

UNIVERSITY AT STONY BROOK

CEAS Technical Report 807

A SHORTER PROOF OF THE TRANSITIVITY OF TRANSFINITE
CONNECTEDNESS

A.H. Zemanian

December 23, 2002

A SHORTER PROOF FOR THE TRANSITIVITY OF TRANSFINITE CONNECTEDNESS

A. H. Zemanian

Abstract — A criterion is established for the transitivity of connectedness in a transfinite graph. Its proof is much shorter than a prior argument published previously for that criterion.

Key Words: Transfinite graphs, transfinite connectedness, nondisconnectable tips.

The objective of this work is provide a proof of a criterion for the transitivity of transfinite connectedness that is much shorter than a previously published proof [1, pages 57-63] for that criterion.

We shall use the terminology defined in that book [1] and will add a few more definitions here. \mathcal{G}^ν will denote a transfinite graph of rank ν with $0 \leq \nu \leq \omega$. Two branches in \mathcal{G}^ν are said to be ρ -connected if there exists a γ -path ($0 \leq \gamma \leq \rho \leq \nu$) that terminates at a 0-node of each branch. Then, \mathcal{G}^ν is called *transfinitely connected* if every two of its branches are ρ -connected for some ρ depending on the branches. Transfinite connectedness is reflexive and symmetric binary relationship between the branches of \mathcal{G}^ν , but it need not be transitive. Examples of such nontransitivity are given in [1, Examples 3.1-5 and 3.1-6] and [2, Example 3.1-1].

However, the transitivity of ρ -connectedness can be assured by restricting the graph \mathcal{G}^ν appropriately. Two criteria, each of which suffices for such transitivity, are [1, Condition 3.2-1 and Condition 3.5-1]. The first criterion is rather lengthy and of limited use. The second one is quite simple to state and not overly restrictive, but the proof of it given in [1] is quite long and complicated.¹ This latter case can be simplified by assuming that

¹Please note the correction for [1, pages 63-69] stated in the Errata for that book given in the URL:

every node of \mathcal{G}^ν is *pristine*, that is, each node does not embrace a node of lower rank—and consequently is not embraced by a node of higher rank either.² Under this condition, the second criterion [1, Condition 3.5-1] works again and with a much shorter proof. It turns out that that latter proof can be modified to apply to nonpristine graphs as well, yielding thereby a more elegant argument for the transitivity of ρ -connectedness. This is the subject of the present work.

We need some definitions. Two nonelementary tips (not necessarily of the the same rank) are said to be *disconnectable* if one can find two representatives, one for each tip, that are totally disjoint. (Any two tips, at least one of which is elementary, are simply taken to be *disconnectable*.) As the negation of “disconnectable,” we say that two tips are *nondisconnectable* if every representative of one of them meets every representative of the other tip infinitely often, that is, at infinitely many nodes. We can state this somewhat differently, as follows. Two tips, t_1^γ and t_2^δ , are called *nondisconnectable* if P_1^γ and P_2^δ meet at least once whenever P_1^γ is a representative of t_1^γ and P_2^δ is a representative of t_2^δ .

Here is the sufficient condition that will ensure the transitivity of connectedness.

Condition 1. *If two tips of ranks less than ν (and not necessarily of the same rank) are nondisconnectable, then either they are shorted together or at least one of them is open.*

Before turning to our principle result, we need to introduce some more specificity into our terminology. The maximal nodes partition all the nodes of \mathcal{G}^ν ; indeed, two nodes are in the same set of the partition if they are embraced by the same maximal node. Thus, corresponding to any path, there is a unique set of maximal nodes that together embrace all the nodes of the path, and that set of maximal nodes becomes totally ordered in accordance with an orientation assigned to the path. Moreover, two maximal nodes are perforce totally disjoint. In order to avoid explicating repeatedly which nodes embrace which nodes, it is convenient to deal only with the maximal nodes of our graph \mathcal{G}^ν .

Let us expand upon this point because it is the key idea that allows us to replace the long proof in [1, pages 57-73] by a shorter, modified version of the proof in [2, pages 31-

www.ee.sunysb.edu/~zeman. Namely, every elementary set of a sequence of any rank is required to be node-distinct. This was implied by the fact that the argument was based on paths, but it should have been explicitly stated.

²See [2, Sec. 3.1] in this regard.

35]. Given any path P^ρ of any rank ρ in \mathcal{G}^ν , let \mathcal{X} be the set of (possibly nonmaximal) nodes encountered in the recursive construction of P^ρ ; that is, \mathcal{X} consists of the nodes in the sequential representation of P^ρ along with the nodes of lower ranks in the sequential representations of the paths between those nodes, and also along with the nodes of still lower ranks in the sequential representations of the paths of lower ranks in those latter paths, and so on down to the 0-nodes of the 0-paths embraced by P^ρ . Each node of \mathcal{X} need not be maximal, but it has a unique maximal node embracing it. Let \mathcal{X}_m denote the set of those maximal nodes. So, given P^ρ and thereby \mathcal{X} and \mathcal{X}_m , each node of \mathcal{X}_m corresponds to a unique set of nodes in \mathcal{X} such that, for every two nodes in that latter set, one embraces the other. The union of those sets is \mathcal{X} . In this way, there is a bijection between \mathcal{X}_m and the collection of the said sets, which in fact comprise a partition of \mathcal{X} . Moreover, when P^ρ has an orientation, that orientation induces a total ordering of \mathcal{X}_m . Thus, when dealing with the nodes of P^ρ , we can fix our attention on the nodes of \mathcal{X}_m . This we shall do. Any node x^β of \mathcal{X}_m will be called a *maximal node for P^ρ* to distinguish it from the corresponding subset of nodes in \mathcal{X} . In general, we may have $\beta > \rho$, but not necessarily always. Also, we can transfer our terminology for the nodes of a path, such as “incident to a node,” “meets a node,” or “terminates at a node,” to those maximal nodes. For example, a path is *incident to* a node $x \in \mathcal{X}_m$ if one of the path’s terminal tips is embraced by x ; this will be so if it has a terminal node embraced by x . In the latter case, we say that the path *meets* x and also *terminates at* x . Thus, two paths *meet at* $x \in \mathcal{X}_m$ if x embraces a node y of one path and a node z of the second path such that y embraces or is embraced by z .

The proof of our main result (Theorem 3) requires another idea, namely, “path cuts.” Let P^ρ be a ρ -path with an orientation. Let \mathcal{Y} be the set of all branches and all (not necessarily maximal) nodes of all ranks in P^ρ . The orientation of P^ρ totally orders \mathcal{Y} . With y_1 and y_2 being two members of \mathcal{Y} , we say that y_1 is *before* y_2 and that y_2 is *after* y_1 if in a tracing of P^ρ in the direction of its orientation y_1 is met before y_2 is met. A *path cut* $\{\mathcal{B}_1, \mathcal{B}_2\}$ for P^ρ is a partitioning of the set of branches of P^ρ into two nonempty subsets, \mathcal{B}_1 and \mathcal{B}_2 , such that every branch of \mathcal{B}_1 is before every branch of \mathcal{B}_2 . Another way of stating this is as follows. The partition $\{\mathcal{B}_1, \mathcal{B}_2\}$ of the branch set of P^ρ comprises a path cut for

P^ρ if and only if, for each branch $b \in \mathcal{B}_1$, every branch of P^ρ before b is also a member of \mathcal{B}_1 .

Here is a result that is easily proven through induction on ranks.

Lemma 2. *For each path cut $\{\mathcal{B}_1, \mathcal{B}_2\}$ for P^ρ , there is a unique maximal node x^γ ($\gamma \leq \rho$) for P^ρ such that every branch $b_1 \in \mathcal{B}_1$ is before x^γ and every branch $b_2 \in \mathcal{B}_2$ is after x^γ .*

We will say that the path cut *occurs at* the maximal node x^γ . As was stated before, a node is called *maximal* if it is not embraced by a node of higher rank. On the other hand, a maximal node for P^ρ may embrace many other nodes. It follows that all the nodes of P^ρ other than the nodes of P^ρ embraced by x^γ are also partitioned into two sets, the nodes of one set being before x^γ and the nodes of the other set being after x^γ .

Here now is our main result.

Theorem 3. *Let \mathcal{G}^ν ($0 \leq \nu \leq \omega$) be a ν -graph for which Condition 1 is satisfied. Let x_a , x_b , and x_c be three different maximal nodes in \mathcal{G}^ν such that, if any one of them is a singleton, its sole tip is disconnectable from every tip in the other two nodes. If x_a and x_b are ρ -connected and if x_b and x_c are ρ -connected ($0 \leq \rho \leq \nu$), then x_a and x_c are ρ -connected.*

Proof. Note that, if any one of x_a , x_b , and x_c is a nonsingleton, then each of its embraced tips (perforce, nonopen) must be disconnectable from every tip embraced by the other two nodes; indeed, otherwise, the tips of that node would be shorted to the tips of another one of those three nodes, according to Condition 1 and our hypothesis, and thus x_a , x_b , and x_c could not be three different maximal nodes.

Let P_{ab}^α ($\alpha \leq \rho$) be a two-ended α -path that terminates at the maximal nodes x_a and x_b and is oriented from x_a to x_b , and let P_{bc}^β ($\beta \leq \rho$) be a two-ended β -path that terminates at the maximal nodes x_b and x_c and is oriented from x_b to x_c . Let P_{ba}^α be P_{ab}^α but with the reverse orientation. P_{ba}^α cannot have infinitely many α -nodes because it is two-ended.

Let $\{x_i\}_{i \in I}$ be the set of maximal nodes at which P_{ba}^α and P_{bc}^β meet, and let \mathcal{X}_1 be that set of nodes with the order induced by the orientation of P_{ba}^α . If \mathcal{X}_1 has a last node x_l , then a tracing along P_{ab}^α from x_a to x_l followed by a tracing along P_{bc}^β from x_l to x_c yields a path

of rank no larger than ρ that connects x_a and x_c . Thus, x_a and x_c are ρ -connected in this case. This will certainly be so when $\{x_i\}_{i \in I}$ is a finite set.

So, assume \mathcal{X}_1 is an infinite, ordered set (ordered as stated). We shall show that \mathcal{X}_1 has a last node. Let Q_1 be the path induced by those branches of P_{ba}^α that lie between nodes of \mathcal{X}_1 (i.e., as P_{ba}^α is traced from x_b onward, such a branch is traced after some node of \mathcal{X}_1 and before another node of \mathcal{X}_1 , those nodes depending upon the choice of the branch.) Let \mathcal{B}_1 be the set of those branches. We can take it that P_{ba}^α extends beyond the nodes of \mathcal{X}_1 , for otherwise \mathcal{X}_1 would have x_a as its last node, and x_a and x_c would be ρ -connected. Therefore, we also have a nonempty set \mathcal{B}_2 consisting of those branches in P_{ba}^α that are not in \mathcal{B}_1 . $\{\mathcal{B}_1, \mathcal{B}_2\}$ is a path cut for P_{ba}^α . Consequently, by Lemma 2., there is a unique maximal node $x_1^{\gamma_1}$ at which that path cut occurs. Thus, Q_1 terminates at $x_1^{\gamma_1}$. Let $t_1^{\rho_1}$ be the ρ_1 -tip through which Q_1 reaches $x_1^{\gamma_1}$. Every representative of $t_1^{\rho_1}$ contains infinitely many nodes of \mathcal{X}_1 (otherwise, \mathcal{X}_1 would have a last node).

Now, consider P_{bc}^β . We can take it that there is a maximal node x_d in P_{bc}^β different from x_c such that the subpath of P_{bc}^β between x_d and x_c is totally disjoint from P_{ba}^α , for otherwise x_a and x_c would have to be the same node according to Condition 1 and our hypothesis again. We can partition the branches of P_{bc}^β into two sets, \mathcal{B}_3 and \mathcal{B}_4 , as follows. Each branch of the first set \mathcal{B}_3 is such that it lies before (according to the orientation of P_{bc}^β) at least one node in \mathcal{X}_1 of each representative of $t_1^{\rho_1}$, this being so for all such representatives.³ The second set \mathcal{B}_4 consists of all the branches of P_{bc}^β that are not in \mathcal{B}_3 . No branch of \mathcal{B}_4 can precede a branch of \mathcal{B}_3 . Thus, we have a path cut $\{\mathcal{B}_3, \mathcal{B}_4\}$ for P_{bc}^β and thereby (according to Lemma 2.) a unique maximal node $x_2^{\gamma_2}$ lying after the branches of \mathcal{B}_3 and before the branches of \mathcal{B}_4 . Let Q_2 be the path induced by \mathcal{B}_3 . It reaches $x_2^{\gamma_2}$ through some tip $t_2^{\rho_2}$. Furthermore, each representative of $t_2^{\rho_2}$ must meet each representative of $t_1^{\rho_1}$ at least once because they meet at at least one node of \mathcal{X}_1 . Thus, $t_1^{\rho_1}$ and $t_2^{\rho_2}$ are nondisconnectable. Moreover, neither of those tips can be open (i.e., be in a singleton node) because the paths P_{ba}^α and P_{bc}^β pass through and beyond their respective nodes $x_1^{\gamma_1}$ and $x_2^{\gamma_2}$. So, by Condition

³Let us note here a correction for [2, page 34, line 19 up]. Replace the sentence on that line by the following sentence: "Let \mathcal{N}_2 be the set of those nodes in $\{n_i\}_{i \in I}$ that lie before (according to the orientation of P_{bc}^β) at least one node of $\{n_i\}_{i \in I}$ in each representative of $t_1^{\rho_1}$, this being so for all such representatives."

1, $x_1^{\gamma_1}$ and $x_2^{\gamma_2}$ must be the same node because the tips $t_1^{\rho_1}$ and $t_2^{\rho_2}$ are shorted together.

This means that \mathcal{X}_1 has a last node, namely, $x_1^{\gamma_1} = x_2^{\gamma_2}$. It follows now that x_a and x_c are ρ -connected. ♣

The last proof has established the following two result.

Corollary 4. *Under the hypothesis of Theorem 3, let P_{ab}^α be a two-ended α -path connecting nodes x_a and x_b , and let P_{bc}^β be a two-ended β -path connecting nodes x_b and x_c . Then, there is a two-ended γ -path ($\gamma \leq \max\{\alpha, \beta\}$) connecting x_a and x_c that lies in $P_{ab}^\alpha \cup P_{bc}^\beta$.*

Corollary 5. *Under the hypothesis of Theorem 3, let $\{x_i\}_{i \in I}$ be the set of maximal nodes at which two two-ended paths meet. Assume that, if any terminal node of either path is a singleton, its sole tip is disconnectable from every tip in the two terminal nodes of the other path. Assign to $\{x_i\}_{i \in I}$ the total ordering induced by an orientation of one of these two paths. Then, $\{x_i\}_{i \in I}$ has both a first node and a last node.*

References

- [1] A.H. Zemanian, *Transfiniteness for Graphs, Electrical Networks and Random Walks*, Birkhauser, Boston, 1996.
- [2] A.H. Zemanian, *Pristine Transfinite Graphs and Permissive Electrical Networks*, Birkhauser, Boston, 2001.