\vec{\omega}}$ if and only if $\{n : d(x_n, y_n) < \omega^\omega\} \in \mathcal{F}$. In this case, we say that $\mathbf{x}$ and $\mathbf{y}$ are *$\vec{\omega}$-limitedly distant*. Here, too, the property of being $\vec{\omega}$-limitedly distant defines an equivalence relation on the set of all hypernodes in ${}^*G^\nu$. So, the nodal $\vec{\omega}$-galaxies partition the set of all hypernodes. Then, the $\vec{\omega}$-galaxy $\Gamma^{\vec{\omega}}$ corresponding to any nodal $\vec{\omega}$-galaxy $\dot{\Gamma}^{\vec{\omega}}$ consists of the hypernodes in $\dot{\Gamma}^{\vec{\omega}}$ along with the hyperbranches whose 0-hypernodes are in $\dot{\Gamma}^{\vec{\omega}}$.

Finally, the ω-galaxies of a nonstandard ω-wgraph ${}^*G^\omega$ are defined just as are the ρ-galaxies of natural number ranks. We now require that $\{n : d(x_n, y_n \leq \omega^\omega \cdot \mu_{xy}\} \in \mathcal{F}$ for some natural number μ_{xy} depending upon $\mathbf{x} = [x_n]$ and $\mathbf{y} = [y_n]$ in order for $\mathbf{x}$ and $\mathbf{y}$ to be in the same nodal ω-galaxy. When this is so, we again say that $\mathbf{x}$ and $\mathbf{y}$ are *ω-limitedly distant.* The same partitioning properties hold.

In general now, let G^ν be a ν-graph where $0 \leq \nu \leq \omega$, possibly $\nu = \vec{\omega}$. The *principal ν-galaxy* Γ_0^ν of ${}^*G^\nu$ is that ν-galaxy whose hypernodes are ν-limitedly distant from a standard hypernode of ${}^*G^\nu$. We shall show later on that ${}^*G^\nu$ either has exactly one ν-galaxy, its principal one, or has infinitely many of them.

Let us now recall another definition concerning standard transfinite wgraphs. A *ρ-wsection* S^ρ of G^ν ($\rho < \nu$) is a maximal ρ-wsubgraph of the ρ-wgraph of G^ν that is ρ-wconnected. This *ρ-wconnectedness* means that, for every two wnodes in S^ρ, there is a two-ended α-walk ($\alpha \leq \rho$) terminating at those wnodes. (When $\rho = \vec{\omega}$, we have that

$\alpha < \vec{\omega}$.) Furthermore, the branch set of any ρ-wsection S^ρ is partitioned by the branch sets of the α-wsections lying in S^ρ.

Consider now any $(\rho-1)$-wsection $S^{\rho-1}$ within a ρ-wsection S^ρ. A ρ-wnode (not necessarily maximal now) will be called *incident* to S^ρ if x^ρ embraces an α-wtip t^α $(\alpha < \rho)$ belonging to $S^{\rho-1}$ (that is, $S^{\rho-1}$ contains a one-ended extended α-walk that is a representative of t^α). This definition includes the case where the α-walk is simply a single branch; we let the extremity of the branch be an (-1)-*tip*. Also, when $\rho = \omega$, $\rho - 1$ denotes $\vec{\omega}$.

Lemma 3.1. *Given any two ρ-wnodes x^ρ and y^ρ incident to a $(\rho-1)$-section, there exists an α-walk $(\alpha < \rho - 1)$ in $S^{\rho-1}$ that reaches x^ρ and y^ρ.*

Proof. Let $W_x^{\alpha_1}$ (resp. $W_y^{\alpha_2}$) be a representative walk for the α_1-wtip (resp. α_2-wtip) embraced by x^ρ (resp. y^ρ). We have that $\alpha_1, \alpha_2 < \rho - 1$. Let u (resp. v) be a wnode of $W_x^{\alpha_1}$ (resp. $W_y^{\alpha_2}$) not embraced by x^ρ (resp. y^ρ). By the $(\rho-1)$-wconnectedness of $S^{\rho-1}$, there is a two-ended walk W_{uv} of rank no larger than $\rho - 1$ that terminates at u and v. Then, the walk that passes first from x^ρ along $W_x^{\alpha_1}$ to u, then along W_{uv} to v, and finally along $W_y^{\alpha_2}$ to y^ρ is the asserted walk. □

Now, each ρ-wsection S^ρ of G^ν is a ρ-wgraph by itself, and therefore the enlargement $^*S^\rho$ of S^ρ has its own principal ρ-galaxy $\Gamma_0^\rho(S^\rho)$.

Theorem 3.2. *If $\alpha < \rho \le \nu$ and if S^α is an α-wsection lying S^ρ, then the enlargement $^*S^\alpha$ of S^α lies within the principal ρ-galaxy $\Gamma_0^\rho(S^\rho)$ of $^*S^\rho$.*

Note. The conclusion means that every hypernode in $^*S^\alpha$ is a hypernode in $\Gamma_0^\rho(S^\rho)$, and consequently every hyperbranch in $^*S^\alpha$ is a hyperbranch in $\Gamma_0^\rho(S^\rho)$.

Proof. Let $\mathbf{x} = [\langle x, x, x, \ldots \rangle]$ be any standard hypernode in $\Gamma_0^\rho(S^\rho)$, and let $\mathbf{y} = [\langle y, y, y, \ldots \rangle]$ be any standard hypernode in the principal α-galaxy $\Gamma_0^\alpha(S^\alpha)$ of $^*S^\alpha$. Since S^α lies in S^ρ, the standard wnodes x and y corresponding to $\mathbf{x}$ and $\mathbf{y}$ are no further apart than the wdistance $\omega^\rho \cdot k$ for some $k \in \boldsymbol{N}$. (Indeed, there is a walk wconnecting them that does not pass through any wnode of rank greater than ρ.) Consequently, $\mathbf{x}$ and $\mathbf{y}$ are ρ-limitedly distant. Also, for every hypernode $\mathbf{z} = [z_n]$ in $^*S^\alpha$, $\mathbf{y}$ and $\mathbf{z}$ are α-limitedly distant, which implies that they are ρ-limitedly distant. So, by the triangle inequality (1), $\mathbf{x}$ and $\mathbf{z}$ are ρ-limitedly distant. Thus, $\mathbf{z}$ is in $\Gamma_0^\rho(S^\rho)$. □

Note that ${}^*S^\alpha$ may (but need not) have other α-galaxies besides its principal one $\Gamma_0^\alpha(S^\alpha)$, and ${}^*S^\rho$ may have still other α-galaxies not in $\Gamma_0^\rho(S^\rho)$. Also, there may be a ρ-galaxy (possibly many of them) consisting of a single λ-hypernode when $\lambda > \rho$.

Furthermore, it is possible of ${}^*G^\nu$ to have exactly one ρ-galaxy. This occurs, for instance, when there is a single node in G^ν to which all the other nodes of G^ν are connected through two-ended ρ-paths, with each such path being connected to the rest of G^μ only at its terminal nodes. When ${}^*G^\nu$ has exactly one ρ-galaxy, then ${}^*G^\nu$ has exactly one σ-galaxy for every σ such that $\rho < \sigma \leq \nu$ because, if two hypernodes are ρ-limitedly distant, then they are also σ-limitedly distant.

In view of all this, we again observe that the galaxies of ${}^*G^\nu$ can have rather complicated structures and dissimilarities.

4 Locally Finite Sections and a Property of Their Enlargements

In this and the next section, the rank ρ is not allowed to be $\vec{\omega}$. We now establish a sufficient (but not necessary) condition under which the enlargement ${}^*S^\rho$ has at least one ρ-galaxy different from its principal galaxy $\Gamma_0^\rho(S^\rho)$. Let us first recall some definitions for standard wgraphs.

Assume initially that ρ is a natural number. Two ρ-wnodes of S^ρ will be called *ρ-wadjacent* if they are incident to the same $(\rho-1)$-wsection. A ρ-wnode will be called a *boundary ρ-wnode* if it is incident to two or more $(\rho-1)$-wsections. A ρ-wsection S^ρ will be called *locally ρ-finite* if each of its $(\rho-1)$-wsections has only finitely many incident boundary ρ-wnodes. These same definitions hold when $\rho = \omega$ except that $\rho - 1$ is understood to be $\vec{\omega}$. The case where $\rho = \vec{\omega}$ is prohibited in the statements of this section.

In the following, we let ρ be a natural number or $\rho = \omega$.

Lemma 4.1. em Let x^ρ be a boundary ρ-wnode. Then, any ρ-walk that passes through x^ρ from one $(\rho-1)$-wsection $S_1^{\rho-1}$ incident to x^ρ to another $(\rho-1)$-wsection $S_2^{\rho-1}$ incident to x^ρ must have a length no less that ω^ρ (resp. when $\rho = \vec{\omega}$, a length no less than ω^ω).

Proof. The only way such a walk can have a length less than ω^ρ is if it avoids traversing a $(\rho-1)$-wtip in x^ρ. But, this means that it passes through two wtips embraced by x^ρ of ranks less than ρ. But, that in turn means that $S_1^{\rho-1}$ and $S^{\rho-1}$ cannot be different $(\rho-1)$-wsections. □

An immediate consequence of Lemma 4.1 is

Lemma 4.2. *Any two ρ-wnodes x^ρ and y^ρ that are ρ-wconnected but not ρ-wadjacent must satisfy $d(x^\rho, y^\rho) \geq \omega^\rho$.*

Theorem 4.3. *Let the ρ-wsection S^ρ of G^ν be locally ρ-finite and have infinitely many boundary ρ-wnodes. Then, given any ρ-wnode x_0^ρ in S^ρ, there is a one-ended ρ-walk W^ρ starting at x_0^ρ:*

$$W^\rho \;=\; \langle x_0^\rho, W_0^{\alpha_0}, x_1^\rho, W_1^{\alpha_1}, \ldots, x_m^\rho, W_m^{\alpha_m}, \ldots \rangle$$

such that there is a subsequence of ρ-wnodes $x_{m_k}^\rho$, $k = 1, 2, 3, \ldots$, satisfying $d(x_0^\rho, x_{m_k}^\rho) \geq \omega^\rho \cdot k$.

The proof of this theorem is just like that of Theorem 10.3 in [3] except that the rank 1 therein is replaced by the rank ρ herein. In the same way, Corollary 10.4 of [3] generalizes into the following assertion.

Corollary 4.4 *Under the hypothesis of Theorem 4.3, the enlargement $^*S^\rho$ of S^ρ has at least one ρ-hypernode not in its principal galaxy $\Gamma_0^\rho(S^\rho)$ and thus has at least one ρ-galaxy Γ^ρ different from its principal ρ-galaxy $\Gamma_0^\rho(S^\rho)$.*

5 When the Enlargement $^*S^\rho$ of a ρ-Wsection Has a ρ-Hypernode Not in the Principal ρ-Galaxy of $^*S^\rho$

As always, we take G^ν to be a ν-wgaph with $2 \leq \nu \leq \vec{\omega}$, possibly $\nu = \vec{\omega}$. We continue to asssume that the rank ρ of a ρ-wsection S^ρ of G^ν is either a natural number or ω, but not $\vec{\omega}$. Let Γ_a^ρ and Γ_b^ρ be two ρ-galaxies in the enlargement $^*S^\rho$ of a ρ-section S^ρ that are different from the principal ρ-galaxy Γ_0^ρ of $^*S^\rho$. We shall say that Γ_a^ρ *is closer to* Γ_0^ρ *than is* Γ_b^ρ and that Γ_b^ρ *is further away from* Γ_0^ρ *than is* Γ_a^ρ if there are a hypernode $\mathbf{y} = [y_n]$ in Γ_a^ρ and a hypernode $\mathbf{z} = [z_n]$ in Γ_b^ρ such that, for some $\mathbf{x} = [x_n]$ in Γ_0^ρ and for every $m_0 \in \boldsymbol{N}$, we have

$$N_0(m_0) \;=\; \{n \colon d(z_n, x_n) - d(y_n, x_n) \geq \omega^\rho \cdot m_0\} \;\in\; \mathcal{F}.$$

(The ranks of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ may have any values no larger than ν other than $\vec{\omega}$.) Any set of ρ-galaxies for which every two of them, say, Γ_a^ρ and Γ_b^ρ satisfy this condition will be said to be *totally ordered according to their closeness to* Γ_0^ρ. That the conditions for a total ordering (reflexivity, antisymmetry, transitivity, and connectedness) are fulfilled are readily shown. For instance, the proof of Theorem 5.2 below establishes transitivity.

These definitions are independent of the representative sequences $\langle x_n \rangle$, $\langle y_n \rangle$, and $\langle z_n \rangle$ chosen for $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$; the proof of this is exactly the same as the proof of Lemma 4.1 of [3] except that the m_k are replaced by $\omega^\rho \cdot m_k$.

We will say that a set A is a *totally ordered, two-way infinite sequence* if there is a bijection from the set $\boldsymbol{Z}$ of integers to the set A that preserves the total ordering of $\boldsymbol{Z}$.

Theorem 5.1. *If the enlargement* $^*S^\rho$ *of a* ρ*-wsection* S^ρ *of* G^ν *has a* ρ*-hypernode that is not in the principal* ρ*-galaxy* Γ_0^ρ *of* $^*S^\rho$*, then there exists a two-way infinite sequence of* ρ*-galaxies totally ordered according to their closeness to* Γ_0^ρ.

Note. Here, too, the proof of this is much like that of Theorem 4.2 of [3], but, since this is the main result of this work, let us present a detailed argument. As always, we choose and fix upon a free ultrafilter $\mathcal{F}$.

Proof. Let $\mathbf{x} = [\langle x, x, x, \ldots \rangle]$ be a standard hypernode in Γ_0^ρ. Also, let $\mathbf{v} = [v_n]$ be the asserted hypernode not in Γ_0^ρ. The ranks of $\mathbf{x}$ and $\mathbf{v}$ can be any ranks other than $\vec{\omega}$ and no larger than ρ. Thus, for each $m \in \boldsymbol{N}$, we have $\{n : d(v_n, x) > \omega^\rho \cdot m\} \in \mathcal{F}$. We can choose a subsequence $\langle y_n \rangle$ of $\langle v_n \rangle$ such that $d(y_n, x) = \omega^\rho \cdot m_n$ where the m_n are natural numbers that increase monotonically toward ∞ as $n \to \infty$. Thus, $\mathbf{y} = [y_n]$ is a hypernode in a ρ-galaxy Γ_b^ρ different from Γ_0^ρ.

There will be a smallest $n_1 \in \boldsymbol{N}$ such that $d(y_n, x) - d(y_0, x) > \omega^\rho$ for all $n \geq n_1$. Set $w_n = y_0$ for $0 \leq n < n_1$. Thus, for $0 \leq n < n_1$, we have that $d(y_n, x) - d(w_n, x) \geq 0$ and $d(w_n, x) \geq 0$.

Again, there will be a smallest $n_2 \in \boldsymbol{N}$ such that $d(y_n, x) - d(y_{n_1}, x) > \omega^\rho \cdot 2$ for all $n \geq n_2$. Set $w_n = y_0$ for $n_1 \leq n < n_2$. Thus, for $n_1 \leq n < n_2$, we have that $d(y_n, x) - d(w_n, x) > \omega^\rho$ and $d(w_n, x) \geq 0$.

Once again, there will be a smallest $n_3 \in \boldsymbol{N}$ such that $d(y_n, x) - d(y_{n_2}, x) > \omega^\rho \cdot 3$

for all $n \geq n_3$. Set $w_n = y_{n_1}$ for $n_2 \leq n < n_3$. Thus, for $n_2 \leq n < n_3$, we have that $d(y_n, x) - d(w_n, x) > \omega^\rho \cdot 2$ and $d(w_n, x) > \omega^\rho$. The last inequality follows from $d(y_{n_1}, x) > d(y_0, x) + 1 \geq \omega^\rho$ for all $n \geq n_1$.

Continuing this way, we will have a smallest $n_k \in \boldsymbol{N}$ such that $d(y_n, x) - d(y_{n_{k-1}}, x) > \omega^\rho \cdot k$ for all $n \geq n_k$. Set $w_n = y_{n_{k-2}}$ for $n_{k-1} \leq n < n_k$. In this general case for $n_{k-1} \leq n < n_k$, we have that $d(y_n, x) - d(w_n, x) > \omega^\rho \cdot (k-1)$ and $d(w_n, x) > \omega^\rho \cdot (k-2)$. The last inequality occurs because $d(y_{n_{k-2}}, x) > d(y_{n_{k-3}}, x) + \omega^\rho \cdot (k-2) > \omega^\rho \cdot (k-2)$ for all $n \geq n_{k-2}$.

Altogether then, w_n is defined for all n. Moreover, $d(w_n, x)$ increases monotonically, eventually becoming larger than m for every $\omega^\rho \cdot m \in \boldsymbol{N}$. Therefore, $\mathbf{w} = [w_n]$ is in a ρ-galaxy Γ_a^ρ different from the principal ρ-galaxy Γ_0^ρ of ${}^*S^\rho$. Furthermore, $d(y_n, x) - d(w_n, x)$ also increases monotonically in the same way. Consequently, the ρ-galaxy Γ_a^ρ containing $\mathbf{w} = [w_n]$ is closer to Γ_0^ρ than is the ρ-galaxy Γ_b^ρ containing $\mathbf{y} = [y_n]$.

We can now repeat this argument with Γ_b^ρ replaced by Γ_a^ρ to find still another ρ-galaxy $\tilde{\Gamma}_a^\rho$ of ${}^*S^\rho$ different from Γ_0^ρ and closer to Γ_0^ρ than is Γ_a^ρ. Continual repetitions yield an infinite sequence of ρ-galaxies indexed by, say, the negative integers and totally ordered by their closeness to Γ_0^ρ.

The conclusion that there is an infinite sequence of ρ-galaxies progressively further away from Γ_0^ρ than is Γ_b^ρ is easier to prove. With $\mathbf{y} \in \Gamma_b^\rho$ as before, we have that, for every $m \in \boldsymbol{N}$, $\{n : d(y_n, x) > \omega^\rho \cdot m\} \in \mathcal{F}$. Therefore, for each $n \in \boldsymbol{N}$, we can choose z_n as an element of $\langle y_n \rangle$ such that $d(z_n, x) \geq d(y_n, x) + \omega^\rho \cdot n$ and also such that $d(z_n, x)$ monotonically increases with n and eventually becomes larger than $\omega^\rho \cdot m$ for every $m \in \boldsymbol{N}$. This implies that $\mathbf{z} = [z_n]$ must be in a ρ-galaxy Γ_c^ρ that is further away from Γ_0^ρ than is Γ_b^ρ

We can repeat the argument of the last paragraph with Γ_c^ρ in place of Γ_b^ρ and with $\mathbf{w} = [w_n]$ playing the role that $\mathbf{y} = [y_n]$ played to find still another ρ-galaxy $\tilde{\Gamma}_c^\rho$ further away from Γ_0^ρ than is Γ_c^ρ. Repetitions of this argument show that there is an infinite sequence of ρ-galaxies indexed by, say, the positive integers and totally ordered by their closeness to Γ_0^ρ. The union of the two infinite sequences yields the conclusion of the theorem. □

By virtue of Corollary 4.4, the conclusion of Theorem 5.1 holds whenever G is locally finite.

In general, the hypothesis of Theorem 5.1 may or may not hold. Thus, ${}^*S^\rho$ either has exactly one ρ-galaxy, its principal one Γ_0^ρ, or has infinitely many ρ-galaxies.

Instead of the idea of "totally ordered according to closeness to Γ_0^ρ," we can define the idea of "partially ordered according to closeness to Γ_0^ρ" in much the same way. Just drop the connectedness axiom for a total ordering.

Theorem 5.2. *Under the hypothesis of Theorem 5.1, the set of ρ-galaxies of ${}^*S^\rho$ is partially ordered according to the closeness of the ρ-galaxies to the principal ρ-galaxy Γ_0^ρ.*

Proof. Reflexivity and antisymmetry are obvious. Consider transitivity: Let Γ_a^ρ, Γ_b^ρ, and Γ_c^ρ be ρ-galaxies different from Γ_0^ρ. (The case where $\Gamma_a^\rho = \Gamma_0^\rho$ can be argued similarly.) Assume that Γ_a^ρ is closer to Γ_0^ρ than is Γ_b^ρ and that Γ_b^ρ is closer to Γ_0^ρ than is Γ_c^ρ. Thus, for any $\mathbf{x}$ in Γ_0^ρ, $\mathbf{u}$ in Γ_a^ρ, $\mathbf{v}$ in Γ_b^ρ, and $\mathbf{w}$ in Γ_c^ρ and for each $m \in \boldsymbol{N}$, we have

$$N_{uv} \;=\; \{n\colon d(v_n, x_n) - d(u_n, x_n) \geq \omega^\rho \cdot m\} \;\in\; \mathcal{F}$$

and

$$N_{vw} \;=\; \{n\colon d(w_n, x_n) - d(v_n, x_n) \geq \omega^\rho \cdot m\} \;\in\; \mathcal{F}.$$

We also have

$$d(w_n, x_n) - d(u_n, x_n) \;=\; d(w_n, x_n) - d(v_n, x_n) + d(v_n, x_n) - d(u_n, x_n).$$

So,

$$N_{uw} \;=\; \{n\colon d(w_n, x_n) - d(u_n, x_n) \geq \omega^\rho \cdot 2m\} \;\supseteq\; N_{uv} \cap N_{vw} \;\in\; \mathcal{F}.$$

Thus, $N_{uw} \in \mathcal{F}$. Since m can be chosen arbitrarily, we can conclude that Γ_a^ρ is closer to Γ_0^ρ than is Γ_c^ρ. □

References

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