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HYPERREAL TRANSIENTS ON TRANSFINITE DISTRIBUTED TRANSMISSION LINES AND CABLES

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Abstract — A prior work showed how nonstandard analysis could be used to derive hyperreal transients in transfinite electrical networks containing lumped inductors, capacitors, resistors, and sources. In this work hyperreal transients are derived for transfinite electrical networks whose parameters are distributed. In particular, explicit expressions are derived for hyperreal transients on uniform transmission lines and cables that "extend beyond infinity" transfinitely. This requires a substantially altered technique as compared to the prior work. The present one uses a different kind of truncation procedure that reduces the transfinite line or cable to a conventionally infinite one and then expands the latter in steps to "fill out" the transfinite line or cable.

Key Words: Distributed transmission lines, distributed cables, hyperreal transients, nonstandard networks.

1 Introduction

This paper is a sequel to [5], which presented for the first time a method for analyzing a transfinite RLC network.[1] All prior works on transfinite networks were restricted to purely resistive ones. The idea in [5] was to represent the transfinite RLC network as the end result of of an expanding sequence of finite RLC networks that "fill out" the transfinite network. Since the solutions of finite networks are available, this provides for each branch of the transfinite network a sequence of time-varying voltages and another sequence of time-varying currents, which can be identified as a hyperreal branch voltage

[1] An electrical network containing lumped resistors, inductors, and capacitors.

and a hyperreal branch current depending upon hyperreal time. Essential to that approach was the requirement that the transfinite network be lumped; that is, the network must consist of a transfinite graph whose branches consist of lumped electrical elements.

The objective of this work is to devise an analysis for transfinite distributed electrical transmission lines and cables. In particular, we take it that the transmission line or cable extends "beyond infinity" in a mannêr specified in Sec. 3. We then determine how a hyperreal traveling wave in the case of a transmission line or a hyperreal diffusion in the case of a cable can pass "through infinity" to produce hyperreal voltages within the transfinite extensions of the line or cable. The difficulty that must be overcome in the present case is that the techniques that worked for lumped networks must be modified in a substantial way in order to make them applicable to distributed lines and cables. We now exploit the fact that the voltages on a conventional, one-way infinite line or cable is known. So, we represent the transfinite line or cable as the end result of an expanding sequence of conventionally infinite lines or cables that "fill out" the transfinite line or cable. This is accomplished by assigning sample points along the transfinite line or cable and then truncating it by removing infinite parts between some of the sample points to obtain a conventionally infinite line or cable. Then, upon reducing the removed parts, that is, expanding the remaining parts in steps, we can obtain a sequence of time-varying voltages at each sample point, which in turn can be identified as a hyperreal voltage variation at that sample point. We thus obtain finally hyperreal voltage transients at all the sample points of the transfinite line or cable.

In the same way, we can determine hyperreal current transients at the sample points, but we skip doing this since the technique is exactly the same.

With regard to the hyperreal numbers, we follow the notation and terminology of [2], a textbook that explains all the concepts of nonstandard analysis we employ herein.[2] We will be using ultrapower constructions of hyperreals. Thus, each sequence of voltages is a representative sequence of an equivalence class of sequences with respect to a chosen nonprincipal ultrafilter $\mathcal{F}$, and that class is by definition a hyperreal. $I\!N$ will denote the set of natural numbers $\{0, 1, 2, \ldots\}$, and $n \in I\!N$ will always be the index for the sequences

[2]Sec. 3 of [5] contains a very brief statement of those concepts. This all that will be needed for a comprehension of this work.

that represent hyperreals. Such a sequence will be denoted by $\langle v_n : n \in I\!N \rangle$ or simply $\langle v_n \rangle$, and $[v_n]$ will denote an equivalence class (modulo $\mathcal{F}$) of such sequences; it is understood that the v_n within $[v_n]$ are the elements of one of the representatives of the class.[3] It is a fact that altering finitely many of the v_n does not change $[v_n]$. Arithmetic and inequalities for hyperreals are defined componentwise on their representative sequences. We will be discussing various kinds of hyperreals, such as "infinitesimal," "limited," "unlimited," and "appreciable" hyperreals and will also mention the "halo" of a real number; all these are defined in [2] and also in [5, Sec. 3].

$I\!R$ (resp. $^*I\!R$) denotes the set of reals (resp. hyperreals).[4] Also, $I\!R_+$ (resp. $^*I\!R_+$) denotes the set of nonnegative reals (resp. nonnegative hyperreals). Hyperreals will be denoted by boldface notation; thus, $\mathbf{v} = [v_n] \in {}^*I\!R$ denotes a hyperreal voltage, and $\mathbf{t} = [t_n] \in {}^*I\!R_+$ denotes hyperreal time, where now $\langle t_n : n \in I\!N \rangle$ is one of the sequences in the equivalence class $\mathbf{t}$ of sequences.

2 Conventionally Infinite, Uniform, Transmission Lines and Cables

During the initial truncation process, we will be reducing the transfinite lines and cables to conventionally infinite ones and will use the known real transient responses of the conventional lines and cables when constructing the hyperreal transient responses of the transfinite lines and cables.

A conventionally infinite (transmission) line is illustrated in Fig. 1. A cable is a special case of a line. For a reason that will become evident later on, we call this an ω-line. We assume throughout that the line is uniform with the distributed series resistance and inductance being r ohms/meter and l henries/meter and the distrubuted shunt conductance and capacitance being g siemens/meter and c farads/meter. The distance along the line from the input in the conventional case is x meters, and the voltage at the distance x and time t seconds is $v(x,t)$. We will have $v(x,t) \in I\!R$ and $x, t \in I\!R_+$. We also assume

[3] This notation differs from that used in [5], wherein a sequence was denoted by $\{v_n\}_{n \in I\!N}$ and a hyperreal by $\langle v_n \rangle$. The present notation seems to be more commonly used and conforms with that of [2].

[4] Conventionally, "real" and "hyperreal" are often said in place of "real number" and "hyperreal number."

throughout that at the input to the line $v(0,t) = 1_+(t)$, where 1_+ denotes the unit step function: $1_+(t) = 1$ for $t \geq 0$ and $1_+(t) = 0$ for $t < 0$.

When r, l, g, c are all positive real numbers, we have the general case of an ω-line, and the Laplace transform $V(x,s)$ of $v(x,t)$ is [3, page 379]

$$V(x,s) = \frac{1}{s}\exp(-\sqrt{lc}\sqrt{(s+\delta)^2-\sigma^2}), \quad \operatorname{Re} s > 0, \tag{1}$$

where

$$\delta = \frac{1}{2}\left(\frac{r}{l}+\frac{g}{c}\right), \quad \sigma = \frac{1}{2}\left(\frac{r}{l}-\frac{g}{c}\right). \tag{2}$$

Taking the inverse Laplace transform of (1), we get [3, page 383]

$$v(x,t) = f_0(x,t) + f_\sigma(x,t), \tag{3}$$

where

$$f_0(x,t) = e^{\delta x\sqrt{lc}}\, 1_+(t - x\sqrt{lc}), \tag{4}$$

and

$$f_\sigma(x,t) = \sigma x\sqrt{lc}\int_{x\sqrt{lc}}^{t} \frac{e^{-\delta\tau}}{\sqrt{\tau^2 - x^2 lc}} I_1(\sigma\sqrt{\tau^2 - x^2 lc})\, d\tau\, 1_+(t - x\sqrt{lc}). \tag{5}$$

Here, I_1 is the modified Bessel function of first kind and order 1 [1, page 374]; it is an entire function. Furthermore, $f_0(x,t)$ is the voltage for the distortionless line occurring when $\sigma = 0$ (i.e., $rc = lg$), and $f_\sigma(x,t)$ is the added distortion occurring when $\sigma \neq 0$. Both $f_0(x,t)$ and $f_\sigma(x,t)$ take on nonnegative values only.

As a special case of the distortionless line, we have the lossless line occurring when $r = g = 0$, $l > 0$, and $c > 0$. Its transient response is simply

$$v(x,t) = 1_+(t)(t - x\sqrt{lc}). \tag{6}$$

A different phenomenon occurs when $g = l = 0$, $r > 0$, and $c > 0$. This corresponds to a cable, for which case the wave response (4) is replaced by a diffusion response. Specifically, the response to the unit-step input $v(0,t) = 1_+(t)$ is [3, page 330]

$$v(x,t) = \operatorname{erfc}\left(\frac{x}{2}\sqrt{\frac{rc}{t}}\right). \tag{7}$$

Here, $\operatorname{erfc}(\cdot)$ is the complementary error function [1, page 297]

$$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-\zeta^2}\, d\zeta. \tag{8}$$

3 The ω^2-Line

The first transfinite structure we wish to explore is a transmission line that extends transfinitely with infinitely many (more precisely, ω-many) ω-lines connected in cascade by having the infinite extremity of each ω-line connected to the input of the next ω-line. We call this an ω^2-line. It is illustrated in Fig. 2. We shall analyze this structure by choosing uniformly spaced sample points Δx meters apart within the entire ω^2-line, taking the input of each ω-line as one of the sample points. We will then determine hyperreal voltage transients $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ at each sample point. The number of sample points within each ω-line does not depend upon the choice of Δx; that number is always ω (more precisely, the cardinality of the set of such points is always $\omega = \aleph_0$). Thus, we obtain the sample-point numbering at the input of each ω-line as shown in Fig. 2, and we therefore number the infinite extremity of the entire ω^2-line as ω^2 (not shown in Fig. 2). Moreover, the indices j of the sample points within the entire ω^2-line first traverse the natural numbers within the initial ω-line, then the ordinals from ω through all those below $\omega 2$ in the next ω-line, and so forth. In general, $\omega k \leq j < \omega(k+1)$ for the kth ω-line ($k \in I\!N$) in Fig. 2.

Two questions may arise when trying to make sense out of the configuration in Fig. 2. First, how can the connection between the infinite extremity of the kth ω-line and the input of the $(k+1)$st ω-line be defined? One way is to think of the line as being "artificial," having discrete series and shunt branches and then to use the definition of 1-nodes [4, page 22]. In this way, the small circles in Fig. 2 represent 1-nodes that connect the 0-tips of the horizontal paths in any artificial ω-line to the input 0-nodes of the next artificial ω-line. Then, the present distributed structure might be viewed as a limiting case arising when the artificial line smooths out into a distributed one. But. perhaps, such elaboration is not needed if one is willing to accept the idea that the infinite extremity of each distributed ω-line "is connected to" the input terminals of the next ω-line.

The second question concerns the continuous spatial variable x. Within the initial ω-line its values are real numbers, but what is its values in the subsequent ω-lines? Even at the sample points we have a problem in interpreting $x =$ "$j\Delta x$" when j is a transfinite ordinal and Δx is a real number. This difficulty will be circumvented in the following way: When setting up a nonstandard model of the ω^2-line of Fig. 2, we will truncate each ω-line within the ω^2-line into a finite line and will thereby reduce the ω^2-line to an ω-line. This will allow us to use the voltage transients cited in Sec. 2 in order to derive hyperreal transients at the sample points of the ω^2-line.

4 A Nonstandard Model for an ω^2-Line

The general idea for analyzing the ω^2-line of Fig. 2 is to reduce it to an infinite cascade of finite lines which together comprise an ω-line; then, the finite lines are expanded in steps to "fill out" the ω^2-line. Since the response of an ω-line to a unit step of voltage is known, we will obtain at each sample point a sequence of voltages depending upon time t, which can then be identified as a hyperreal voltage depending upon hyperreal time $\mathbf{t}$.

So, consider the nth sample point ($n \in \mathbb{N}$) within each ω-line of the ω^2-line. Remove that part of the ω-line beyond that nth sample point and then connect that nth sample point to the input of the next ω-line. What is left is a cascade of finite lines, each having n sample points (not counting the input node) and together comprising an ω-line because the number of finite lines is ω. We shall refer to this structure as the nth truncation of the ω^2-line. Now, consider the sequence of such nth truncations as $n \to \infty$. Any fixed sample point of the ω^2-line will eventually appear in those nth truncations for all n sufficiently large and will have a voltage in accordance with (4). That fixed sample point will be absent in no more than finitely many nth truncations, and in these cases we can set the voltage equal to 0 without disturbing the hyperreal voltage that will arise as $n \to \infty$.

To be more specific, consider the jth sample point in the ω^2-line of Fig. 2. We can set $j = \omega k_1 + k_0$ ($k_0, k_1 \in \mathbb{N}$), where k_1 is the number of ω-lines to the left of the ω-line in which the jth sample point appears (but not counting that ω-line) and k_0 is the number of sample points to the left of the jth sample point in the ω-line in which the jth sample point appears. Then, for all n sufficiently large, the jth sample point appears in the nth truncation, and the distance from the input to the jth sample point in the nth truncation is

$$x_{j,n} = (nk_1 + k_0)\Delta x \in \mathbb{R}_+.$$

where Δx is the distance between sample points, as before. In this case, the voltage at the jth sample point is

$$v(x_{j,n}, t) = f_0(x_{j,n}, t) + f_\sigma(x_{j,n}, t). \tag{9}$$

As $n \to \infty$, we obtain a sequence $\langle x_{j,n} : n \in I\!N \rangle$ of distances to the fixed jth sample point, which we take to a representative sequence for a hyperreal distance $\mathbf{x}_j = [x_{j,n}]$ to the jth sample point. Furthermore, the voltage wave on the original ω^2-line will require an infinite amount of time in order to reach the ω-lines beyond the initial one. So, in order to examine this, we will need to use hyperreal time $\mathbf{t} = [t_n]$. Altogether then, by replacing x by $\mathbf{x}_j$ and t by $\mathbf{t}$ and using (3), (4), and (5), we obtain the following hyperreal voltage as the response of the ω^2-line at its jth sample point:

$$\mathbf{v}(\mathbf{x}_j, \mathbf{t}) = [v(x_{j,n}, t_n)] = [f_0(x_{j,n}, t_n) + f_\sigma(x_{j,n}, t_n)] \tag{10}$$

Because of the presence of the unit-step function $1_+(t - x\sqrt{lc})$ in (4) and (5), we see that $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ equals 0 for $\mathbf{t} = [t_n] < [\sqrt{lc}\Delta x(nk_1 + k_0)]$ and is positive for $\mathbf{t} \geq [\sqrt{lc}\Delta x(nk_1 + k_0)]$. Moreover, because of the factor $\exp(-\delta x\sqrt{lc})$ in (4), the distortionless term $[f_0(x_{j,n}, t_n)]$ is an infinitesimal for unlimited $t = [t_n]$ if $k_1 > 0$ (i.e., at all sample points at and beyond $j = \omega$).

More generally, the total response $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ is also an infinitesimal when k_1, r, g, l, c are all positive. (That $k_1 > 0$ means that the jth sample point is beyond the initial ω-line.) To show that the total response is infinitesimal, first apply the final-value theorem to (1):

$$\begin{aligned} \lim_{t\to\infty} v(x,t) &= \lim_{s\to 0+} \exp\left(-x\sqrt{lc}\sqrt{(s+\delta)^2 - \sigma^2}\right) \\ &= \exp\left(-x\sqrt{lc}\sqrt{\delta^2 - \sigma^2}\right) \\ &= \exp\left(-x\sqrt{rg/lc}\right). \end{aligned} \tag{11}$$

Now, at the jth sample point, we have $x_{j,n} = \Delta x(k_1 n + k_0)$. Because of the unit-step function, we need only consider the case where $t_n \geq x_{j,n}\sqrt{lc}$. So, the character of $\mathbf{v}(\mathbf{x}_j, \mathbf{t}) = [v_{j,n}, t_n)]$ can be determined by letting $n \to \infty$ but keeping j fixed. From (11) we have that

$$\lim_{n\to\infty} v(x_{j,n}, t_n) = \lim_{n\to\infty} \exp(-\Delta x(k_1 n + k_0)\sqrt{rg/lc}) = 0.$$

Whence our assertion; in fact, $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ is a positive infinitesimal for $\mathbf{t} \geq [x_{j,n}\sqrt{lc}]$. Note also that, since the distortionless term and the total response are both infinitesimals when $k_1 > 0$, so also is the additional distortion term $[f_\sigma(x_{j,n}, t_n)]$ occurring when $\sigma \neq 0$.

However, when $g = 0$ and r, l, c are positive, the same analysis shows that the limit in (11) is 1. This means that $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ is infinitesimally close to 1 when $\mathbf{t} \geq [x_{j,n}\sqrt{lc}]$ and $k_1 > 0$. To say this another way, at any sample point beyond the initial ω-line, the hyperreal voltage $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ remains equal to 0 for $\mathbf{t} < [x_{j,n}\sqrt{lc}] = \mathbf{x}_j\sqrt{lc}$, and then it jumps to within the halo of 1 and remains therein for $\mathbf{t} > \mathbf{x}_j\sqrt{lc}$. This is very different from the voltage response within the first ω-line ($k_1 = 0$) wherein the transition toward 1 is gradual; see [3, page 384, Fig. 8.2].

In the still more special case of a lossless line ($r = g = 0, l > 0, c > 0$), $\mathbf{v}(\mathbf{x}_j\mathbf{t}) = 0$ for $\mathbf{t} < \mathbf{x}_j\sqrt{lc}$ and is exactly equal to 1 for $\mathbf{t} \geq \mathbf{x}_j\sqrt{lc}$.

Finally, let us point out that we have obtained a hyperreal transient response for the ω^2-line by specifying a particular way of truncating it into an ω-line and then expanding the ω-line in steps to fill out the ω^2-line. However, there are many ways of doing this by using, say, nonuniformly spaced sample points and different truncations and subsequent expansions among the ω-line. In this way, we can generate many nonstandard versions of the ω^2-line. We have presented the one method for which the hyperreal transient has the simplest expression.

5 Nonstandard Models for Transfinite Lines of Higher Ranks

By connecting ω-many ω^2-lines in cascade, we will obtain an ω^3-line. Recursively, we can construct in this way an ω^μ-line, where μ is any natural number; the ω^μ-line consists of ω-many $\omega^{\mu-1}$-lines in cascade. To analyze this in a nonstandard way, we again choose sample points with Δx spacing. Then, a typical sample point has the index

$$j \;=\; \omega^{\mu-1}k_{\mu-1} + \omega^{\mu-2}k_{\mu-2} + \ldots + \omega k_1 + k_0, \tag{12}$$

where $k_{\mu-1}$ is the number of $\omega^{\mu-1}$-lines to the left of the $\omega^{\mu-1}$-line in which the sample point j appears, $k_{\mu-2}$ is the number of $\omega^{\mu-2}$-lines within the $\omega^{\mu-1}$line in which the sample point j appears and to the left of the $\omega^{\mu-2}$-line in which the sample point j appears, and so on. In general, for $1 \leq \alpha \leq \mu - 1$, k_α is the number of ω^α-lines within the $\omega^{\alpha+1}$-line in which the sample point j appears and to the left of the ω^α-line in which the sample point j appears. Finally, k_0 is the number of sample points within the ω-line in which the sample point appears and to the left of that sample point.

Next, we create the nth truncation ($n \in \mathbb{N}, n \geq 1$) of this ω^μ-line as follows. Consider each $\omega^{\mu-1}$-line in the ω^μ-line. Remove all the $\omega^{\mu-2}$-lines in the $\omega^{\mu-1}$-line beyond the nth $\omega^{\mu-2}$-line, and then connect the output of each so-truncated $\omega^{\mu-1}$-line to the input of the next truncated $\omega^{\mu-1}$-line. What is left is a cascade of ω-many truncated $\omega^{\mu-1}$-lines each having n $\omega^{\mu-2}$-lines, and thus ω-many $\omega^{\mu-2}$-lines altogether. Next, within each of those $\omega^{\mu-2}$-lines, remove all the $\omega^{\mu-3}$-lines beyond the nth $\omega^{\mu-3}$-line, and then connect the output of each so-truncated $\omega^{\mu-2}$-line to the input of the next truncated $\omega^{\mu-2}$-line. What is left in each truncated $\omega^{\mu-2}$-line is a cascade of n-many $\omega^{\mu-3}$-lines. Altogether, we now have a cascade of ω-many truncated $\omega^{\mu-2}$-lines each being a cascade of n-many $\omega^{\mu-3}$-lines, and thus ω-many $\omega^{\mu-3}$-lines altogether. In general, having recursively made truncations to get a cascade of ω-many truncated ω^p-lines ($3 \leq p \leq \mu - 1$) each having n-many ω^{p-1}-lines, remove within each ω^{p-1}-line all the ω^{p-2}-lines beyond the nth one, and then connect the output of the so-truncated ω^{p-1}-line to the input of the next truncated ω^{p-1}. What is left is a cascade of ω-many truncated ω^{p-1}-lines each having n-many ω^{p-2}-lines, and thus ω-many ω^{p-2}-lines altogether. Finally, we will have reached a cascade of ω-many truncated ω^2-lines each having n-many ω-lines. In each of those ω-lines, remove that part of the line to the right of the nth sample point, and connect the output of the so-truncated ω-line to the input of the next truncated ω-line. The final result will be a cascade of ω-many finite lines each having $n^{\mu-1}$ sample points (not counting the input node of the original ω^μ-line).[5] That result is simply an ω-line, whose solution was displayed in Sec. 2. We call this the nth truncation of the ω^μ-line. As $n \to \infty$, that truncation shrinks, and the ω-line expands to ultimately fill out the original ω^μ-line.

[5]For an ω^2-line, we get ω-many finite lines of n sample points each. For an ω^3-line, we get ω-many finite lines of n^2 sample points each. For an ω^4-line, we get ω-many finite lines of n^3 sample points each. Continue this way.

Now any fixed sample point of index j on the ω^μ-line will eventually appear as a sample point of the nth truncation of the ω^μ-line for all n sufficiently large. When this happens, the distance $x_{j,n}$ from the input of that truncated ω^μ-line to the jth sample point will be

$$x_{j,n} \;=\; \Delta x(n^{\mu-1}k_{\mu-1} + n^{\mu-2}k_{\mu-2} + \ldots + nk_1 + k_0) \qquad (13)$$

where the natural numbers k_p ($p = 0, \ldots, \mu - 1$) have the same meanings as before in (12). Since the nth truncation of the ω^μ-line is an ω-line, we can invoke (3), (4), and (5) to obtain the voltage at $x_{j,n}$. This gives us a sequence $\langle v(x_{j,n}, t) : n \in I\!N\rangle$ of voltage values at the jth sample point for any given time t. In order to obtain a hyperreal time $\mathbf{t} = [t_n]$ when analyzing the original ω^μ-line, we choose a sequence $\langle t_n : n \in I\!N\rangle$ of time values in place of t. All this yields a hyperreal voltage $\mathbf{v}(\mathbf{x}_j, \mathbf{t}) = [v(x_{j,n}, t_n)]$ on this nonstandard version of the ω^μ-line. More specifically, we have

$$v(x_{j,n}, t_n) \;=\; f_0(x_{j,n}, t_n) \;+\; f_\sigma(x_{j,n}, t_n) \qquad (14)$$

where $x_{j,n}$ is given by (13). Here, too, $\mathbf{v}(\mathbf{x}_j, \mathbf{t}) = 0$ for $0 \leq \mathbf{t} = [t_n] < \mathbf{x}_j\sqrt{lc} = [x_{j,n}]\sqrt{lc}$ and $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ is a positive hyperreal for $\mathbf{t} \geq \mathbf{x}_j\sqrt{lc}$.

As in the case of an ω^2-line, the presence of the factor $e^{-\Delta x\sqrt{lc}}$ in (4) in the distortionless case (i.e., $\sigma = 0$) and the expression (13) for $x_{j,n}$ insures that the distortionless term $[f_0(x_{j,n}, t_n)]$ is infinitesimal when the jth sample point is beyond the initial ω-line (that is, when at least one of the k_p, $p = 1, \ldots, \mu - 1$, is positive. Also, the argument employing the final-value theorem as in the preceding section shows that the total response $[v(x_{j,n}, t_n)]$ at any sample point beyond the initial ω-line is also infinitesimal whenever r, g, l, c are all positive. Therefore, so also is the added distortionless term $[f_\sigma(x_{j,n}, t_n)]$. On the other hand, when $g = 0$ and r, l, c are positive, $\mathbf{v}(\mathbf{x}_j, \mathbf{t})]$ is infinitesimally close to 1 for all $\mathbf{t} > [x_{j,n}\sqrt{lc}]$ at every sample point after the initial ω-line. Altogether then, the response of the ω^μ-line is much the same as that of the ω^2-line.

The next rank for transfinite transmission lines beyond those of the natural-number ranks is the ω^ω-line. This can be viewed as a cascade of an ω-line, followed by an ω^2-line, followed by an ω^3-line, and so forth indefinitely. We can reduce the ω^ω-line to an ω-line through a somewhat different set of truncations than those for ω^μ-lines. Specifically, we reduce the initial ω-line to a finite line of n sample points by truncating that line beyond the nth sample point. We then reduce the ω^2-line to a finite line of n^2 sample points by first truncating the ω^2-line beyond the nth ω-line and then truncating each of the remaining ω-lines beyond their nth sample points (not counting their input nodes). In a similar way, the next ω^3-line is first truncated down to its initial n ω^2-lines, which in turn are each truncated down to their initial n ω-lines, which again are each truncated down to n sample points—to obtain finally n^3 sample points for that ω^3-line. Continuing in this way indefinitely, we obtain n sample points followed by n^2 sample points, followed in turn by n^3 sample points, and so on indefinitely to get ω-many sample points on an ω-line. We call this the nth truncation of the ω^ω-line.

Now, the standard formulas of Sec. 2 can be applied once more to get a nonstandard analysis of an ω^ω-line. To be specific, let the jth sample point of the ω^ω-line (again counting the input as the 0th sample point) be in the ω^μ-line of the cascade of ω^α-lines ($\alpha = 1, 2, 3, \ldots$) defining the ω^ω-line. In the nth truncation of the ω^ω-line, the number of sample points before that ω^μ-line is

$$n + n^2 + n^3 + \ldots + n^{\mu-1}. \tag{15}$$

Within the truncation of that ω^μ-line, the number of sample points to the left of the jth sample point of the ω^ω-line can be written as

$$n^{\mu-1}k_{\mu-1} + n^{\mu-2}k_{\mu-2} + \ldots + nk_1 + k_0$$

where the $k_{\mu-1}, k_{\mu-2}, \ldots, k_0$ are defined exactly as in (12). (Remember that input nodes are indexed by 0.) So altogether, the number $N_j(n)$ of sample points within the nth truncation of the ω^ω-line at and to the left of the jth sample point for the ω^ω-line (not counting the input node) is

$$N_j(n) \;=\; n^{\mu-1}(1 + k_{\mu-1}) + n^{\mu-2}(1 + k_{\mu-2}) + \ldots + n(1 + k_1) + k_0. \tag{16}$$

Here, it is understood that μ depends upon j as stated: That is, the jth sample point of the ω^ω-line lies in the ω^μ-line of the cascade of transfinite lines defining the ω^ω-line. So, with $\mathbf{x}_j = [x_{j,n}] = [\Delta x N_j(n)]$, $\mathbf{t} = [t_n]$, $\mathbf{v}(\mathbf{x}_j, \mathbf{t}) == [v(x_{j,n}, t_n)]$, and $v(x_{j,n}, t_n)$ defined again by (14), we have the hyperreal voltage response $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ of the ω^ω-line at its jth sample point. Again, we can identify $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ as being either exactly equal to 0 or equal to a positive infintesimal or infinitesimally close to 1 just as before at any sample point beyond the initial ω-line.

Having treated all the ω^μ-lines ($\mu = 1, 2, 3, \ldots$) and then the ω^ω-line, we can now treat the $\omega^{\omega+\mu}$-lines ($\mu = 1, 2, 3, \ldots$) and then the $\omega^{\omega 2}$-line in much the same way. So also can the $\omega^{\omega 2+\mu}$-lines and the $\omega^{\omega 3}$-lines be treated similarly—and so on.

For example, consider the $\omega^{\omega+1}$-line. It is a cascade of ω-many ω^ω-lines. This time, let us truncate each of the ω-many ω^ω-lines by deleting all the ω^ν-lines ($\nu = n+1, n+2, \ldots$) beyond the ω^n-line in each ω^ω-line and then connecting the output of that ω^n-line to the input of the next so-truncated ω^ω-line. Then, we truncate each ω^α-line ($\alpha = 1, 2, \ldots, n$) as before to get n sample points followed by n^2-sample points, ..., and finally n^n sample points. Thus, we will have $n + n^2 + \ldots + n^n$ sample points in each of the ω-many truncated ω^ω-line. Altogether, we will have ω-many sample points in an ω-line. We will call this the nth truncation of the $\omega^{\omega+1}$-line. Next, consider the jth sample point in the $\omega^{\omega+1}$-line. Let k_ω be the finite number of ω^ω-lines to the left of the ω^ω-line in which the jth sample point appears. Assume n is so large that the jth sample point appears within the nth truncation of the $\omega^{\omega+1}$. Thus, the number of sample points in the nth truncation of the $\omega^{\omega+1}$-line up to the jth sample point is

$$M_j(n) \;=\; k_\omega(n + n^2 + \ldots + n^n) \;+\; N_j(n) \tag{17}$$

where $N_j(n)$ is given by (16), namely, it is the number of sample points in the truncated ω^ω-line in which the jth sample point appears at and to the left of the jth sample point. So, with $\mathbf{x}_j = [x_{j,n}] = [\Delta x M_j(n)]$ and $\mathbf{t} = [t_n]$, we have as before

$$\mathbf{v}(\mathbf{x}_j, \mathbf{t}) \;=\; [f_0(x_{j,n}, t_n)] \;+\; [f_\sigma(x_{j,n}, t_n)]. \tag{18}$$

We can identify what kind of hyperreal $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ is again as before.

Finally, we point out again that many other nonstandard models can be constructed by varying the ways we have taken truncations and subsequent expansions.

6 Nonstandard Models for Transfinite Cables

For a cable we have $g = l = 0$, $r > 0$, $c > 0$, and the standard voltage response $v(x,t)$ on an ω-cable due to a unit step of voltage at the input to the cable is given by (7). This represents a diffusion phenomenon.

For a transfinite cable of rank greater than ω, we truncate it down to an ω-cable, with the truncations depending upon the natural number n, and then let $n \to \infty$ to obtain a representative sequence for the hyperreal voltage response. Thus, for an ω^{μ}-cable with $\mu \in I\!N, \mu > 1$, we choose the nth truncation exactly as in Sec. 5. In particular, the jth sample point on the ω^{μ}-cable is given by (12), and for all n sufficiently large the distance $x_{j,n}$ of the jth sample point from the input of the nth truncation of the ω^{μ}-cable is given by (13). Then, with $\mathbf{x}_j = [x_{j,n}]$ being hyperreal distance and $\mathbf{t} = [t_n]$ being hyperreal time as before, we have the hyperreal voltage $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ on the ω^{μ}-cable due to a unit step of voltage at the input as

$$\mathbf{v}(\mathbf{x}_j, \mathbf{t}) \;=\; \left[\operatorname{erfc} \frac{x_{j,n}}{2}\sqrt{\frac{rc}{t_n}}\right] \tag{19}$$

We can invoke the properties of the complementary error function erfc$(\cdot)$ to assert the following. In the initial ω-cable of the ω^{μ}-cable ($\mu > 1$), $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ is infinitesimal (resp. appreciable, resp. unilimited) when $\mathbf{t}$ is infinitesimal (resp. appreciable, resp. unlimited). However, for subsequent ω-cables within the ω^{μ}-cable, the following properties hold. As $\mathbf{t}$ increases through all limited and then through some initial unlimited values, $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ increases through some but not all infinitesimal values. Then, as $\mathbf{t}$ increases through larger unlimited values, $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ continues to increase first through larger infinitesimal values, then through appreciable values, and eventually gets infinitesimally close to 1 but never reaches 1. This variation of $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ is strictly monotonic.

At the next rank of transfiniteness, we have the ω^{ω}-cable. This time we construct the nth-truncation of the ω^{ω}-cable exactly as we constructed the nth-truncation of the ω^{ω}-line. Then, at the jth sample point of the ω^{ω}-cable, we have the number $N_j(n)$ of sample points within the nth-truncation up to that jth sample point. $N_j(n)$ is given by (16) as before. Finally, we have $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ given by (19) with $x_{j,n} = \Delta x N_j(n)$ now.

Continuing on to the next higher rank of transfiniteness, we have the $\omega^{\omega+1}$-cable. This time we truncate as we did the $\omega^{\omega+1}$-line to get the following exparessions for the hyperreal voltage at the jth sample point in the $\omega^{\omega+1}$-cable. Let $M_j(n)$ be given by (17). Once again, $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ is given by (19) but with $x_{j,n} = \Delta x M_j(n)$.

This analysis can be continued to still higher ranks of transfinite cables. The voltage $\mathbf{v}(\mathbf{x}_j, \mathbf{t})$ behaves monotonically as before with respect to increasing hyperreal $\mathbf{t}$, but now a diffusion replaces a wave.

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Figure Captions

Fig. 1. An ω-line or ω-cable.

Fig. 2. An ω^2-line or ω^2-cable.

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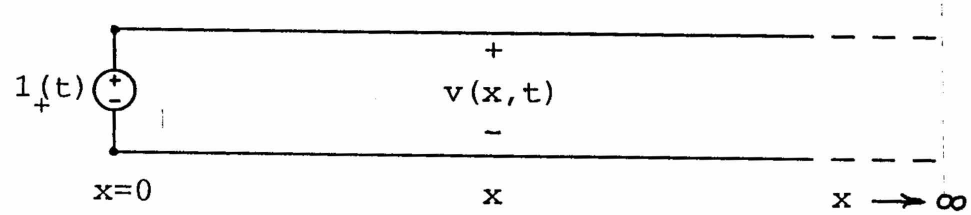

Fig. 1 ZEMANIAN

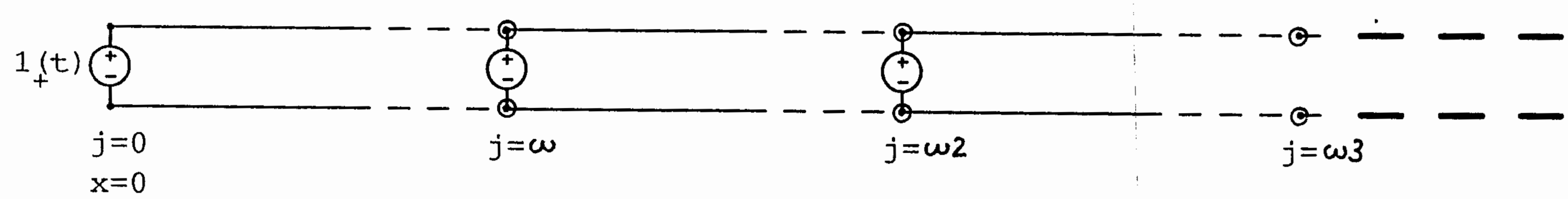

Fig. 2 ZEMANIAN