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IRROTATIONAL, INCOMPRESSIBLE, FREE-FALLING JET

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Stony Brook, New York

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ABSTRACT

A matched asymptotic expansion type of solution is developed for the case of an irrotational free-falling jet emanating from a slit at a bottom of a large tank. The terms in the asymptotic expansion which represent the flow upstream are obtained as series solutions of potential problems with assigned boundary values. The coefficients in the series are governed by a set of infinite algebraic equations which are solved by inverting the associated matrix. The resulting solution for the flow yields a formula which relates the volumetric flow rate to the relative strength of gravity.

1. INTRODUCTION

A large class of two dimensional potential flows can be easily solved when the so-called Helmholtz condition holds on the free streamlines. However, with non zero gravity, the boundary condition that is imposed there is highly non-linear. This non-linearity is troublesome when the fluid particles on the free surface undergo a finite vertical drop. The situation is ever so more difficult when that vertical drop is infinite and the fluid speed increases indefinitely, even if gravity is comparatively ineffective. It is noted that the flow pattern for a free falling jet is in no way similar to that which prevails in the absence of gravity. Modification of the classical mapping sequence technique or perturbation around a Helmholtz type flow is therefore not expected to succeed if the flow downstream is gravity controlled. It thus seems that the only way to deal with such problems is to recognize the distinctness of such domain, develop an appropriate expression for the solution there, and match it with the solution characterizing the inertia or pressure controlled flow. This approach was suggested by Clarke (1965), who applied it to what is known as the waterfall problem. Ackerberg (1968) follows suit. In this work the author adheres to this general philosophy. However a slightly more complicated case is considered and a somewhat less familiar analysis is employed.

Both Clarke and Ackerberg consider cases in which the flow pattern upstream is (very nearly) uniform and of constant-width. The volumetric flow rate and axial flux of momentum are therefore predetermined, and the a-priori knowledge of these quantities makes the analyses easier. Indeed, the fact that these quantities should be equal in terms of the upstream and downstream expansions simplifies the matching procedure considerably. Typically, Clarke evaluates the flux of momentum using the far upstream conditions and employs this quantity as a parameter in the downstream asymptotic expansion. It therefore seems appropriate to consider a case in which gravity not only affects the geometry of the free falling jet downstream, but also influences the volumetric flow rate throughout. One such case is that of a jet emerging from a slit at the horizontal bottom of a large tank (see Fig. 1). This case is solved here.

Both in the works cited and in the present paper the terms in the upstream expansion are the solutions of a sequence of potential problems. The usual way to solve these is in terms of integrals over the surface, or parts of it. In the sequence under discussion the boundary values for the higher order solutions are sums of products of the lower order ones. Therefore, although in principle a solution of almost any order can be obtained by carrying out repeatedly integrations over the boundary, the results obtained by such process are often difficult to evaluate. Since in the matching procedure coefficients in the asymptotic expansions have to be evaluated numerically this computational difficulty can become a serious shortcoming. Therefore, rather than obtain solutions for the potential problems in terms of integrals, these will be solved by the series method which was used by the author (1963) in the treatment of Levi Civitas' problem. The adaptability of this latter technique to the complicated boundary conditions encountered here will soon become apparent.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

The complex variable z associated with the physical plane (Fig. 1) is chosen so that gravity acts along the positive real axis. The complex potential w-plane is shown in Fig. 2, and it is assumed that there exists an analytic function w(z) which maps the entire flow into an infinite strip of width $M\pi$. The origin of the w plane is chosen so that plane of symmetry in the z plane, the Re(z) axis, coincides with $\psi = Im(w) = 0$. The curve $\varphi = Re(w) = 0$ is assumed to pass through the edges of the slit.

The problem under discussion is solved by letting (φ, ψ) play the role of independent variables while (β, α), which are defined as follows,

$$\zeta = \beta + i\alpha = \ln(dz/dw),$$

are taken to be the dependent variables. Hence $(-\beta)$ is the logarithm of the flow-speed while α is the angle between the direction of flow and the positive $Re(z)$ axis. If $\zeta(w)$ is analytic the following Cauchy Riemann equations hold,

$$\partial\beta/\partial\varphi = \partial\alpha/\partial\psi, \qquad \partial\beta/\partial\psi = -\partial\alpha/\partial\varphi, \qquad (1) \quad (2)$$

whence α and β are harmonic in the domain $-M\pi/2 < \psi < M\pi/2$, $-\infty < \varphi < \infty$. On the boundaries these quantities satisfy the conditions

$$\alpha = \mp \pi/2 \qquad \psi = \pm M\pi/2, \qquad \varphi < 0, \tag{3}$$

$$\frac{\partial}{\partial\varphi}\left(\frac{1}{3}\exp(-3\beta)\right) = \varepsilon\cos\alpha, \quad \psi = \pm M\pi/2, \qquad \varphi > 0. \tag{4}$$

Here, the first condition represents the solid boundaries, or the bottom of the tank, while the latter is the Bernoullis' equation when the pressure along the free streamlines is constant. The parameter ε is the inverse of the Froude number defined thus:

$$\varepsilon \equiv g\ell/V^2$$

where g is the gravitational acceleration. All variables are non-dimensionalized with respect to half the width of the slit and the flow-speed at the points where the free streamlines emerge from the slit.

It follows from the choice of characteristic speed, V, that the following condition should hold

$$\beta(0, \pm M\pi/2) = 0. \tag{5}$$

The choice of characteristic length ℓ gives

$$\int_{-iM\pi/2}^{iM\pi/2} \exp(\zeta(0,\psi))\, d(i\psi) = 2i. \tag{6}$$

Since the last two relationships should be satisfied by expansions, every term in these will be made to satisfy equation (5). However, equation (6) cannot be as easily satisfied. M will therefore be left as an unknown until the solution $\zeta(w)$, or a suitable approximation for it, is obtained. Finally, by requiring that equation (6), should hold, M will be obtained. This procedure reflects the fact that the volumetric discharge cannot be evaluated without solving for the flow pattern.

3. SOLUTION BY MATCHED EXPANSIONS

The asymptotic expansions will now be developed in terms of the appropriately scaled independent variables. These are taken to be

$$(\Phi, \Psi) = (\varphi/M, \psi/M)$$

$$(\tilde{\bar{\Phi}}, \Psi) = (\varepsilon\varphi/M, \psi/M)$$

upstream and downstream, respectively. The role of the scale factor ε in the definition of these variables has been amply discussed. The introduction of the factor M is convenient but not essential.

The following expansions are assumed to hold downstream

$$\beta^{(d)}(\tilde{\bar{\Phi}}, \Psi) = \sum_{i=0}^{\infty} \beta_i^{(d)} \varepsilon^i \quad , \quad \alpha^{(d)}(\tilde{\bar{\Phi}}, \Psi) = \sum_{i=0}^{\infty} \alpha_i^{(d)} \varepsilon^i$$

as indicated by the bracketed superscript 'd'. When these expressions are substituted into the Cauchy-Riemann equations and like powers of ε are equated, the following relationships are obtained:

$$\partial \alpha_i^{(d)} / \partial \Psi = \partial \beta_{i-1}^{(d)} / \partial \tilde{\tilde{\Phi}} \qquad , \quad \partial \beta_i^{(d)} / \partial \Psi = -\partial \alpha_{i-1}^{(d)} / \partial \tilde{\tilde{\Phi}} \ . \qquad (7_i)(8_i)$$

It follows from (7_o) and the antisymmetry of α with respect to $\Psi = 0$ that $\alpha_o^{(d)}$ vanishes. When use is made of that result, the Bernoullis' equation yields

$$\frac{\partial}{\partial \tilde{\tilde{\Phi}}} \left(\frac{1}{3} \exp\left(-3\beta_o^{(d)}\right) \right) = M , \qquad (9_o)$$

$$- \frac{\partial}{\partial \tilde{\tilde{\Phi}}} \left(\beta_1^{(d)} \exp\left(-3\beta_o^{(d)}\right) \right) = 0 , \qquad (9_1)$$

$$\frac{\partial}{\partial \tilde{\tilde{\Phi}}} \left(\left(\frac{3}{2}\left(\beta_1^{(d)}\right)^2 - \beta_2^{(d)} \right) \exp\left(-3\beta_o\right) \right) = - M \frac{\left(\alpha_1^{(d)}\right)^2}{2} , \qquad (9_2)$$

etc. The first of these equations together with equation (8_o) yields

$$\beta_o^{(d)} = -1/3 \ln \left(C_1 + 3M \tilde{\tilde{\Phi}} \right)$$

where C_1 is a constant. Then, by considering equations (7_i) (9_i) and (8_i), in this order, for $i = 1$ and $i = 2$ one obtains

$$\beta_1^{(d)} = C_2 \left(C_1 + 3M \tilde{\tilde{\Phi}} \right)^{-1} , \qquad \alpha_1^{(d)} = - M \Psi \left(C_1 + 3M \tilde{\tilde{\Phi}} \right)^{-1} ,$$

$$\alpha_2^{(d)} = - 3 M C_2 \Psi \left(C_1 + 3M \tilde{\tilde{\Phi}} \right)^{-2} , \qquad (10)$$

$$\beta_2^{(d)} = \left[\frac{3}{2} \left(C_2\right)^2 - \frac{M^2}{6} \left(9\Psi^2 - 8\left(\frac{\pi}{2}\right)^2 \right) \right] \left(C_1 + 3M \tilde{\tilde{\Phi}} \right)^{-2} + C_3 \left(C_1 + 3M \tilde{\tilde{\Phi}} \right)^{-1} .$$

This process can be continued, but within the framework of this study only three terms in each expansion will be evaluated. The constants of integrations C_i are obtained by matching with the upstream expansion.

As indicated by the bracketed superscript the assumed upstream expansion is

$$\beta^{(u)}(\Phi,\Psi)=\sum_{i=0}^{\infty}\beta_i^{(u)}\varepsilon^i \quad , \quad \alpha^{(u)}(\Phi,\Psi)=\sum_{i=0}^{\infty}\alpha_i^{(u)}\varepsilon^i \; .$$

This form is combined with the Bernoullis' equation to yield

$$\frac{\partial}{\partial\Phi}\left(\frac{1}{3}\exp\left(-3\beta_0^{(u)}\right)\right) = 0 \; , \qquad (11_0)$$

$$-\frac{\partial}{\partial\Phi}\left(\beta_1^{(u)}\exp\left(-3\beta_0^{(u)}\right)\right) = M\cos\alpha_0^{(u)} \; , \qquad (11_1)$$

$$\frac{\partial}{\partial\Phi}\left(\left(\frac{3}{2}\left(\beta_1^{(u)}\right)^2-\beta_2^{(u)}\right)\exp\left(-3\beta_0^{(u)}\right)\right) = -M\alpha_1^{(u)}\sin\alpha_0^{(u)} \qquad (11_2)$$

For the solid part of the boundary equation (3) yields

$$\alpha_i^{(u)} = \mp\delta_{0i}\,\pi/2 \, , \qquad \Psi=\pm\pi/2 \, , \quad \Phi<0 \, , \quad (12_i)$$

where δ_{jk} is the Kronecker delta. Inside the domain $-\frac{\pi}{2}<\Psi<\frac{\pi}{2}$, $-\infty<\Phi<\infty$, the Cauchy-Riemann equations hold. One notes that in the definition of the independent variables associated with the upstream domain the scale factor ε is absent. Therefore, $(\beta_i^{(u)},\alpha_i^{(u)})$ unlike $(\beta_i^{(d)},\alpha_i^{(d)})$ form complex conjugate pairs. Hence $\beta_i^{(u)}$

and $\alpha_i^{(n)}$ and harmonic in $(\bar{\Phi}, \bar{\Psi})$.

It follows from equations (10_0) (12_0) and (5) that the zeroth complex pair is the so-called Helmholtz solution, given by

$$\zeta_0^{(n)} = (\beta_0^{(n)} + i\,\alpha_0^{(n)}) = \ln\left\{ \exp(-W) + \left(1 + \exp(-2W)\right)^{1/2} \right\} \tag{13}$$

where

$$W = \bar{\Phi} + i\bar{\Psi} .$$

By expressing $\beta_0^{(n)}(\bar{\Phi}, \pm\pi/2)$ in terms of $\tilde{\bar{\Phi}}$ and $\beta_0^{(d)}(\tilde{\bar{\Phi}}, \pm\pi/2)$ in terms of $\bar{\Phi}$, expanding and comparing terms of $0(\varepsilon^0)$ one obtains the (known) result

$$C_1 = 1 .$$

Use is then made of the fact that since $\beta_0^{(n)}$ vanishes the following holds

$$Re\left\{ \exp\left(\beta_0^{(n)} + i\,\alpha_0^{(n)}\right) \right\} = \cos\alpha_0^{(n)} \qquad \bar{\Psi} = \pm\frac{\pi}{2} , \quad \bar{\Phi} > 0 ,$$

so that equation (11_1) can be rewritten thus

$$\frac{\partial}{\partial\bar{\Phi}}\left(\beta_1^{(n)}\exp\left(-3\beta_0^{(n)}\right)\right) = -M\left(1 - \exp(-2\bar{\Phi})\right)^{1/2} \tag{11_1}$$

This can be integrated, subject to condition (5), to yield

$$\beta_1^{(n)} = M\left\{ \frac{1}{2}\ln\left[\frac{1 - (1 - \exp(-2\bar{\Phi}))^{1/2}}{1 + (1 - \exp(-2\bar{\Phi}))^{1/2}}\right] + \left(1 - \exp(-2\bar{\Phi})\right)^{1/2} \right\} \tag{14}$$

The last relationship, together with equation (12_1) constitute the boundary conditions which determine $(\beta_1^{(u)}, \alpha_1^{(u)})$. However, without solving for these one can match along the free streamlines $\beta_0^{(u)} + \varepsilon\beta_1^{(u)}$ with $\beta_0^{(d)} + \varepsilon\beta_0^{(d)}$. This yields the following result

$$C_2 = M(1 - \ln 2)$$

As mentioned above $\zeta_1^{(u)}$ and $\zeta_2^{(u)}$ are solved for by a series technique. This can be explained by noting that $\zeta_0^{(u)}$ can be expanded as follows:

$$\zeta_0^{(u)} = -W + \sum_{n=0}^{\infty}(-1)^n B_n^0 \exp(2nW) \qquad \Phi < 0,\ -\frac{\pi}{2} < \Psi < \frac{\pi}{2},$$

$$= \sum_{m=0}^{\infty}(-1)^m A_m^0 \exp(-(2m+1)W) \qquad \Phi > 0,\ -\frac{\pi}{2} < \Psi < \frac{\pi}{2}$$

and that this form could have been deduced from the following considerations. Far upstream the flow pattern is that of sink of (unknown) strength M so that the following relationship holds there

$$w = -M\ln z + iM\pi, \qquad \zeta = -W - \ln M.$$

The contribution to $\zeta_0^{(u)}$ which represents the deviation from a sink-type flow must vanish for $\Phi \to -\infty$. Moreover, the imaginary part of that contribution should vanish on $\Psi = \pm\pi/2$, where the sink flow satisfies the conditions imposed on $\alpha_0^{(u)}$. The most general analytic function satisfying these two requirements is the even series with arbitrary coefficients B_n^0. Similarly, the odd series with the arbitrary coefficients A_m^0 is the most general form for an analytic function which is finite for $\Phi \to \infty$ and has a vanishing real part along $\Psi = \pm\pi/2$.

A similar consideration leads one to choose the following form of solution for $\zeta_1^{(u)}(W)$:

$$\zeta_1^{(u)} = M\left\{(1-\ln 2) - W - \sum_{n=1}^{\infty}\frac{(2n-2)!}{2^{2n}(n!)^2}(-1)^n \exp(-2nW)\right.$$

$$\left.+\sum_{m=0}^{\infty}(-1)^{m}A'_m \exp(-(2m+1)W)\right\}, \qquad \Phi > 0, \quad -\frac{\pi}{2} < \Psi < \frac{\pi}{2},$$

$$= M\sum_{n=0}^{\infty}(-1)^n B'_n \exp(2nW), \qquad \Phi < 0, \quad -\frac{\pi}{2} < \Psi < \frac{\pi}{2}. \quad (16)$$

The odd and even series have vanishing real and imaginary parts where equation (14) and (12_1), respectively, should hold. Furthermore, these are finite for $\Phi \to -\infty$ and $\Phi \to \infty$. The contribution to the expression for $\zeta_1^{(u)}$, which has known coefficients satisfies the inhomogeneous equation (14). To complete the solution for $\zeta_1^{(u)}$, one is thus left with the task of evaluating the coefficients A'_m and B'_n. This is achieved by requiring that $\beta_1^{(u)}$ and $\partial\beta_1^{(u)}/\partial\Phi$ (or else $\alpha_1^{(u)}$, and its Φ derivative) should be continuous along $\Phi = 0$. As explained in an earlier work by the author (1963) these requirements yield two equalities between Fourier cosine (or sine) series of different types. When these equalities are multiplied by $\cos(2n\Psi)$ and integrated with respect to Ψ from $\Psi = -\frac{\pi}{2}$ to $\Psi = +\pi/2$ one gets

$$B_0'\pi = (1 - \ln 2)\pi + 2\sum_{m=0}^{\infty} A_m' (2m+1)^{-1}, \qquad (17_0)$$

$$0 = -\pi - 2\sum_{m=0}^{\infty} A_m', \qquad (18_0)$$

$$B_n'\frac{\pi}{2} + \frac{(2n-2)!}{2^{2n}\,(n!)^2}\frac{\pi}{2} = 2\sum_{m=0}^{\infty} A_m' \frac{(2m+1)}{(2m+1)^2 - (2n)^2}, \qquad (17_n)$$

$$2n B_n'\frac{\pi}{2} - \frac{(2n-2)!}{2^{2n-1}\,n!\,(n-1)!}\frac{\pi}{2} = -2\sum_{m=0}^{\infty} A_m' \frac{(2m+1)^2}{(2m+1)^2 - (2n)^2}. \qquad (18_n)$$

These relationships provide as many algebraic equations as unknowns. However, solution by straightforward transaction is unreliable because the coefficients in equations (18_n), $n = 0, 1, 2 \ldots$ do not decrease for $m \to \infty$. Therefore, both this system, as well as that encountered in the solution for $\zeta_2^{(n)}$ require a more sophisticated treatment, which is explained in the next section. First a solution for the next term in the expansion, $\zeta_2^{(n)}$ will be developed, assuming that $\zeta_1^{(n)}$ and hence the coefficients A_m' and B_n' are known.

In order to evaluate $\zeta_2^{(u)}$ and C_3 the boundary value of $\beta_2^{(u)}$ along the free streamline is obtained by integrating equation (11_2). When use is made of equations (11_2), (16), and the identity

$$\operatorname{Im} \exp(\zeta_0^{(u)}) = \sin(\alpha_0^{(u)}) = \mp \exp(-\Phi), \qquad \Psi = \pm \pi/2 ,$$

the following result is obtained:

$$\begin{aligned}
\beta_2^{(u)} = M^2 \Big\{ & 1/2 \sum_{m=0}^{\infty} (m+1)^{-1} A_m' \left(1 - \exp(-(2m+1)\Phi)\right) + (\pi/2)(1 - \exp(-\Phi)) \\
& + (3/2)(1 - \ln 2)^2 + (3/2)\Phi^2 + 3(\ln 2 - 1)\Phi \\
& + \frac{3}{2} \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} \frac{(2n-2k-2)!}{2^{2n-2k}((n-k)!)^2} \, \frac{(2k-2)!}{2^{2k}(k!)^2} \right) \exp(-2n\Phi) \\
& 3 \sum_{n=1}^{\infty} \frac{(2n-2)!}{2^{2n}(n!)^2} \Phi \exp(-2n\Phi) + 3(\ln 2 - 1) \sum_{n=1}^{\infty} \frac{(2n-2)!}{2^{2n}(n!)^2} \exp(-2n\Phi) \Big\} . \qquad (19)
\end{aligned}$$

This result is used in the well known matching process. Retaining terms of $0(\varepsilon^2)$ in both the upstream and downstream expansions one gets

$$\begin{aligned}
\beta_0^{(u)} + \varepsilon \beta_1^{(u)} + \varepsilon^2 \beta_2^{(u)} \sim \; & - \varepsilon M (\ln 2 - 1) \\
& - M\tilde{\Phi} + \varepsilon^2 M^2 (1/2) \sum_{m=0}^{\infty} (m+1)^{-1} A_m' + (\pi/2)\varepsilon^2 M^2 \\
& + (3/2)\varepsilon^2 M^2 (1 - \ln 2)^2 + (3/2) M^2 \tilde{\Phi}^2 + \\
& + 3\varepsilon M^2 (\ln 2 - 1)\tilde{\Phi} + O(\varepsilon e^{-1/\varepsilon}) \qquad (20_u)
\end{aligned}$$

and

$$\beta_0^{(d)} + \varepsilon\beta_1^{(d)} + \varepsilon^2\beta_2^{(d)} \sim -M\varepsilon\bar{\Phi} + (3/2)\,\varepsilon^2 M^2\bar{\Phi}^2$$

$$-\varepsilon M(\ln 2 - 1) + 3\varepsilon^2 M^2(\ln 2 - 1)\bar{\Phi} + \varepsilon^2 C_3$$

$$+\varepsilon^2 M^2\left[(3/2)(1-\ln 2)^2 - (1/6)(\pi/2)^2\right] + O(\varepsilon^3) \,. \qquad (20d)$$

Therefore, C_3 is given by

$$C_3 = M^2\left[(\pi/2) + (1/6)(\pi/2)^2 + (1/2)\sum_{m=0}^{\infty}(m+1)^{-1}A_m'\right].$$

It follows from arguments similar to those expounded earlier that (ζ_2^u / M^2) can be expressed by the following form

$$\zeta_2^u/M^2 = 3(\ln 2 - 1)\Psi + (3/2)\left(\Psi^2 + (\pi/2)^2\right) - \Psi\exp(-\Psi)$$

$$+ \sum_{n=0}^{\infty}(-1)^n C_n \exp(-2n\Psi) + \sum_{m=0}^{\infty}(-1)^m A_m^2\exp(-(2m+1)\Psi)$$

$$+ 3\sum_{n=1}^{\infty}(-1)^n\frac{(2n-2)!}{2^{2n}(n!)^2}\,\Psi\exp(-2n\Psi)\,, \qquad \bar{\Phi} > 0$$

$$= \sum_{n=0}^{\infty}(-1)^n B_n^2\exp(2n\Psi)\,, \qquad \bar{\Phi} < 0\,. \qquad (21)$$

Here the coefficients in the odd and even series, A_m^2 and B_n^2 are still unknown, while the constants C_n are given by:

$$C_0 = (1/2)\sum_{m=0}^{\infty}(m+1)^{-1}A'_m + (3/2)(1-\ln 2)^2 + (\pi/2)$$

$$C_1 = -(1/2)A'_0 + (3/4)(\ln 2 - 1)$$

$$C_n = -\frac{A'_{n-1}}{2n} + \frac{3}{2}\sum_{k=1}^{n-1}\frac{(2n-2k-2)!}{2^{2n-2k}((n-k)!)^2}\cdot\frac{(2k-2)!}{2^{2k}(k!)^2} + 3(\ln 2-1)\frac{(2n-2)!}{2^{2n}(n!)^2}$$

The requirement that $\zeta_2^{(n)}$ and its ϕ derivative should be continuous along $\phi = 0$ yields the algebraic relationships governing the unknowns. These read

$$B_0^2\pi = C_0\pi + (1/4)\pi^3 + 2$$

$$- 3\pi\sum\frac{(2k-2)!}{2^{2k+1}\,k!\,(k+1)!} + \sum_{m=0}^{\infty}A_m^2, \qquad (22_0)$$

$$0 = 3(\ln 2-1)\pi + 3\pi\sum_{k=1}^{\infty}\frac{(2k-2)!}{2^{2k}(k!)^2} - 2\sum_{m=0}^{\infty}A_m^2, \qquad (23_0)$$

$$B_n^2(\pi/2) = C_n(\pi/2) - (3/2)\, n^{-2}(\pi/2) - 2(4n^2+1)(4n^2-1)^{-2}$$

$$+\ 3\,\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{(2k-2)!}{2^{2k}\,(k!)^2}\begin{cases} k(n^2-k^2)^{-1} & n\neq k \\ -(4k)^{-1} & n=k \end{cases}$$

$$+\ 2\sum_{m=0}^{\infty} A_m^2\,(2m+1)\left[(2m+1)^2-(2n)^2\right]^{-1}, \qquad (22_n)$$

$$(2n)\,B_n^2(\pi/2) = -(2n)\,C_n(\pi/2) + 3(\pi/2)\,\frac{(2n-2)!}{2^{2n-1}\,n!\,(n-1)!}$$

$$-\ 3\,\frac{\pi}{2}\sum_{k=1}^{\infty}\frac{(2k-2)!}{2^{2k-1}\,k!\,(k-1)!}\begin{cases} k(n^2-k^2)^{-1} & n\neq k \\ -(4k)^{-1} & n=k \end{cases}$$

$$16n^2(4n^2-1)^{-2} - 2\sum_{m=0}^{\infty} A_m^2\,(2m+1)^2\left[(2m+1)^2-(2n)^2\right]^{-1}. \qquad (23_n)$$

The coefficients in this system of equations and in the system consisting of equations (17_n) and (18_n) are identical. Moreover, it is the same set of coefficients which form the matrix associated with the solution for $\beta_3^{(n)}$, $\beta_4^{(n)}$ ... etc.

4. Solution for the Unknowns A_m^i and B_n^i

The system of equations governing the unknowns (A_m^i, B_n^i) can be reduced to the following form,

$$(\pi/2)\,D_n^i = \sum_{m=0}^{\infty} A_m^i\,(2m+1)\Big/(2m+1-2n), \qquad n=0,1,2\ldots \qquad (24_n)$$

Equation (24_0) is simply another way of writing equations (18_0) and (23_0). Equations (24_n), $n > 0$, are derived by linearly combining equation (17_n) with (18_n) or (22_n) with (23_n). ~~In this~~ process the constants B^i_n, $n > 0$, are eliminated. Of course, once the set (24_n) is solved and the A^i_n are known, the constants B^i_n can be evaluated by using equations (17_n) and (22_n), $n = 0, 1, 2 \cdots$. The inhomogeneous terms D^i_n are known for $i = 1, 2$ and can be obtained, without too much difficulty, for the set governing the coefficients in the higher terms of the upstream asymptotic expansion. It is noted that the reduction to the form (24_n) is possible for all functions $\zeta(W)$ which are expressible in the form containing odd and even expansions for $\bar{\Phi} > 0$ and $\bar{\Phi} < 0$, respectively. The correspondence between such functions and the sets (24_n) plays an important role in the inversion of the matrix associated with the latter. Furthermore, once achieved, the inversion of this matrix allows the refinement of the present solution by the evaluation of (A^i_m, B^i_n) $i > 2$.

The solution for $\zeta^{(n)}_0(W)$ contains a clue leading to the inversion scheme proposed here. One finds from equation (13) that the coefficients in the odd power series representing $\zeta^{(n)}_0(W)$ are given by

$$A^0_m = (2m+1)^{-1}(2m)!\, 2^{-2m}(m!)^{-2} \tag{25}$$

On the other hand these constants satisfy equations (24_n) when the inhomogeneous terms are

$$D^0_n = \delta_{n0},$$

i. e. the first is the only non-trivial one. In other words the terms on the right hand side of equation (25) are $(\pi/2)$ times the members of the first column in the inverted matrix sought. This leads one to seek the general $(l+1)$ column in the inverted matrix by constructing the appropriate sequence of functions ζ_l, $l = 1, 2, 3\ldots$. These must be analytic in $-\infty < \Phi < \infty$, $-\pi/2 < \Psi < \pi/2$, expressible by a form containing an odd series in the domain $\Phi > 0$, with coefficients satisfying equations (24_n) in which all but the $(l+1)$ inhomogeneous term vanish. One such class of functions is given by

$$\zeta_l = (-1)^l \exp(-2lW) + \sum_{m=0}^{\infty} a_m^l (-1)^m \exp(-(2m+1)W), \quad \Phi > 0,$$

$$= \sum_{n=0}^{\infty} b_n^l (-1)^n \exp(2nW), \qquad \Phi < 0 . \tag{26}$$

Indeed, it can easily be shown that the constants a_m^l satisfy the equations under discussion:

$$(\pi/2)(2l)\,\delta_{ln} = \sum_{m=0}^{\infty} a_m^l (2m+1)/(2m+1-2n) \tag{$24'_n$}$$

and that the terms on their left hand side have the desired properties. Furthermore, it is possible to obtain a closed-form expression for ζ_l from which the constants a_m^l can be calculated; these are proportional to the elements in the $(l+1)$ column of the inverted matrix. The function $\zeta_l(W)$ is uniquely determined from the boundary conditions implied by the right hand side of equation (26)

$$\mathrm{Re}(\zeta_\ell) = \exp(-2\ell\Phi), \qquad \Phi > 0, \quad \Psi = \pm\pi/2, \tag{27}$$

$$\mathrm{Im}(\zeta_\ell) = 0, \qquad \Phi < 0, \quad \Psi = \pm\pi/2 \tag{28}$$

Without going into the uninteresting detailed solution of the boundary value problem at hand, one can verify that it is given by

$$\zeta_\ell(W) = (-1)^\ell \exp(-2\ell W)$$

$$-(-1)^\ell\left(1+\exp(-2W)\right)^{1/2}\sum_{k=0}^{\ell-1}(-1)^k\frac{(2k)!}{2^{2k}(k!)^2}\exp\left(-(2\ell-1-2k)W\right). \tag{29}$$

From the finite summation in the above equation together with equations ($24_n'$) and (25) one can determine each and every term in the inverted matrix. Its upper left corner is tabulated below

1	-1/2	-1/8	-1/16	-5/128	...
1/6	1/4	-3/16	-5/96	-7/256	...
3/40	1/16	9/64	-15/128	-35/1026	...
5/112	1/32	5/128	25/256	-175/4096	...
35/1152	5/256	21/1024	175/6144	1225/16384	
...	...	...	...	...	

Table 1. $(\pi/2)$ times upper left part of the inverse of the matrix with elements $(2m+1)(2m+1-2n)^{-1}$, $(m, n) = 0, 1, 2 \ldots$

The constants A_n^i can be obtained by multiplying the inverse matrix with the vector formed by the sequence of elements D_n^i , $n = 0, 1, 2 \ldots$ The latter are easily obtained for $i = 1$. However, the infinite series appearing on the left hand sides of equation (22_n) and (23_n) makes the evaluation of D_n^2 a somewhat more complicated process. The summation contained in the expression for D_0^2 is given by

$$\sum_{k=1}^{\infty} \frac{(2k-2)!}{2^{2k}(k!)^2} = \int_0^1 \frac{1}{x}\left(1 - (1-x^2)^{1/2}\right) dx = (1 - \ln 2)$$

from which it follows that D_0^2 vanishes. The infinite summations contained in the expression for D_n^2, $n > 0$, can be similarly calculated by considering a finite integral which contains $(1 - x)^{1/2}$ rather than $(1 - x^2)^{1/2}$. The values of the first four constants for i = 1,2 are listed below

$A_0^1 = 1.27$	$A_1^1 = .07$	$A_2^1 = .02$	$A_3^1 = .01$
$A_0^2 = 2.69$	$A_1^2 = 1.18$	$A_2^2 = .37$	$A_3^2 = .19$

5. RESULTS AND CONCLUSION

The properties of the flow under discussion can be studied either by examining the upstream and downstream expressions separately, or else by forming a composite solution. Many of these typical features are known. However, a property which is of interest and which cannot be obtained from other solutions for flow under gravity obtained to date, is the influence of that effect on the volumetric discharge. This is obtained by approximating the integrand of equation (6) in the following manner:

$$\exp\left(\zeta^{(4)}(0+,\Psi)\right) \simeq \left[\exp(-i\Psi) + \left(1+\exp(-2i\Psi)\right)^{1/2}\right]\times$$

$$\times\left(1 - \varepsilon M\left\{\sum_{m=0}^{\infty}(-1)^m A'_m \exp(-(2m+1)i\Psi) - i\Psi + \right.\right.$$

$$\left.\left. + (1-\ln 2) - \sum_{n=1}^{\infty}(-1)^n \frac{(2n-2)!}{2^{2n}(n!)^2}\exp(-2ni\Psi)\right)\right) + O(\varepsilon^2)$$

Integration and rearrangement yield

$$M = \left(\frac{2}{2+\pi}\right)\left(1 + \varepsilon(2\cdot 33)\left(\frac{2}{2+\pi}\right)^2\right)$$

which is correct to within an error of $O(\varepsilon^2)$. This shows that, as could have been expected, the rate of volumetric discharge increases with gravity.

One notes that the matrix with elements $(2m+1)(2m+1-2n)^{-1}$ is encountered not only here, but also in the author's previous work (1963). There, however, the matrix was first truncated and then inverted - and this is reflected in the accuracy of the numerical results. These can now be improved by making use of the results contained in Table 1. In addition, by adopting the procedure of Section 4 one can probably find correspondence between other infinite matrices and known, or easily available, solutions of boundary value problems. Correspondence of this type, should it be found, will not only make the two-series technique a more attractive method for solving boundary value problems but perhaps will also contribute to an understanding of infinite matrix inversion. It therefore seems advisable to record in the Appendix the interrelationship between known results in that field of study and those obtained here.

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APPENDIX

Consider the form

$$\zeta = \sum_{m=0}^{\infty}(-1)^m(2m+1)^{-1}\, v_{-2m-1}\exp\left(-(2m+1)W\right)$$

$$+ \sum_{n=1}^{\infty}(-1)^n(2n)^{-1}\,u_{-2n}\exp(-2nW)\ , \quad \Phi > 0\ ,$$

$$= -\sum_{m=0}^{\infty}(-1)^m(2m+1)^{-1}\, v_{2m+1}\exp\left((2m+1)W\right) + u_0 W$$

$$+ \sum_{n=1}^{\infty}(-1)^n(2n)^{-1}\,u_{2n}\exp(2nW)\ , \quad \Phi < 0\ , \qquad \text{(A1)}$$

which is a somewhat more general version of that used repeatedly above. Again u_p and v_q are real coefficients which can be related by imposing the requirements that $Re(\zeta)$ and its Φ derivative should be continuous along $\Phi = 0$. However, there are two ways to satisfy these. One can multiply the equalities expressing these requirements by $\cos(2n\Psi)$ and thus get u_o, $(u_{2n} + u_{-2n})$, $(u_{2n} + u_{-2n})$, in terms of the constants v_q. This yields

$$u_p = \pi^{-1}\sum_{q=-\infty}^{\infty} v_q\left(1 - (-1)^{p-q}\right)(p-q)^{-1} \qquad \text{(A2)}$$

where p is even and q is odd. This procedure is the one used to derive equations (24_n). In fact the latter system can be deduced from the more general form (A2). However, it is also possible to multiply the equalities by $\cos(q\Psi)$ and integrate. This would yield relationships expressing each of the constants v_q in terms of the u_p. When use is made of the relationship

$$\sum_{n=1}^{\infty} (2n)^{-1} \left(u_{2n} - u_{-2n} \right) = 0 \tag{A3}$$

the following inverse of (A2) is obtained:

$$v_q = (\pi)^{-1} \sum_{p=-\infty}^{\infty} u_p \left(1 - (-1)^{q-p}\right) (q-p)^{-1} \tag{A4}$$

This result was obtained by Duffin (1956). Equation (A3) implies that $R_e(\zeta)$ is continuous at the end points $\Psi = \pm\pi/2$, $\Phi = 0$. Indeed, this must be so if this function and its Φ derivative are continuous in the open range $-\frac{\pi}{2} < \Psi < \frac{\pi}{2}$.

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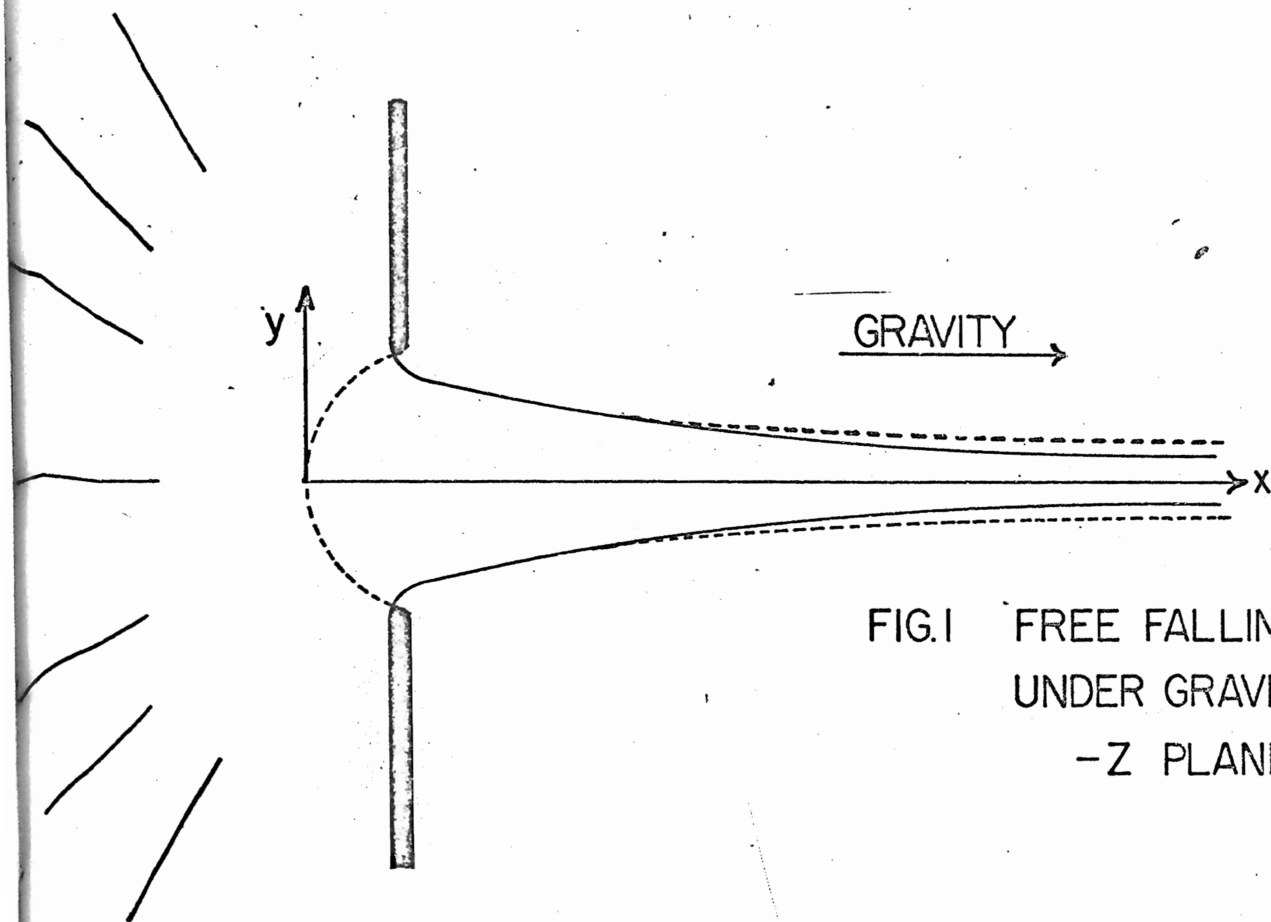

FIG. I FREE FALLING JET
UNDER GRAVITY
-Z PLANE.

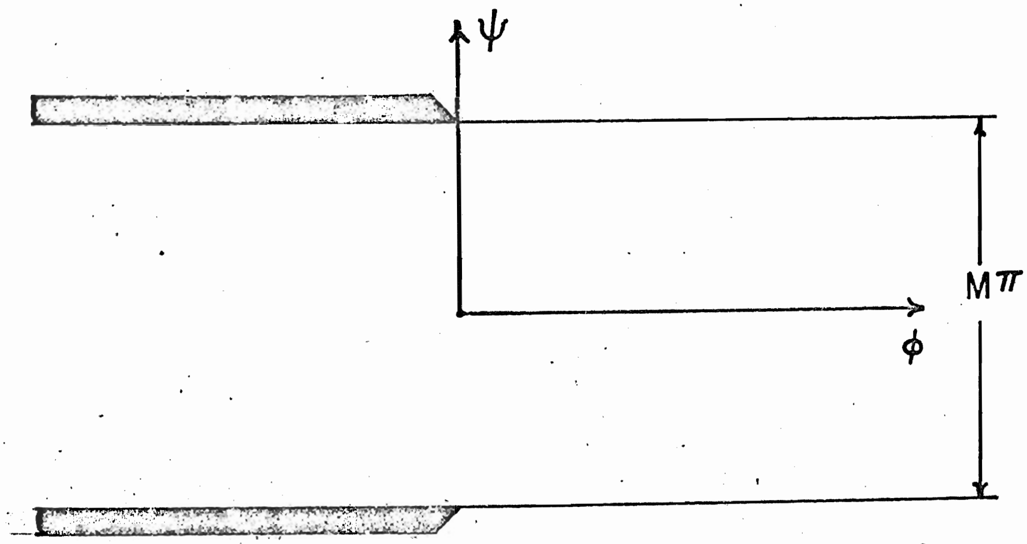

FIG. 2 COMPLEX POTENTIAL PLANE.