STATE UNIVERSITY OF NEW YORK
AT STONY BROOK

COLLEGE OF
ENGINEERING

Report No. 129

VISCOUS INCOMPRESSIBLE FLOW DUE TO A FINITE PLATE MOVING AT ZERO INCIDENCE PARALLEL TO A SOLID PLANE

by

Michael Bentwich

March 1969

VISCOUS INCOMPRESSIBLE FLOW DUE TO

A FINITE PLATE MOVING AT ZERO

INCIDENCE PARALLEL TO A

SOLID PLANE(*)

Michael Bentwich

Mechanics Department
State University of New York at Stony Brook
Stony Brook, New York

March 1969

(*) An earlier version was presented before the First Canadian Congress of Applied Mechanics in Quebec, May, 1967.

ABSTRACT

When a finite-sized obstacle moves through an infinite body of viscous incompressible fluid, then the momentum flux, integrated over a control volume surrounding the source of disturbance, is constant and equal to the drag force. However, it is shown here that for the corresponding flow produced by an obstacle moving parallel to an infinite solid plane, the system of forces on the obstacle is balanced by an equal and opposite one exerted by the plane. Therefore, the total momentum imparted to the semi-infinite body of fluid is zero. Apart from being physically significant, this result makes it possible to use the computational procedure explained herewith.

The title problem is solved by a finite difference technique. The half space is replaced by a large finite grid and the conditions imposed at its outer boundaries simulate a uniform parallel flow. This is possible because the wake is known to decay with the distance from the plate. The non-linearity inherent in the governing equation is circumvented by an iteration scheme based on Newton's method. It is found that the plate in question experiences appreciable positive pitching moment as well as drag.

INTRODUCTION

An obstacle which is symmetric with respect to its axis and moves with zero incidence through an infinite body of viscous incompressible fluid experiences drag but no lift or pitching moment. However, when such body moves parallel to a solid plane and not too far from it, the pattern is no longer symmetric. The interaction between the motion produced by the moving obstacle and the confining influence of the plane may give rise to a transverse force and a pitching moment. In as much as these occur they are of practical importance, and the purpose of this work is to provide a numerical method by which such streaming problems could be tackled. As an example the author considers the case of a finite plate moving near a wall as shown in Fig. 1. Reynolds number is taken to be small, so that the influence of the moving plate is not confined to a thin boundary layer, and the effect of the solid plane is significant.

With respect to a two dimensional co-ordinate system (x,y) moving with the plate, a solution for the steady stream function, $\psi(x,y)$ is obtained. The non-linearity of the Navier Stokes equations is tackled by forming the iteration scheme suggested by the author [1]. The 'n'th approximation of the stream function is governed by a fourth order differential equation, in which the co-effieients and inhomogeneous term depend on the (n-1) approximation. The boundary conditions are derived from the requirements that the fluid on the wet surfaces should not move with respect to the solid, and that the flow at infinity should be uniform and parallel. Numerical solutions for the resulting differential systems are obtained by replacing the theoretically semi-infinite domain with a suitably large grid.

Note that it would have been impossible to apply this procedure to the case of a plate, or other obstacle, in an unbounded stream. In the absence of the solid plane the drag on the obstacle is balanced by the momentum flux through a contour surrounding the body. If the contour is large the momentum flux at any of the far grid points is small and the local discrepancy between the imposed condition, simulating uniform parallel flow, and the correct one would be minute. Nevertheless, the sum total of these discrepencies, for all outer grid points, would be sizable. Moreover the resulting solution would lack one of the essential qualitive features of the flow. However, it is shown that in the case at hand the total momentum flux can be made arbitrarily small by chosing a sufficiently large contour. Consequently, no error is committed by requiring that the flow at infinity should be uniform and parallel. The vanishing of the wake is understandable on physical grounds. In as much as the obstacle disturbs the otherwise uniform velocity field there will be stresses on the solid plane. These will act in the direction tending to restore the stream's uniformity. Therefore, over a solid plane of infinite span the momentum flux will be completely absorbed.

MATHEMATICAL DEVELOPMENT

The governing equation in the case at hand is

$$\left([(\partial\psi/\partial y)\partial/\partial x - (\partial\psi/\partial x)\partial/\partial y] - Re^{-1}\nabla^2\right)\nabla^2\psi = 0 \quad (1)$$

where Re is Reynolds number, and ∇^2 is the two-dimensional Laplace operator. The independent variables are non-dimensionalized with respect to half the plate's axial length b. The stream function is non-dimensionalized with respect to this length and the free stream speed U. The boundary conditions are

$$\psi = 0 \qquad \partial\psi/\partial y = 1 \qquad y = 0, \quad -\infty < x < \infty \quad (2)$$

$$\partial\psi/\partial y = 0 \qquad \psi = \text{const.} \qquad y = h+, h-, \quad -1 < x < 1 \quad (3)$$

$$\psi \longrightarrow y \qquad (\pm x, y) \longrightarrow \infty \quad (4)$$

where h+ and h- designate the top and bottom surfaces of the plate.

Since the domain under discussion is multiply - connected equations (1) - (4) do not define a unique solution for ψ. The solution sought is that for which the pressure, p, is single valued. With s as the arch length, measured along a contour surrounding the plate, and lying in the half space $y>0$ the requirement of single-valuedness is expressed thus:

$$\oint (\partial p/\partial s)\, ds = 0 \quad (5)$$

Note that for flows which are symmetric with respect to x=o or some other axis, this requirement is automatically satisfied.[2] In the absence of such symmetry, as in the case at hand,[3] equation (5) yields a non trivial relationship without which the differential system (1) - (4) is underdeterminate.

Let $\psi^{(n)}$, $n = 1, 2, 3$ and so on be a sequence of approximations for $\psi(x,y)$, each satisfying boundary conditions (2) - (4). As has been shown, two consecutive approximations $\psi^{(n)}$ and $\psi^{(n-1)}$ are related by

$$[(\partial\psi^{(n-1)}/\partial y)(\partial/\partial x) - (\partial\psi^{(n-1)}/\partial x)(\partial/\partial y)]\nabla^2\psi^{(n)} +$$

$$+ [(\partial\psi^{(n)}/\partial y)\partial/\partial x - (\partial\psi^{(n)}/\partial x)\partial/\partial y]\nabla^2\psi^{(n-1)} - Re^{-1}\nabla^4\psi^n =$$

$$= [(\partial\psi^{(n-1)}/\partial y)\partial/\partial x - (\partial\psi^{(n-1)}/\partial x)\partial/\partial y]\nabla^2\psi^{(n-1)} \qquad (6)$$

At the 'n'th iteration a solution for $\psi^{(n)}$ is sought which satisfies equation (6) and is associated with a single-valued approximate pressure field, $p^{(n)}$. The latter requirement is obtained by combining the relationship

$$[(\partial\psi^{(n)}/\partial y)\partial/\partial x - (\partial\psi^{(n)}/\partial x)\partial/\partial y](\partial\psi^{(n-1)}/\partial y) + [(\partial\psi^{(n-1)}/\partial y)\partial/\partial x$$

$$-(\partial\psi^{(n-1)}/\partial x)\partial/\partial y](\partial\psi^{(n)}/\partial y) - [(\partial\psi^{(n-1)}/\partial y)\partial/\partial y$$

$$-(\partial\psi^{(n-1)}/\partial x)\partial/\partial y](\partial\psi^{(n-1)}/\partial y) - Re^{-1}\nabla^2(\partial\psi^{(n)}/\partial y) = -\rho^{-1}(\partial p^{(n)}/\partial x), \qquad (7)$$

and the corresponding relationship for $\partial p^{(n)}/\partial y$ with equation (5). Here ρ designates the density of the fluid. If the iteration scheme converges the following holds

$$\psi^{(n)} \longrightarrow \psi, \quad p^{(n)} \longrightarrow p \qquad as \quad n \longrightarrow \infty \qquad (8)$$

and the three groups of non-linear terms in equations (6) and (7) become equal. The groups with opposite signs cancel one another and these equations reduce to the regular vorticity and momentum equations.

Once the zeroth approximation is chosen the iteration scheme can be started. The convergence and rapidity of convergence, depend markedly on this starting point. In the case at hand the following choice is made

$$\psi^{(0)} = y, \qquad (9)$$

so that $\psi^{(1)}$ is the Oseen's solution. Such solution is often taken as a good approximation even when the obstacle is blunt and it is even better in the present case. For a blunt body the flow field given by equation (9) both penetrates through and slips along the boundary, while in the case at hand the assumed field violates only the no slip condition. The solution procedure is stopped not after the first iteration but when the three groups of non-linear terms associated with the momentum or vorticity in equations (7) and (6), respectively are (essentially) equal.

Approximate solutions for $\psi^{(n)}(x,y)$ are obtained by evaluating $\psi^{(n)}(i\Delta x, j\Delta y)$ for discrete and finite values of i and j. At the outer boundaries of the grid as well as at $y = 0$ ψ is prescribed, so that at these points $\psi^{(n)}$ is known. The unknowns of the problem are the values of $\psi^{(n)}$ one or more mesh-lengthes away from the outer boundaries, and the value of $\psi^{(n)}$ along the plate. The differential system provides an equal number of linear algebraic equations. Corresponding to every point in the flow there is a finite difference counterpart of equation (6). An additional relationship is provided by equation (5), which by virtue of equation (7) can be expressed in terms of the values of $\psi^{(n)}$ and $\psi^{(n-1)}$ at mesh points around and on the plate. The finite difference - counterpart of equations (6) and (7) contain error of $O(\Delta x^2)$ or $O(\Delta y^2)$. The conditions imposed on the derivative of $\psi^{(n)}$ normal to the outer boundary or the plate, are incorporated in the finite difference scheme in the usual manner.

THE VANISHING OF THE MOMENTUM FLUX AT INFINITY

Application of the singular perturbation method to the problem at hand would yield the following far field expansion

$$\psi \sim y + \Delta_1(Re)\psi_1 \qquad (10)$$

where

$$\Delta_1(Re) \rightarrow 0 \qquad as \qquad Re \rightarrow 0 \qquad (11)$$

Here ψ_1 is a solution of the Oseen equation which is finite at infinity and matches the near field expansion. However, it is evidently difficult to construct such matched asymptotic expansion type of solution. (In fact, this is why the author resorts to numerical technique.) Hence the far field behavior of ψ_1 will be investigated without evaluating this function exactly. This limited aim is achieved by imposing suitable conditions on ψ_1 near the origin, rather than by matching with the near field expansion. In a sense the author follows Oseen's original treatment of flow past a circular cylinder. There too the conditions which hold on the obstacle surface are not exactly satisfied. Nevertheless, as was proved long after its publication, Oseen's solution adequately represent the general features of the far field behavior.

It follows from the previous remarks that ψ_1 is governed by

$$(\bar{\nabla}^2 - \partial/\partial X)\bar{\nabla}^2\psi_1 = 0$$

where the dependent variables and Laplacian operators are scaled thus

$$(x, y) \equiv Re^{-1}(X, Y) \quad , \quad \bar{\nabla}^2 \equiv \partial^2/\partial X^2 + \partial^2/\partial Y^2 = Re^{-2}\nabla^2$$

In the context of singular perturbation solution the implication of this scaling is that the region $-1<x<1$, $0\ y<h$, shrinks to a point. Hence conditions (2) - (4) are simulated by

$$\psi_1 = \text{finite} \qquad (\pm X, Y) \rightarrow \infty$$

$$\psi_1 = 0 \qquad \partial\psi_1/\partial Y = - M[\Delta_1(Re)Re]^{-1}\delta(X)$$

where δ is Dirac's delta function. The factor M reflects the possible discrepency between the value of ψ_1, which could be obtained by the matching process, and that derived by imposing the simulated conditions (13) and (14). The constants appearing on the right hand side of equation (14) are supressed by normalizing the relevant component of the stream-function as follows

$$\Psi = - M[A_1(Re)Re]^{-1}\psi_1$$

In view of the semi infinite nature of the domain a solution of the following Fourier Integral form is sought

$$\Psi = \int_{-\infty}^{\infty} F(Y,\alpha)\exp(i\alpha X)\,d\alpha$$

It follows from the governing equation that the transform function F satisfies

$$(d^2/dY^2 - (\alpha^2 + i\alpha))(d^2/dY^2 - \alpha^2)F = 0$$

Of the four solutions of the last equation two increase exponentially with Y and are therefore ruled out. Hence Ψ is given by

$$\Psi = \int_0^{\infty}[A(\alpha)\exp(-\alpha Y) + B(\alpha)\exp(-\beta Y)]\exp(i\alpha X)\,d\alpha$$

where

$$\beta \equiv (\alpha^2 + i\alpha)^{1/2}$$

and the real part of β is non-negative. From the two conditions imposed at $Y=0$ A and B are evaluated and the solution sought is found to be

$$\Psi = \alpha^{-1}\int_0^{\infty}(\beta-\alpha)^{-1}[\exp(-\alpha Y) - \exp(-\beta Y)]\exp(i\alpha X)\,d\alpha$$

Note that by writing β as follows

$$\beta = 2^{-1/2}\{[\alpha^2 + \alpha(1+\alpha^2)^{1/2}]^{1/2} + i[-\alpha^2 + \alpha(1+\alpha^2)^{1/2}]^{1/2}\}$$

Ψ can be rewritten as the sum of a real Fourier sine transform plus a real Fourier cosine transform. This rearranged form can be easily used in computing Ψ. However, the complex form (17), and a related one (19), derived below, have more easily recognized order properties for large X and Y.

So as to deduce the asymptotic behavior at infinity use is made of the following relationship

$$(\beta - \alpha)^{-1} = (i)^{-1}\left[1 + (\alpha + i)^{1/2}\alpha^{-1/2}\right]$$

where the real parts of both $(\alpha+i)^{1/2}$ and $\alpha^{1/2}$ is positive. For fixed X and large Y the expression (17) is treated as the sum of two components. The integrand in the component satisfying the Laplace equation consists of the product of $\exp(-\alpha Y)$ times a function of α. The other term, hereafter referred to as the Oseen component, can be rearranged in the form of an integral with respect to β with $\exp(-\beta Y)$ times a function of β as the integrand. For both integrals Watson's Lemma[4] is applicable. Hence the following relationship holds

$$\Psi \sim \sigma^{-1/2}\, Y^{-1/2} \qquad X = \text{const} \qquad Y \to \infty$$

In the case of fixed Y and large positive X, Ψ is rewritten thus:

$$\Psi = \sigma^{-1}\int_0^\infty \left[1 + (k+1)^{1/2}k^{-1/2}\right]\left(\exp(-ikY) - \exp\left(-ik^{1/2}(1+k)^{1/2}Y\right)\right)\exp(-kX)\,dk$$

This rearrangement is achieved by treating k as the real part of the complex variable $(\alpha + ik)$ and invoking the Cauchy Theorem. Watson's Lemma is clearly applicable to the form (19). Hence the following result is obtained

$$\Psi \sim \sigma^{-1}\, Y\, X^{-1} \qquad Y = \text{const} \qquad X \to \infty$$

It is of interest to note that the harmonic and Oseen components, have the dominant contributions to the asymptotic expressions (18) and (20), respectively. Understandably the former has the major influence in the transverse direction while the latter leaves its mark downstream. The influence of both combined vanishes at infinity, so that condition (4) is justified.

RESULTS AND CONCLUSIONS

Calculations were carried out with the following choice of constants

$$Re = 1 \qquad \Delta x = \Delta y = 1/2 \qquad h = 2$$

The number of mesh points in the grid was about 400. The unknown values of $\psi^{(n)}$ at the mesh points were calculated by the Gaus - Sidel iteration scheme[5]. Allowing 2% error in the values of $\psi^{(n)}$ it was found unnecessary to go beyond n = 2, i.e. just one step beyond the Oseen approximation. The resulting flow pattern is plotted in Fig. 1. Note that the disturbance created by the plate is negligible beyond the semi circle $(x^2+y^2)^{1/2} = 4$. The grid used in the calculation was considerably larger, (only its central part is shown in the figure). In agreement with the analysis of the last section the disturbance created by the obstacle decays faster with increasing x than with increasing y.

The stresses σ_{xy} and σ_{yy} at the solid plane were computed by utilizing the numerical solution for ψ. The former is proportioned to $(\partial^2\psi/\partial y^2)_{y=0}$ and can therefore be expressed as a linear combination of $\psi(x, \Delta y)$ and $\psi(x, 2\Delta y)$. The stress σ_{yy} at the solid plane is equal to the pressure p and must, therefore, be computed by integrating along y = 0 the x momentum equation. This process involves accumulative error and hence the resulting values of σ_{yy} plotted in Fig. 2 are less reliable.

Since the disturbance created by the plate decays at infinity the sum total of the forces acting on the solids must vanish. Therefore, the drag, lift and pitching moment on the plate can be obtained by integration over the solid infinite plane y = 0. It follows from the results plotted in Fig. 2 that the plate experiences drag and a positive pitching moment. However, there is no significant transverse force on the plate. These results can be explained qualitatively. Due to the relatively high viscosity of the fluid the plate drags a significant amount of fluid in the transverse direction as it moves axially. Since motion in the -y direction is hindered by the solid plane the fluid particle - pathes curve around the plate in the concave manner (see Fig.1.). Associated with the (+ y) and (-y) directed acceleration which is experienced by the fluid in front and behind the plate there is high and low pressure zones, respectively. These give rise to the positive pitching moment. However, the deviation of the pattern (Fig. 1) from symmetry or the pressure field (Fig. 2) from antisymmetry is slight. Hence there is no appreciable transverse y- directed resultant.

REFERENCES

1. M. Bentwich, Application of Newton's method to various non-linear problems in fluid mechanics. Proc. 8th Israeli Conf. on Avia. and Astro. pp 34 - 37 (1966)

2. S.C.R. Dennis & J. Dunwoody, Steady flow of fluid past a flat plate. J. Fluid Mech. 24 pp 577 - 595 (1966)

3. M. Bentwich & C. Elata, Eddy formation in an eccentric onnular domain. Phys. of Fluids 8 pp 2204 - 2210 (1965)

4. H. Jeffreys & B.S. Jeffreys "Method of Mathematical Physics" Cambridge Un. Press. §17·03 (1956)

5. S.S. Kuo, "Numerical Methods and Computers" Addison Wesley § 8·10 (1965)

ACKNOWLEDGEMENT

Thanks are due to Professor S.C.R. Dennis and my colleague Professor E.E. O'Brien. Their remarks left a clear impression on this work. The solution presented was computed by Mr. Nehemia Miller.

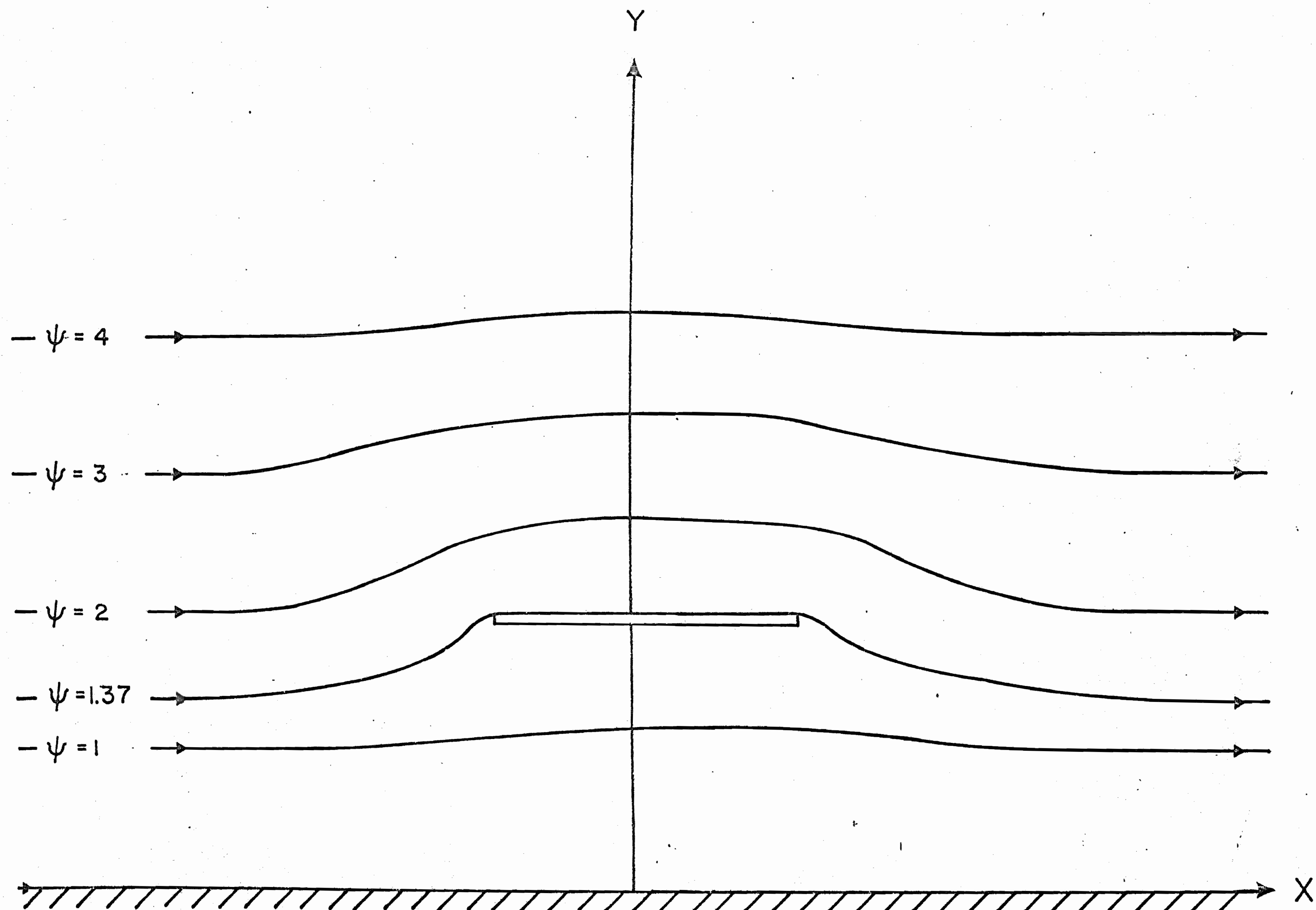

Fig. 1. Flow pattern produced by a plate moving parallel to a stationary solid plane.

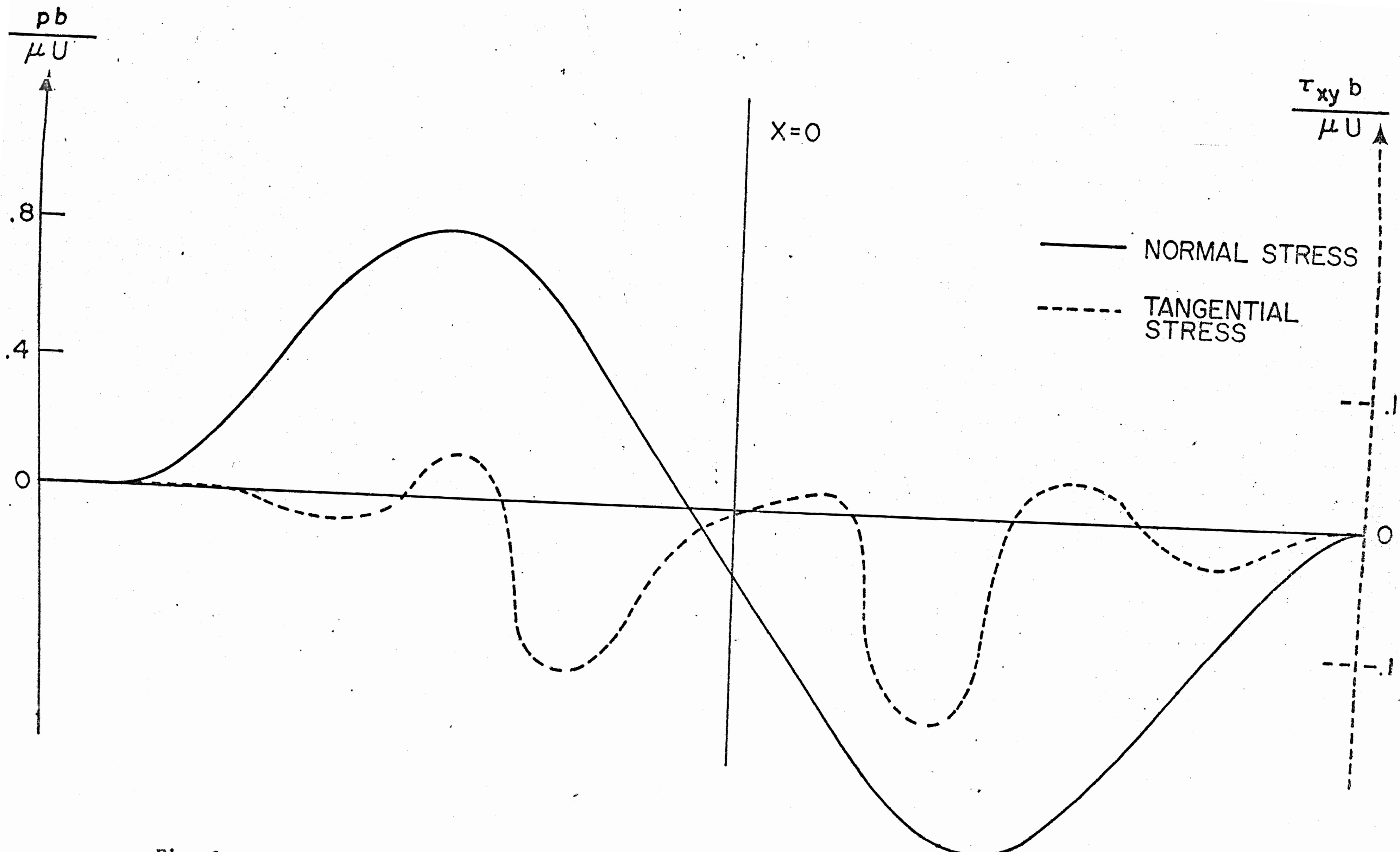

Fig. 2. Normal and tangential stresses on the solid plane.