STATE UNIVERSITY OF NEW YORK
AT STONY BROOK

COLLEGE OF
ENGINEERING

Report No. 113

ON THE HEAT TRANSFER IN TWO PHASE COCURRENT AND COUNTERCURRENT FLOWS

By

Michael Bentwich

May 1968

ON THE HEAT TRANSFER IN
TWO PHASE COCURRENT AND COUNTERCURRENT FLOWS

Michael Bentwich

May 1968

Mechanics Department
State University of New York
Stony Brook

Report no. 113

ABSTRACT

A new type of analysis is proposed for the cases of heat transfer in two phase cocurrent and countercurrent exchange systems. This is done by splitting the expression for the temperature distribution in each phase into two components. One represents the temperature distribution when the partition is insulative, the other expresses the effect of the heat transfer across the partition. The flux across the partition is found to be governed by an integral equation of the Volterra and Fredholm type in respectively, cocurrent and countercurrent cases. These equations can be easily solved by Laplace Transform methods. Examples of cocurrent and countercurrent cases are considered and the results obtained are found to be compatible with existing results.

I. INTRODUCTION

Most of the analyses of heat transfer in two phase flows reported to date are based on the form suggested by Graetz. Thus, the temperature (or mass) distribution in the entire domain is obtained as an infinite summ of products of eigenfunctions and exponential functions. The argument in the exponential function is proportional to the axial distance. The eigenfunctions are generated by a Sturm-Liouville system defined over the range representing the entire transverse cross section. In the cocurrent[1] case the presence of two phases gives rise to minor difficulties in the solution of the Sturm-Liouville system. However, once the eigenfunctions are known one can utilize their orthogonality and completeness and thus satisfy the inlet condition. Consequently, the classical form is easily applicable to this case. However, for the countercurrent case the classical, but somewhat modified,[2] form is not very useful although it is mathematically valid. Assuming negligible axial conduction, the inlet temperature must beprescribed wherever the fluid enters the system, i.e., on parts of two different cross sections. The eigenfunctions, however, are complete and orthogonal only over the range representing the entire cross section of the exchanger and not over its sections. Nunge and Gill[3,4] attempt to overcome this difficulty, and evaluate the coefficients in the expansion by reducing the inlet conditions to an infinite system of algebraic equations. The latter is truncated and a finite number of coefficients ~~have been~~ are evaluated by a computer. In the present work the author proposes a more sophisticated analysis which is not based on Graetz's classical form.

The quest for mathematical elegance is not the only motivation behind this study. There is a slight error in Ref. 3 where the derivation of the algebraic equations is explained, although in Ref. 4 the correct equations are presented and used without derivation. It is pointed out in this latter work that in the limitting case of zero axial length exchanger, all but one term in the summation of equation (38) vanish. Indeed in such exchanger the inlet conditions are prescribed over the range which represents the entire cross-section and for which the orthogonality holds. Nevertheless this test is not satisfied by the corresponding relationship, equation (22) of Ref. 3. Consequently, the available analysis of the countercurrent case is less than adequate. Furthermore, Stein's[5] work indicates that the numerical solution of the resulting algebraic equations is not always straightforward.

The analysis presented here is based on the assumption that the temperature distribution in each phase is given by a form containing two expressions. The first represents the distribution that would prevail when the partition acts as an insulator. This is obtained by applying the standard Graetz procedure to each stream and making use of the inlet condition. The second term in each of these forms represents the distribution when the temperature difference across the partition is some a priori unknown function of the axial distance z, f(z). The requirement that the temperature should be continuous across the entire cross-section yields an integral equation for f(z). This equation is of the Volterra (and Fredholm) type for cocurrent (and countercurrent) exchanger(s). This distinction has physical significance which will be explained later. The comparatively short treatment of the cocurrent case is included here only to emphasize this distinction, and most of this paper will be devoted to the countercurrent case.

The system considered in this work is depicted in Figure 1. Here y represents the distance across the duct, which is taken to be very broad in the x direction. In the range $t < y < a$, phase 1, the fluid is assumed to flow in the $+z$ direction. In the range $b < y < -t$, ~~the~~ phase 2, the flow is in the $+z$ and $-z$ directions in the cocurrent and countercurrent cases, respectively. The range $-t < y < t$ represents a solid conductive partition.

The solutions are derived without specifying the velocity distributions $u_i(y)$, $(i = 1,2)$ in phases 1 and 2. At the outer boundaries, $y = a$ and $y = -b$, the general linear homogeneous boundary conditons are imposed. Thus, the general solution obtained here is applicable to the variety of two phase exchange systems which may have the above mentioned geometry. Moreover, extension of the proposed technique to the double pipe case is obvious. We also present a numerical example so as to explicitly demonstrate the application of the method under consideration.

II. THEORETICAL DEVELOPMENT

The boundary conditions and differential equation governing the non-dimensional temperature $\theta_i(y,z)$ can be written as follows:

$$p_i\left(\partial^2\theta_i/\partial y^2\right) = u_i(y)\left(\partial\theta_i/\partial z\right), \tag{1}$$

$$-M_2\,\theta_2(-b,z) + N_2\frac{\partial\theta_2}{\partial y}(-b,z) = 0, \qquad 0<z<\ell, \tag{2}$$

$$M_1\,\theta_1(a,z) + N_1\frac{\partial\theta_1}{\partial y}(a,z) = 0, \qquad 0<z<\ell, \tag{3}$$

$$k_2\frac{\partial\theta_2}{\partial y}(-t,z) + \frac{k}{2t}\left(\theta_2(-t,z) - \theta_1(t,z)\right), \qquad 0<z<\ell, \tag{4}$$

$$k_1\frac{\partial\theta_1}{\partial y}(t,z) + \frac{k}{2t}\left(\theta_2(-t,z) - \theta(t,z)\right), \qquad 0<z<\ell. \tag{5}$$

Here the subscript $i = 1, 2$ indicates the phase. The independent variables (y, z) as well as the constants N_i and p_i have the dimensions of length. The constants M_i and the variables u_i are dimensionless. The constants k and k_i are the conductivities of the solid and the liquids in phase i. In the cocurrent case the inlet conditions are

$$\theta_1(y,0) = h_1(y). \qquad \theta_2(y,0) = h_2(y) \tag{6),(7}$$

In the countercurrent case the inlet conditions are

$$\theta_1(y,0) = g_1(y), \qquad \theta_2(y,\ell) = g_2(y). \tag{8),(9}$$

We start by developing a solution for the later case.

Equations (4) and (5) reflect the fact that the temperature distribution inside the partition is a linear function of y. Therefore, we can express the physical requirements implied by this linearity in the following manner

$$k_2 \frac{\partial \theta_2}{\partial y}(-t, z) = \frac{k}{2t} f(z), \tag{10}$$

$$k_1 \frac{\partial \theta_1}{\partial y}(t, z) = \frac{k}{2t} f(z), \tag{11}$$

$$f(z) \equiv \theta_1(t, z) - \theta_2(-t, z). \tag{12}$$

This suggests a solution form θ_i of the form

$$\theta_1 = \sum_{n=1}^{\infty} C_n \varphi_n(y) \exp(-p_1 \gamma_n^2 z) + \chi_1, \tag{13}$$

$$\theta_2 = \sum_{m=1}^{\infty} B_m \psi_m(y) \exp(-p_2 \beta_m^2 (\ell - z)) + \chi_2, \tag{14}$$

where the expansions are the solutions when the ~~left~~ right hand sides of equations (10) and (11) vanish identically. Accordingly, the eigenfunctions and corresponding eigenvalues are generated by the following Sturm-Liouville systems

$$\left.\begin{aligned} &\varphi_n'' + \gamma_n^2 u_1(y) \varphi_n = 0 & & t < y < a \\ &M_1 \varphi_n(a) + N_1 \varphi_n'(a) = 0 & & \varphi_n'(t) = 0 \end{aligned}\right\} \tag{15}$$

and

$$\left.\begin{aligned} &\psi_m'' + \beta_m^2 |u_2(y)| \psi_m = 0, && -b < y < -t, \\ &-M_2 \psi_m(-b) + N_2 \psi_m'(-b) = 0, && \psi_m'(-t) = 0, \end{aligned}\right\} \quad (16)$$

while the coefficients in the expansions (13) and (14) are given by

$$C_n \int_t^a u_1 \varphi_n^2 \, dy = \int_t^a u_1 g_1 \varphi_n \, dy, \qquad B_m \int_{-b}^{-t} u_2 \psi_m^2 \, dy = \int_{-b}^{-t} u_2 g_2 \psi_m \, dy. \quad (17), (18)$$

The function χ_i represents the distributions that would prevail in phase i when the heat flux across the interface is $k f(z)/2t$ and the inlet temperature for that phase is zero.

It is possible to split the expression for θ_i' in the above manner because the differential system at hand is linear. This property also enables us to construct χ_i as the superposition of the contributions due to the variations in f at all points upstream of the position (y,z) in the following form

$$\chi_1 = \hat{\theta}_1(z, y; 0) f(0) + \int_0^z \hat{\theta}_1(z, y; \zeta) f'(\zeta) \, d\zeta, \quad (19)$$

$$\chi_2 = \hat{\theta}_2(z, y; \ell) f(\ell) - \int_z^\ell \hat{\theta}_2(z, y; \zeta) f'(\zeta) \, d\zeta. \quad (20)$$

Here $\hat{\theta}_i$ are the temperature distributions which vanish at the inlets to the phase i and satisfy equations (1), (2), (3), (10) and (11) when f(z) is given by

$$f(z) = H(z - \zeta) \quad i = 1, \qquad f(z) = H(\zeta - z) \quad i = 2$$

where H denotes Heaviside's step function. One can easily show that the $\hat{\theta}_i$ are given by

$$\hat{\theta}_1 = -\frac{k}{2k_1 t}\left\{\left[(a + N_1/M_1) - y\right] + \sum_{q=1}^{\infty} D_q \varphi_q(y) \exp\left(-p_1 \gamma_q^2 (z-\zeta)\right)\right\} H(z-\zeta) \quad (21)$$

$$\hat{\theta}_2 = \frac{k}{2k_2 t}\left\{\left[(b + N_2/M_2) + y\right] + \sum_{r=1}^{\infty} E_r \psi_r(y) \exp\left(-p_2 \beta_r^2 (\zeta - z)\right)\right\} H(\zeta - z) \quad (22)$$

Since the distributions represented by the $\hat{\theta}_i$ have to be continuous across $\zeta = z$, the expressions in the curly brackets vanish for this value of z. From this condition the coefficients D_q and E_r can be evaluated by making use of the Fourier Theorem. In the case at hand this Theorem insures that $\hat{\theta}_i(z,y;z)$ should vanish not only inside the ranges $-b < y < -t$, $t < y < a$ but also at the end points $y = \pm t$. This feature plays an important role in the ensuing development; this is discussed below.

The expressions (13) and (14) represent distributions which satisfy the conditions of the problem in each phase. If these are the solutions for the distribution in the combined system they must also reflect the fact that the flux and temperature are continuous across the partition. The first requirement holds automatically, because $kf(z)/2t$ is used as the local value of flux for both phases. The second requirement yields the relationship from which $f(z)$ can be determined. Once this is done the solution for θ in the entire system is completed. Use is now made of equations (12), (13), (14), (19) and (20) which after integration by parts yield

$$
\begin{aligned}
f(z) = \; & \sum_{n=1}^{\infty} C_n \varphi_n(t) \exp(-p_1 \gamma_n^2 z) \\
& - \sum_{m=1}^{\infty} B_m \Psi_m(-t) \exp(-p_2 \beta_m^2 (\ell - z)) \\
& + \frac{k}{2k_1 t} \int_0^z \Big[\sum_{q=1}^{\infty} D_q \varphi_q(t)(p_1 \gamma_q^2) \exp(-p_1 \gamma_q^2 (z - \zeta)) \Big] f(\zeta)\, d\zeta \\
& + \frac{k}{2k_2 t} \int_z^{\ell} \Big[\sum_{r=1}^{\infty} E_r \Psi_r(-t)(p_2 \beta_r^2) \exp(-p_2 \beta_r^2 (\zeta - z)) \Big] f(\zeta)\, d\zeta
\end{aligned} \tag{23}
$$

Note that since $\hat{\theta}_1(z,y;z)$ vanish the process of integration leaves no boundary terms involving $f(z)$ $f(0)$ or $f(\ell)$. The resulting relationship is known as the Fredholm integral equation of the second kind. If there is direct contact between the two phases one lets t and the left hand side of equation (23) vanish and the limit of $(f/2t)$ is then governed by Fredholm equation of the first kind.

It can be similarly shown that in the cocurrent case the ; distribution in each phase is given by

$$\theta_1 = \sum_{n=1}^{\infty} K_n \varphi_n(y) \exp(-p_1 \gamma_n^2 z) + \tilde{\theta}_1(z,y;0) f(0) + \int_0^z \tilde{\theta}_1(z,y;\zeta) f'(\zeta)\, d\zeta \tag{24}$$

$$\theta_2 = \sum_{m=1}^{\infty} L_m \Psi_m(y) \exp(-p_2 \beta_m^2 z) + \tilde{\theta}_2(z,y;0) f(0) + \int_0^z \tilde{\theta}_2(z,y;\zeta) f'(\zeta)\, d\zeta \tag{25}$$

The coefficients K_n and L_m are obtained from equations (6) and (7). The functions $\tilde{\theta}_1$ and $\hat{\theta}_1$ are identical while $\tilde{\theta}_2$ is given by the right hand side of equation (22) with $(z - \zeta)$ substituted for $(\zeta - z)$.

In this case f(z) is governed by

$$f(z) = \sum_{n=1}^{\infty} K_n \varphi_n(t) \exp(-p_1 \gamma_n^2 z) - \sum_{m=1}^{\infty} L_m \psi_m(-t) \exp(-p_2 \beta_m^2 z)$$

$$+ \frac{k}{k_1 2t} \int_0^z \Big[\sum_{q=1}^{\infty} D_q \varphi_q(t) (p_1 \gamma_q^2) \exp(-p_1 \gamma_q^2 (z-\zeta)) \Big] f(\zeta)\, d\zeta$$

$$+ \frac{k}{k_2 2t} \int_0^z \Big[\sum_{r=1}^{\infty} E_r \psi_r(t) (p_2 \beta_r^2) \exp(-p_2 \beta_r^2 (z-\zeta)) \Big] f(\zeta)\, d\zeta \,. \quad (26)$$

This is Volterra's integral equation. It is of the first and second kind for vanishing and finite t , respectively.

EXAMPLES

There are numerous methods by which integral equations of the Fredholm and Volterra types can be solved. The choice will depend on the features of the particular case which is to be analyzed. In practice heat exchangers are designed so that the transfer across the partition is rapid. Therefore, for such systems the ratios $(b/2t)$ $(a/2t)$ and k/k_i $(i = 1, 2)$ are large. Consequently, the integrals of equations (23) and (26) are large compared with the terms on the left hand sides. This feature all but rules out Neumann's method of repeated substitutions and other techniques which draw heavily on the dominance of the left hand side. In fact, the most representative cases seem to be those in which the left hand sides vanish and equations (23) and (26) become Fredholm and Volterra equations of the first kind. Examples in which this is so are presented below. It will become apparent that the method used in these examples is applicable also to the case in which $t \neq 0$ and the left hand sides do not vanish.

Consider the cocurrent case in which the inlet temperature of each stream is uniform. Without much loss of generality one can let $h_1(y)$ be ΔT and let $h_2(y)$ vanish identically. It is further assumed that the following holds:

$$
\begin{aligned}
&u_1 = -u_2 = 1, \qquad a = b = \ell/5 \qquad t = 0, \qquad N_i = 0 \\
&p_1 = a, \qquad p_2 = 2a
\end{aligned}
\tag{27}
$$

In view of equations (15) and (16) the eigenfunctions and corresponding eigenvalues are given by

$$\varphi_n = \cos\left(\frac{2n-1}{2}\pi\frac{y}{a}\right) , \qquad \gamma_n^2 = \left(\frac{2n-1}{2}\frac{\pi}{a}\right)^2 ,$$

$$\psi_m = \cos\left(\frac{2m-1}{2}\pi\frac{y}{a}\right) , \qquad \beta_m^2 = \left(\frac{2m-1}{2}\frac{\pi}{a}\right)^2 , \tag{28}$$

whence

$$K_n = \Delta T\,\frac{4}{\pi}\frac{(-1)^{n+1}}{(2n-1)} , \quad D_q = \frac{-8a}{\pi^2}\left(\frac{1}{2q-1}\right)^2 , \quad E_r = \frac{-8a}{\pi^2}\left(\frac{1}{2r-1}\right)^2 , \quad L_m = 0 , \tag{29}$$

It is then noted that equation (26) contains a convolution- integral. Therefore, by Laplace transforming this equation the equation is reduced to the product of the transform of the kernel times the transform of $f(z)$, $\bar{f}(s)$. The latter is therefore given by

$$\frac{k\bar{f}(s)}{2t} = \frac{k_2\left(\frac{2\Delta T}{\pi}\right)\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}\left(sa + \frac{p_1}{a}\left(\frac{(2n-1)\pi}{2}\right)^2\right)^{-1}}{\frac{k_2 p_1}{k_1 a}\sum\limits_{q=1}^{\infty}\left(sa + \frac{p_1}{a}\pi^2\left(\frac{2q-1}{2}\right)^2\right)^{-1} + \frac{p_2}{a}\sum\limits_{r=1}^{\infty}\left(sa + \frac{p_2}{a}\left(\frac{(2r-1)\pi}{2}\right)^2\right)^{-1}} \tag{30}$$

This expression has simple pole singularities on the negative real s axis. Its inverse, the heat flux at the interface, is therefore given by the form

$$k f(z)/2t = (4/\pi)(k_1\,\Delta T/a)\sum_{j=1}^{\infty} A_j \exp(-\alpha_j z/a) . \tag{31}$$

The result is in qualitative agreement with that of Reference 1. Here (k_1/k_2) ~~and (p_1/p_2)~~ can attain any finite non-zero value. When th~~e~~s~~e~~ ratios ~~are~~ is 4 ~~and 10~~ ~~respectively~~ the constants are given by

$$
\begin{array}{lll}
A_1 = .114 & A_2 = .600 & A_3 = -.220 \\
\alpha_1 = -.27 & \alpha_2 = -1.71 & \alpha_3 = -2.61
\end{array}
\qquad (32)
$$

Note that if t is non-zero one gets an expression for $k\overline{f(s)}/2t$ which differs from that given by equation (30) in that it has unity added to the denominator. This modification does not change the order properties of $\overline{f(s)}$ for large s, or the numerical value of $f(z)$.

An example in which the flow is countercurrent is now considered, assuming, as before, that t approaches zero, that the inlet temperatures are uniform and that the various parameters are given by equation (27). It therefore follows that equation (28) holds, and that D_q and E_r are given again by equation (29). In this case, the B_m vanish while the C_n are equal to K_n. The heat flux $k f(z)/2t$ is solved by the following iteration scheme. At every stage an approximate value of $f(z)$ is substituted in the second integral of equation (23) and a new approximation of $f(z)$ is obtained by solving the resulting Volterra type of equation. The solution of the latter is carried in the manner explained herewith. Designating by $f^{(u)}(z)$ the value of $f(z)$ obtained in the 'u'th iteration and letting $f^{(0)}(z)$ vanish identically one gets

$$\frac{k\,\overline{f^{(1)}(s)}}{2t} = \frac{\left(\left(\frac{4}{\pi}\right)k_1\,\Delta T\right)\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{2n-1}\right)\left(sa + \left(\frac{p_1}{a}\right)\left(\frac{2n-1}{2}\right)^2\right)^{-1}}{2\left(\frac{p_1}{a}\right)\sum_{q=1}^{\infty}\left(sa + \left(\frac{p_1}{a}\right)\pi^2\left(\frac{2q-1}{2}\right)^2\right)^{-1}} \qquad (33)$$

The inverse of this relationship has the form

$$\frac{k\,f^{(1)}(z)}{2t} = \left(\frac{a}{p_1}\right)\left(\frac{4\,k_1\,\Delta T}{\pi a}\right)\sum_{k=1}^{\infty}\Gamma_k\exp\left(-\lambda_k z/a\right) \qquad (34)$$

where

$\Gamma_1 = 3.154$	$\Gamma_2 = -0.005$	$\Gamma_3 = 3.151$	$\Gamma_4 = -.0020$
$\lambda_1 = -9.95$	$\lambda_2 = -39.80$	$\lambda_3 = -89.55$	$\lambda_4 = -159.2$

The second iteration yields the following expression for the transform of the correction ($f^{(2)} - f^{(1)}$).

$$k\,\frac{\left(\overline{f^{(2)}(s)} - \overline{f^{(1)}(s)}\right)}{2t} = -\left(\frac{p_2}{p_1}\cdot\frac{k_1}{k_2}\right)\left(\frac{4\,k_1\,\Delta T}{\pi}\right)\left(\frac{a}{p_1}\right)\sum_{r=1}^{\infty}\sum_{k=1}^{\infty}\frac{\Gamma_k}{\lambda_k + \frac{p_2}{a}\left(\frac{2r-1}{2}\right)^2\pi^2}\times$$

$$\times\left\{\frac{1}{sa+\lambda_k} - \exp\left(-\lambda_k\frac{\ell}{a}\right)\exp\left(-2\left(\frac{2r-1}{2}\right)^2\pi^2\frac{\ell}{a}\right)\frac{1}{sa - \frac{p_2}{a}\left(\frac{2r-1}{2}\right)^2\pi^2}\right\}\times$$

$$\times\left[\sum_{q=1}^{\infty}\frac{1}{sa + \frac{p_1}{a}\left(\frac{2q-1}{2}\right)^2\pi^2}\right]^{-1}. \qquad (35)$$

This expression has second order poles at $s = -\lambda_k/a$ and first order poles at $s = p_2(2r-1)^2\pi^2(\ell a)^{-2}$. Therefore the inverse has the form

$$k\left(f^{(1)}(z) - f^{(1)}(z)\right)/2t = \left(\frac{k_2}{p_1}\right)\left(\frac{k_1}{k_2}\right)\left(\frac{4k_1\Delta T}{a}\right)\frac{a}{p_1}\left\{\sum_{k=1}^{\infty} R_k \exp(-\lambda_k z/a)\right.$$

$$\left.+\sum_{k=1}^{\infty} S_k(z/a)\exp(-\lambda_k z/a) + \sum_{r=1}^{\infty} T_r \exp\left(-\frac{p_2}{a}\left(\frac{2r-1}{2}\right)^2\pi^2\left(\frac{\ell-z}{a}\right)\right)\right\} \quad (36)$$

where

$$R_1 = -.965 \qquad S_1 = 6.837 \qquad T_1 \text{ is } 0\,(10^{-10}) \quad (37)$$

while higher terms in all three sequences are numerically insignificant. Note that equations (34) and (36) imply that ($k f(z)/2t$) increases sharply as one gets close to the two ends. Nunge and Gill have shown that this characterizes the heat flux between the two counter-current streams. From a mathematical standpoint ($k f/2t$) is singular at $z = 0$ and $z = \ell$ and this is reflected by the divergence of the series of exponents containing z and $(\ell - z)$ in their arguments.

In the treatment of the co-current case the ratios (k_1/k_2) and (p_1/p_2) could take any value, but in the present case one has to be more careful. For example, if these ratios are set to be 4 and 2, as before, the result (36) contains a multiplier 8. This factor recurs in the ensuing iterations and this considerably slows the convergence of the proposed scheme. However, the situation can be improved by setting the problem up so that the C_n vanish but the B_m do not, and reversing the roles played by two integrals of equation (23). Under such rearrangement the recurring multiplier is $(k_2/k_1) \times (p_1/p_2)$ which is only 1/8. It will be shown later that the role of these integrals are related to the nature of transfer in the corresponding phases. In the meantime one notes that from a purely mathematical standpoint the convergence is the slowest when the multiplier is unity. Since the maximum value of $(z/a) \exp(-\lambda_1 z/a)$ is $(e\lambda_1)^{-1}$, even in such a limiting case the contribution of the correction ($f^{(2)} - f^{(1)}$) is considerably smaller than that of the first approximation $f^{(1)}$. This suggests that for a fairly large class of problems an adequate solution can be obtained by going through two iterations.

DISCUSSION

The new mathematical procedusre proposed in this work follows closely the essential features of the exchangers under consideration. The basic assumption implicit in equation (1) is that heat is transferred either downstream or across the current. Accordingly, the temperature difference across the partition, at any point z , f(z) , is affected by the values of the flux ($k\, f(z)/2t$) at every point upstream. The sum total of these effects is represented by an integral over the appropriate range. In these integrals ($k\, f(z)/2t$) appears in a linear form because the problem (1) - (5) is linear. The kernels in these integrals reflect the down-and cross-stream directed transfer in the appropriate phase. Thus, ~~however,~~ in both cases the transfer affecting f(z) is through both phases. In the co-current case it is expressed by the two integrals over the range $0 < \zeta < z$. In the countercurrent case the transfer through the parts of phases 1 and 2 which are upstream of z is expressed by the integrals over the ranges $0 < \zeta < z$ and $z < \zeta < \ell$, respectively.

The last interpretation of equations (23) and (26) serves as a guide to their solution. Neumann's method of repeated substitutions was not used because its convergence would depend on the dominancy of the left-hand sides over the integrals. However, as pointed out, the cases of interest are those in which the temperature drop f is small, yet the total effect of the flux ($k\, f(z)/2t$) upstream of the point in question is large. Mathematically this is expressed by the largeness of ($a/2t$), ($b/2t$) and (k/k_i), and products of these ratios appear as recurring multipliers when Neumann's method is employed. Therefore, had this method been used, convergence would have ~~to be~~ been, at best, slow. Similar mathematical and corresponding physical considerations lead one to let the first integral of equation (23) play the dominant role when $(k_1/k_2) \times (p_2/p_1)$ is less than unity. As shown, the product of these ratios has the role of a recurring multiplier. Therefore, it slows the convergence in the iteration scheme which is used here, if it is large. Physically, the smallness of (k_1/k_2) and (p_2/p_1) implies that the transfer of heat in phase 2 is relatively slow along the stream yet fast across it. Therefore the relative smallness of ~~that~~ their product is an indication that the heat transferred into phase 1 over the range $0 < \zeta < z$ stays closer to the partition and hence affect the value of f more than the heat transferred into phase 2 over the range $z < \zeta < \ell$. Under such circumstances the first integral is dominant both physically and mathematically.

ACKNOWLEDGEMENT

The author gratefully acknowledges Dr. Stewart Harris' help in the preparation of this paper.

REFERENCES

1. Bentwich, M., and Sideman, S., J. Heat Transfer, 86, 476 (1964).

2. M. Bocher, Bull. Amer. Math Soc. 21, 6 (1914).

3. Nunge, R.J. and Gill, W.N., Int. J. Heat Mass Transfer, 8, 873, (1965).

4. Nunge, R.J. and Gill, W.H., AIChE. J., 12, 279 (1966).

5. Stein, R.P., AIChE. J., 12, 1217 (1966).

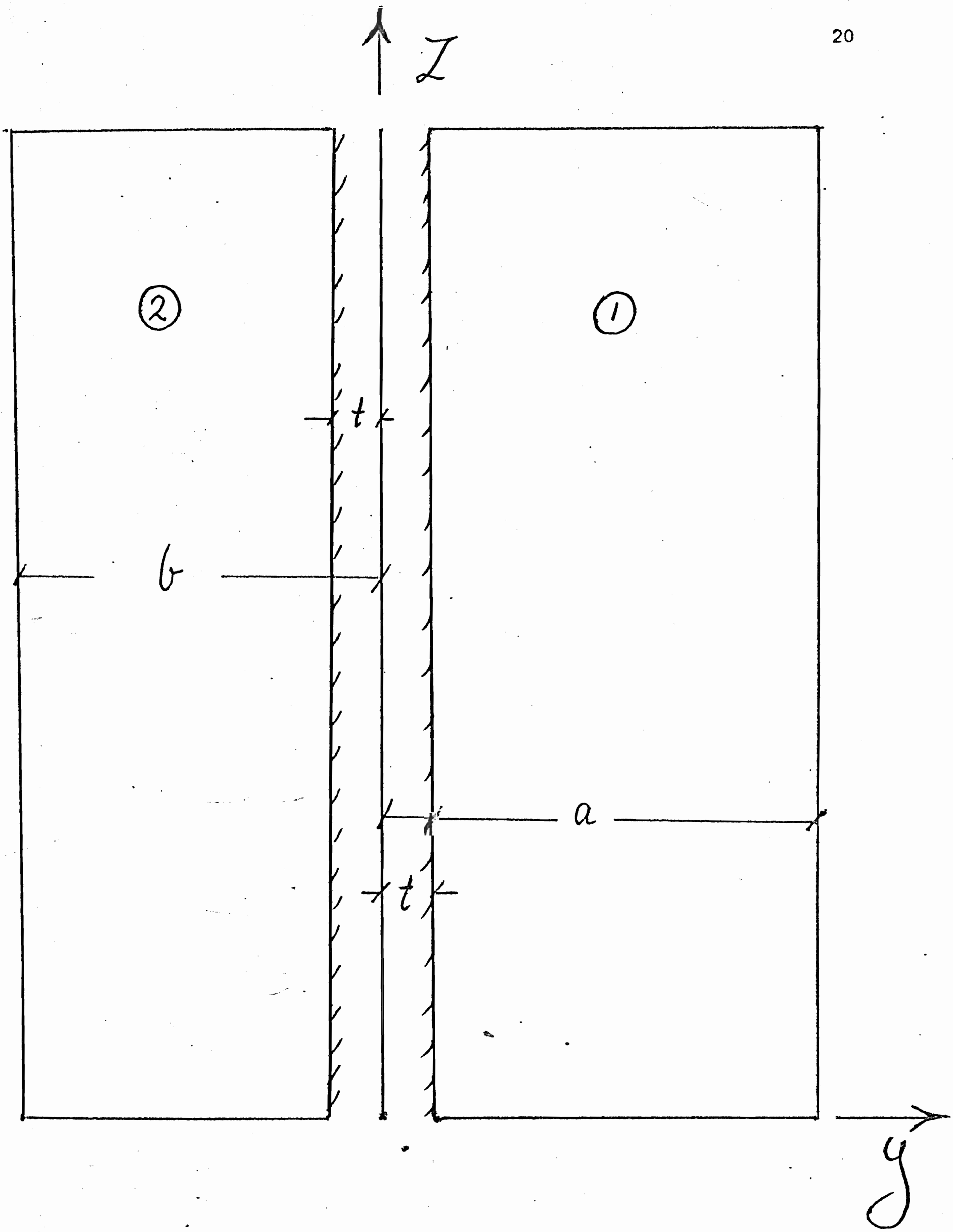

FIGURE 1. SCHEMATIC DIAGRAM OF THE TWO - PHASE FLOW