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TWO-VARIABLE EXPANSIONS FOR SINGULAR PERTURBATIONS

by

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TWO-VARIABLE EXPANSIONS FOR SINGULAR PERTURBATIONS*

A. Erdélyi

Summary. The author applies the two-variable expansion procedure to an example in singular perturbations and shows how the validity of this procedure can be established in many cases (including some nonlinear problems).

1. The boundary value problem

$$\epsilon y'' + y' = h'(t) \qquad 0 \le t \le 1 \tag{1.1}$$

$$y(0) = \alpha, \; y(1) = \beta \;, \tag{1.2}$$

in which h is a given infinitely differentiable function, ϵ is a small positive parameter, and α and β are given numbers independent of ϵ, has often been used to illustrate features of boundary layer theory or, more generally, of certain techniques for singular perturbations.

Some time ago [2] I used this example to illustrate the use of the technique developed by P. A. Lagerstrom and his associates, in particular S. Kaplun [4]. According to Kaplun and Lagerstrom one obtains first an "outer expansion" of the form

$$y^o \sim \Sigma\, f_n(t)\, \epsilon^n \tag{1.3}$$

for the solution of (1.1), (1.2). This expansion is supposed to hold for $t > 0$, and it can be made to satisfy the second, but it will in general fail to satisfy the first, of the boundary conditions (1.2). In the present case

$$y^o \sim \beta + \sum_0^\infty [h^{(n)}(t) - h^{(n)}(1)]\; (-\epsilon)^n. \tag{1.4}$$

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One then introduces the "stretching transformation"

(1.5) $$t = \epsilon\tau$$

and obtains an "inner expansion" of the form

(1.6) $$y^i \sim \Sigma\, g_n(\tau)\, \epsilon^n$$

holding for bounded τ. This expansion can be made to satisfy the first, but not the second, boundary condition (1.2), and it will contain undetermined constants. In the present case,

(1.7) $$y^i \sim \alpha + \sum_1^\infty h^{(n)}(0)\, \frac{\epsilon^n}{n!} \int_0^\tau e^{u-\tau} u^n du + \sum_0^\infty A_n \epsilon^n (1-e^{-\tau}),$$

where the A_n are the undetermined constants. A matching process which need not be described further here leads to the conclusion that the outer and inner expansions will be consistent if

$$A_0 = \beta - \alpha + h(0) - h(1),\quad A_n = (-1)^{n-1} h^{(n)}(1) \quad n \geq 1$$

is chosen.

Neither the outer nor the inner expansion is valid throughout the interval $0 \leq t \leq 1$, and for this reason Kaplun and Lagerstrom propose a "composite expansion" which in the present case takes the form

(1.8) $$y^c \sim \alpha e^{-t/\epsilon} + \beta(1-e^{-t/\epsilon}) + \sum_0^\infty (-\epsilon)^n \{h^{(n)}(t) - h^{(n)}(0) + [h^{(n)}(0) - h^{(n)}(1)](1-e^{-t/\epsilon})\}$$

and can be shown to hold uniformly throughout the interval $0 \leq t \leq 1$.

2. Once the possibility of a uniform expansion has been recognized, and the general character of such an expansion is known, one might try to construct the expansion by the so-called two-variable expansion technique (see for instance [1] Chapter 3) without going through the preliminary stages of inner and outer expansions and matching. At this point it is

necessary to discuss briefly the mathematical character of the various expansions.

In all three expansions $\epsilon \to 0+$. Both the outer and inner expansions are asymptotic expansions in the classical (Poincaré) sense, with coefficients depending on a parameter. In (1.4), t may be regarded as a fixed parameter, $0 < t \leq 1$, while in (1.7), τ may be considered fixed, $\tau \geq 0$. Now, the coefficient of ϵ^n in (1.8) depends on both t and τ. These parameters are related by the stretching transformation $t = \epsilon\tau$ and cannot both be fixed as $\epsilon \to 0+$. Thus, (1.8) is not an asymptotic expansion in the classical sense.

The composite expansion (1.8) is a general asymptotic expansion in the following sense. Given an asymptotic scale $\{\varphi_n(\epsilon)\}$ (in our case $\{\epsilon^n\}$), an asymptotic expansion

$$F(\epsilon) \sim \Sigma\, F_n(\epsilon) \qquad \{\varphi_n\} \tag{2.1}$$

with respect to this scale is characterized by the property that

$$F(\epsilon) - \sum_{n \leq N} F_n(\epsilon) = o(\varphi_n) \tag{2.2}$$

for each N. This definition does not demand that F_n be a multiple of φ_n; it does not even demand that F_n be of the same order of magnitude as φ_n (although the latter property happens to hold in (1.8) and many other general asymptotic expansions).

The most conspicuous feature of general asymptotic expansions is an obvious and fundamental lack of uniqueness. If a function F possesses an asymptotic expansion, in the sense of (2.2), with respect to an asymptotic scale $\{\varphi_n\}$, then it will possess an infinity of such expansions (with respect to the same scale). Mathematically, all such expansions are of the same standing; they all lead to the same error estimate (2.2), although the first approximation and often ϵ^n for higher approximations.

Now, it can be proved under fairly general conditions that u will be an approximation to a solution y of the boundary value problem (3.1) in the sense that

$$y - u = O(\eta), \quad y' - u' = O((1+\epsilon^{-1} e^{-\varphi/\epsilon})\eta) \tag{3.3}$$

provided that $\eta = O(\epsilon)$, and

$$\begin{aligned} \epsilon u'' + F(t,u,u',\epsilon) &= O(\eta) + O(\epsilon^{-1}\eta e^{-m\varphi/\epsilon}) \\ \alpha(\epsilon) - u(0,\epsilon) = O(\eta), &\quad \beta(\epsilon) - u(1,\epsilon) = O(\eta) \end{aligned} \tag{3.4}$$

with $m > 1$. This result will be established, and the conditions for its validity will be given in full detail, in a forthcoming paper [3]. Earlier, a similar but more elaborate theorem was proved by Willett [5] whose result is more general in that it requires only $\eta = o(1)$ in place of the more stringent condition $\eta = O(\epsilon)$.

Because of the nonlinear character of F in (3.1) it does not appear to be feasible to relax the conditions in (3.4) - even if one is willing to accept a weaker result than (3.3). However, in the case under consideration here, it is possible to achieve further progress by making use of the linearity of (1.1).

Let $u(t,\epsilon)$ be a tentative approximate solution of (1.1), (1.2), and set

$$\int_0^t \{h'(s) - \epsilon u''(s) - u'(s)\}\, ds = k(t,\epsilon) \tag{3.5}$$

and

$$y = u + z. \tag{3.6}$$

Then z satisfies

$$\epsilon z'' + z' = k'(t,\epsilon) \tag{3.7}$$

$$z(0) = \alpha_1 = \alpha - u(0,\epsilon), \quad z(1) = \beta_1 = \beta - u(1,\epsilon). \tag{3.8}$$

This system can be solved explicitly.

$$(3.9) \qquad z(t) = \frac{1}{\epsilon}\int_0^t e^{-(t-s)/\epsilon}\, k(s,\epsilon)ds + \alpha_1 \frac{e^{-t/\epsilon} - e^{-1/\epsilon}}{1 - e^{-1/\epsilon}}$$

$$+ \{\beta_1 - \frac{1}{\epsilon}\int_0^1 e^{-(1-s)/\epsilon}\, k(s,\epsilon)ds\} \frac{1 - e^{-t/\epsilon}}{1 - e^{-1/\epsilon}}$$

$$(3.10) \qquad \epsilon z'(t) = \int_0^t e^{-(t-s)/\epsilon}\, k'(s,\epsilon)ds - \frac{\alpha_1 e^{-t/\epsilon}}{1 - e^{-1/\epsilon}}$$

$$+ \{\beta_1 - \frac{1}{\epsilon}\int_0^1 e^{-(1-s)/\epsilon}\, k(s,\epsilon)ds\}\frac{e^{-t/\epsilon}}{1 - e^{-1/\epsilon}} .$$

We now make the assumption that u is an approximate solution satisfying

$$(3.11) \qquad \int_0^1 |\ \epsilon u'' + u' - h'\ |\ dt = O(\eta)$$

$$(3.12) \qquad \alpha_1 = \alpha - u(1) = O(\eta), \quad \beta_1 = \beta - u(1) = O(\eta).$$

Under these conditions clearly $k = O(\eta)$ and from (3.9) also $z = O(\eta)$. Moreover, by integration by parts,

$$\int_0^t e^{-(t-s)/\epsilon}\, k'(s)ds = k(t) - \frac{1}{\epsilon}\int_0^t e^{-(t-s)/\epsilon}\, k(s)ds = O(\eta),$$

and hence also $\epsilon z' = O(\eta)$. Thus, under the assumptions (3.11), (3.12) we have

$$(3.13) \qquad y - u = O(\eta), \quad y' - u' = O(\epsilon^{-1}\eta),$$

a result that is weaker than (3.3) but is acceptable in many cases.

If we make the more demanding assumptions

$$(3.14) \qquad \epsilon u'' + u' - h' = O(\eta) + O(\tfrac{\eta}{\epsilon}\, e^{-mt/\epsilon}) \qquad m > 0$$

and (3.12), then clearly (3.11) is satisfied and hence (3.13) holds. However, we have better estimates in this case for the first term on the right hand side of (3.10).

$$\int_0^t e^{-(t-s)/\epsilon}\, ds = \epsilon(1-e^{-t/\epsilon})$$

and

$$\frac{1}{\epsilon}\int_0^t e^{-(t-s)/\epsilon} e^{-ms/\epsilon}\, ds = \begin{matrix} (m-1)(e^{-t/\epsilon}-e^{-mt/\epsilon}) & \text{if } m \neq 1. \\ (t/\epsilon)\, e^{-t/\epsilon} & \text{if } m = 1. \end{matrix}$$

If we then set

$$\begin{aligned} \Phi(t,\epsilon) &= 1 + \epsilon^{-1} e^{-mt/\epsilon} \text{ if } 0 < m < 1 \\ &= 1 + \epsilon^{-1}(t/\epsilon)\, e^{-t/\epsilon} \text{ if } m = 1 \\ &= 1 + \epsilon^{-1} e^{-t/\epsilon} \text{ if } m > 1, \end{aligned}$$

we have the improved results

(3.15) $$y - u = O(\eta),\ y' - u' = O(\eta\Phi)$$

under the conditions (3.14), (3.12). This coincides with (3.3) if $m > 1$ but is a new result if $0 < m \le 1$. Moreover, (3.13) and (3.15) hold without the restriction $\eta = O(\epsilon)$.

4. We are now ready to construct asymptotic expansions of the solution of (1.1), (1.2) by the two variable method. In addition to the geometrical variable t, we use the boundary layer variable $\tau = t/\epsilon$, and will construct an approximate solution of the form

(4.1) $$u(t,\epsilon) = \sum_{n=o}^{N} f_n(t,\tau)\epsilon^n.$$

At once we notice that this form is not unambiguous since, for instance, $t/\tau = \epsilon$. We shall think of f_n as bounded functions with bounded partial derivatives. Partial differentiation will be indicated by subscripts, so that

$$\frac{df_n}{dt} = f_{nt} + \frac{1}{\epsilon} f_{n\tau},\ \frac{d^2 f_n}{dt^2} = f_{ntt} + \frac{2}{\epsilon} f_{nt\tau} + \frac{1}{\epsilon^2} f_{n\tau\tau}.$$

By straightforward substitution,

(4.2) $$\begin{aligned} \epsilon u'' + u' - h' &= (f_{o\tau\tau} + f_{o\tau})\,\epsilon^{-1} + (f_{1\tau\tau} + f_{1\tau} + 2f_{ot\tau} \\ &+ f_{ot} - h') + \sum_{n=1}^{N-1} (f_{n+1,\tau\tau} + f_{n+1,\tau} + 2f_{nt\tau} + f_{nt} + f_{n-1,tt})\,\epsilon^n \\ &+ (2f_{Nt\tau} + f_{Nt} + f_{N-1,tt})\,\epsilon^N + f_{N,tt}\epsilon^{N+1}. \end{aligned}$$

Since (4.1) is a partial sum of a general asymptotic expansion for which there is lack of uniqueness, we cannot conclude that for each n the coefficient of ϵ^n on the right hand side of (4.2) must vanish. Nevertheless, it is plausible to determine $f_o, \ldots, f_N$ recurrently from

$$f_{o\tau\tau} + f_{o\tau} = 0 \tag{4.3}$$

$$f_{1\tau\tau} + f_{1\tau} = h' - 2f_{ot\tau} - f_{ot} \tag{4.4}$$

$$f_{n+1,\tau\tau} + f_{n+1,\tau} = -2f_{nt\tau} - f_{nt} - f_{n-1,tt} \quad n = 1, 2, \ldots, N-1. \tag{4.5}$$

At once several circumstances are noticed. We have here a system of n + 1 partial differential equations which fail to determine the n + 1 functions $f_o, f_1, \ldots, f_N$. If these equations are satisfied, we have from (4.2) $\epsilon u'' + u' - h' = O(\epsilon^N)$ which is not good enough, since we must have $\eta = o(\epsilon^N)$ to make (4.1) the partial sum of a general asymptotic expansion; yet there is no assurance of our being able to satisfy also

$$2f_{Nt\tau} + f_{Nt} + f_{N-1,tt} = 0.$$

In order to clarify the situation, let us consider the simplest case, N = 0. From (4.3),

$$f_o(t,\tau) = A_o(t)\, e^{-\tau} + B_o(t) \tag{4.6}$$

while from (1.2),

$$A_o(0) + B_o(0) = \alpha, \quad B_o(1) = \beta. \tag{4.7}$$

A_o and B_o are not further determined at this stage. We now set

$$u_o(t,\epsilon) = f_o(t,\tau)$$

and obtain

$$\epsilon u_o'' + u_o' - h' = (\epsilon A_o'' - A_o')e^{-\tau} + \epsilon B_o'' + B_o' - h'. \tag{4.8}$$

Clearly, $\eta = \epsilon$ in this case, and in order to satisfy (3.11) or (3.14) we must have $B_o' = h'$ or

$$B_o(x) = \beta + h(x) - h(1) \tag{4.9}$$

in view of (4.7). Also, from (4.7)

$$A_0(0) = \alpha - \beta + h(1) - h(0), \tag{4.10}$$

and A_0 is not further determined at this stage. Equation (3.14) is then satisfied with $\eta = \epsilon$ and $m = 1$, so that with any A_0 conforming to (4.10),

$$y - u_0 = O(\epsilon),\ y' - u_0' = O(\epsilon + \frac{t}{\epsilon} e^{-t/\epsilon}). \tag{4.11}$$

Equation (4.4) now becomes

$$f_{1\tau\tau} + f_{1\tau} = A_0' e^{-\tau}.$$

This equation will have a particularly simple solution,

$$f_1(t,\tau) = A_1(t)e^{-\tau} + B_1(t)$$

if we make use of the freedom left in the choice of A_0 by choosing $A_0' = 0$, or

$$A_0(x) = \alpha - \beta + h(1) - h(0). \tag{4.12}$$

It must be emphasized that this is an arbitrary choice leading to a simple computation but not influencing the accuracy of the approximations. With $A_0' = 0$, the right hand side of (4.8) becomes $O(\epsilon)$, so that we may take $m > 1$ in (3.15) and obtain

$$y - u_0 = O(\epsilon),\ y' - u_0' = O(\epsilon + e^{-t/\epsilon}) \tag{4.13}$$

but this estimate is no better than (4.11).

We can now repeat with

$$u_1(t,\epsilon) = [A_0(t) + \epsilon A_1(t)]e^{-\tau} + B_0(t) + \epsilon B_1(t)$$

the process carried out earlier with u_0 and obtain $B_1'(t) = -B_0''(t)$, while A_1 will be undetermined except for the value of $A_1(0)$. In order to simplify the equation for f_2, we may choose $A_1' = 0$, but this is again a matter of convenience rather than one of principle.

It will now be shown that (4.5) can be satisfied by

$$f_n(t,\tau) = A_n(t)e^{-\tau} + B_n(t) \tag{4.14}$$

and then determines A_n and B_n for $1 \le n \le N - 1$ but not for $n = N$. It must be stressed that the determination of A_n and B_n is due to our assuming that f_n is of the simple form (4.14). The system (4.5) possesses solutions in which $e^{-\tau}$ is multiplied by a polynomial in τ with coefficients depending on t, and these solutions are highly indeterminate. In fact, at every stage of the construction there arises the same lack of determination as the one we encountered in relation to f_o and are going to encounter again in regard to f_N.

Substitution of (4.14) in (4.5) yields

$$0 = A'_n e^{-\tau} - B'_n - A''_{n-1} e^{-\tau} - B''_{n-1} \quad n = 1, 2, \ldots, N - 1.$$

Since this holds identically in t and τ, we must have

$$(4.15) \qquad A'_n = A''_{n-1},\ B'_n = -B''_{n-1} \quad n = 1, 2, \ldots, N - 1,$$

while from the boundary conditions

$$(4.16) \qquad A_n(0) + B_n(0) = 0,\ B_n(1) = 0 \quad n = 1, 2, \ldots, N.$$

Starting with (4.9) and (4.12), we now see by induction that

$$(4.17) \qquad \begin{aligned} A_n(t) &= (-1)^n[h^{(n)}(1) - h^{(n)}(0)] \\ B_n(t) &= (-1)^n[h^{(n)}(t) - h^{(n)}(1)] \end{aligned} \qquad n = 1, 2, \ldots, N - 1.$$

Apart from the boundary conditions, A_N and B_N are not determined at this stage.

Substituting

$$u = \sum_{n=o}^{N} \{A_n(t)e^{-\tau} + B_n(t)\}$$

and taking account of (4.3) to (4.5), we obtain from (4.2),

$$(4.18) \qquad \epsilon u'' + u' - h' = (A'_N e^{-\tau} + B'_N + B''_{N-1})\epsilon^N + (A''_N e^{-\tau} + B''_N)\, \epsilon^{N+1}.$$

For this to be $O(\epsilon^{N+1}) + O(\epsilon^N e^{-\tau})$, it is both necessary and sufficient that $B'_N + B''_{N-1} = 0$, so that the second equation (4.17) must hold also for $n = N$. We may also take $A_N' = 0$, and A_N according to the first equation (4.17), for convenience but there is no compelling reason to make this choice.

We now have

$$y = \alpha e^{-t/\epsilon} + \beta(1-e^{-t/\epsilon}) + \sum_{n=0}^{N} (-\epsilon)^n \{[h^{(n)}(1) - h^{(n)}(0)]e^{-t/\epsilon} + h^{(n)}(t) - h^{(n)}(1)\} + O(\epsilon^{N+1}).$$

This is precisely the approximation by partial sums of the composite expansion of Kaplun and Lagerstrom. It is seen to be one of many possible expansions of the solution of (1.1), (1.2).

In conclusion, it may be noted that the degree of arbitrariness in the choice of A_n depends on the availability in this case of the estimate (3.15) under the conditions (3.14). The alternative conditions (3.4) are of wider application and by the same token not the best possible conditions in particular circumstances. Had we depended on the latter for the validation of u, it would have been necessary to enforce $m > 1$ and hence apparently necessary to have $A_n' = 0$. Thus, the apparent degree of freedom in the choice of asymptotic expansion depends essentially on the tools one uses.

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