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CONVECTION ENFORCED BY SURFACE
AND TIDAL WAVES

by

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CONVECTION ENFORCED BY SURFACE AND TIDAL WAVES

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Abstract

The time-dependent diffusion equation is solved for an infinite layer of fluid in which there are tidal or surface waves. The hydrodynamical disturbances are found to produce perturbations in the otherwise time-independent temperature distribution. The perturbations are found to propagate, while retaining their form or else getting dispersed, and in general be similar to the hydrodynamic waves. Propagation of the heat associated with the perturbation in the temperature distribution is then shown to give rise to convection.

Introduction

The concept of forced convection is normally associated with streaming through a duct or past an obstacle. In such mechanisms the cold (or solute absorbing) fluid at the vicinity of the heat (or mass) sources is being continually replenished. Less attention has been paid to convection mechanisms in which the mean velocity of the fluid is small and convection results from a wave travelling through the essentially stagnant fluid. It is shown here that according to the accepted incompressible fluid dynamics and diffusion laws this form of transfer can exist. The hydrodynamical wave gives rise to a perturbation in the otherwise time independent temperature (or concentration) distribution. The perturbation propagates as fast as the travelling mechanical waves. It is also similar to these in various other senses. This phenomenon will be therefore referred to as "heat wave". Furthermore, it is treated here by extending the well known perturbation type of analysis of hydrodynamic waves.

Attention is focused on heat waves in an infinite layer of fluid. When undisturbed its depth h is uniform. The temperature of the solid bottom and the possibly distorted free surface, T_- and T_+ are also taken to be uniform. The temperature (or concentration) distribution is no longer linear with the height when surface or tidal waves are present. This deviation is partly due to the geometrical distortion of the otherwise uniform layer and partly due to convection. In the case of surface waves it is possible to make a clear distinction

between the two effects. Convection, in such case is thus shown to be produced by horizontal drift of isothermal fluid particles. Again drift is shown to be closely linked with convection which is produced by tidal waves. It is significant that, in theory, so long as the above mentioned conditions hold no convection takes place when the motion of the fluid is uniform and parallel. This subject could be of practical interest, because tidal waves are normally faster than ocean currents. Furthermore, the two often do not appear at the same time and place. Hence heat waves may be relevant to weather studies.

General Perturbation Analysis

In the problems under consideration the governing diffusion equation

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \kappa \nabla^2 T, \quad (1)$$

is solved together with the boundary conditions

$$T = T_+ \quad \text{at} \quad y = \eta(x, t), \quad (2)$$

$$T = T_- \quad \text{at} \quad y = -h. \quad (3)$$

Here $T(x, y, t)$ is the temperature (or the concentration of the solute) and κ is the diffusivity of the fluid. Conventional cartesian co-ordinates (x, y) are used here and ∇^2 is the Laplacian operator in these. The velocity components are (u, v) , η denotes the elevation of the free surface above its undisturbed position $y = 0$, and t is the time.

As mentioned, the variables u, v and η are of $O(\varepsilon)$. Though different physical significance is attached to the parameter ε in the cases of surface and tidal waves, in both cases $\varepsilon \ll 1$. The dependent variable can therefore be expanded thus:

$$(u, v) = c \sum_{n=1}^{\infty} (U_n, V_n) \varepsilon^n, \quad \eta = L \sum_{n=1}^{\infty} E_n \varepsilon^n, \quad (4)$$

where c is the velocity of wave propagation and λ is a characteristic length. Since T does not vanish when ε is zero, or when there are no waves, this variable is assumed to be expandable in the following form:

$$T = \sum_{n=0}^{\infty} \theta_n(x, y, t) \varepsilon^n \quad (5)$$

When the expansions (4) and (5) are inserted into equations (1) and (3) and like powers of ε are equated, the following relationships are obtained

$$\left(\frac{\partial}{\partial t} - k \nabla^2\right) \theta_n = c \sum_{i=0}^{n-1} \left(u_{n-i} \frac{\partial \theta_i}{\partial x} + v_{n-i} \frac{\partial \theta_i}{\partial y} \right) \quad (1_n)$$

$$\theta_n(x, -h, t) = \delta_{n0} T_+ \quad (3_n)$$

where δ_{mn} is the Kronecker delta. The boundary condition for the exposed surface is derived by expanding $T(x, \eta, t)$ in Taylor series, as shown

$$T(x, \eta, t) = \sum_{j=0}^{\infty} \frac{\partial^j T}{\partial y^j}(x, 0, t) \frac{\eta^j}{j!} = T_+ \quad (2')$$

and substituting into equation (2') from (4) and (5). The resulting boundary conditions imposed on θ_n are therefore

$$\theta_0(x, 0, t) = T_+ \quad (2_0)$$

$$\theta_1(x, 0, t) = -L \frac{\partial \theta_0}{\partial y}(x, 0, t) E_1(x, t) \quad (2_1)$$

$$\theta_2(x, 0, t) = -L^2 \frac{\partial^2 \theta_0}{\partial y^2} \frac{E_1^2}{2!} - L \frac{\partial \theta_1}{\partial y} E_2 - L \frac{\partial \theta_0}{\partial y} E_3 \quad (2_2)$$

and so on. Every term in the expansion (5) is therefore governed by a determinate second order differential system. These can be solved consecutively for $n = 0, 1, \dots$ providing the u_n, v_n and E_n which appear in (1_n) and (2_n) are known. The solution for θ_0 is obviously

$$\theta_0 = T_1 + (T_2 - T_1)(y+h)/h \quad (6)$$

Since when $t=0$ the temperature is given by this term only, it represents the steady distribution which is perturbed when waves are present.

Convection by Surface Waves

In such case u and v are the x and y derivatives of the wave potential [1] [2]

$$\varphi = m^{-1}c \left[\varepsilon \frac{\cosh[m(y+h)]}{\sinh(mh)} \sin[m(x-ct)] + \varepsilon^2 \frac{3}{2} \frac{\cosh[2m(y+h)]}{\tanh^2(mh) \sinh^2(2mh)} \sin[2m(x-ct)] \dots \right] \quad (7)$$

The elevation of the free surface is

$$\eta = m^{-1} \left[\varepsilon \cos[m(x-ct)] + \varepsilon^2 \frac{2 + \cosh(2mh)}{\tanh^2(mh) \sinh(mh)} \cos[2m(x-ct)] \dots \right] \quad (8)$$

The characteristic length L is taken to be m^{-1} which is $(2\pi)^{-1}$ times the wave length. From the last expression one finds that εm^{-1} is very nearly equal to the amplitude of the disturbance of the free surface, so that ε is a measure of its slope.

The solution for θ_1 is in this case governed by

$$\left(\frac{\partial}{\partial t} - \kappa \nabla^2\right) \theta_1 = -c \frac{T_+ - T_-}{h} \frac{\sinh[m(y+h)]}{\sinh(mh)} \sin[m(x-ct)] \quad (1_1)$$

$$\theta_1(x, 0, t) = - \frac{T_+ - T_-}{(mh)} \cos[m(x-ct)] \quad (2_1)$$

$$\theta_1(x, -h, t) = 0 \quad (3_1)$$

where the right hand sides of equations (1₁) and (2₁) are evaluated by making use of (6), (7) and (8). One can easily verify that the solution for θ_1 is

$$\theta_1 = - \frac{T_+ - T_-}{mh} \frac{\sinh[m(y+h)]}{\sinh(mh)} \cos[m(x-ct)] \quad (9)$$

When use is made of equations (6) - (9) equations (1₂) and (2₂) are found to read

$$\left(\frac{\partial}{\partial t} - \kappa \nabla^2\right) \theta_2 = -c \frac{T_+ - T_-}{h} \frac{\coth^2(mh) \sinh[2m(y+h)]}{\sinh^2(2mh)} \sin[2m(x-ct)], \quad (1_2)$$

$$\begin{aligned} \theta_2 = & - \frac{1}{2} \frac{T_+ - T_-}{mh} \frac{2 + \cosh(2mh)}{\tanh^2(mh) \sinh(2mh)} \cos[2m(x-ct)] + \\ & + \frac{T_+ - T_-}{(mh)} \coth(mh) \cos^2[m(x-ct)] \quad (2_2) \end{aligned}$$

These together with (3₂) yield

$$\theta_2 = \frac{T_+ - T_-}{mh} \left\{ \frac{1}{2} \coth(mh) \frac{y+h}{h} - \frac{3}{4} \cdot \frac{\coth(mh) \sinh[2m(y+h)] \cos[2m(x-ct)]}{\sinh^2(mh) \sinh(2mh)} \right\} \quad (10)$$

Function θ_j for $j > 2$ will not be evaluated within the framework of this treatise because the mechanical disturbance which is $O(\epsilon^3)$ propagates at a velocity different than c . Therefore, though the method employed here is in principle applicable the algebra involved in the solutions for θ_j becomes prohibitively cumbersome for $j > 2$.

In this solution $(\partial \theta_j / \partial t)$ is equal to the right hand side of equations (1_j) and $\nabla^2 \theta_j$ vanishes, for $j = 0, 1, 2$. Physically this means that the temperature of every fluid particle is fixed and that equal amounts of heat are conducted into and out of every unit volume. The processes of diffusion and conductions thus do not interact. This is not, by any means, an unknown phenomenon. Distributions which have the same property could be produced by letting the flow between the isothermal surfaces $y=0$ and $y=h$ be parallel and uniform. Such case is also treated by O'Brien⁽³⁾.

In view of the last remarks and the form of the solution, T satisfies the Laplace equation in (x, y) as well as (x', y) where $x' = x - ct$. The conditions (2_j) and (3_j) for $j = 0, 1, 2$ can also be expressed in terms of x' . The solution for T

is therefore the steady state distribution in a solid strip bounded by a wavy (stationary!) top surface and a plane bottom. It is to be expected that the increase in the area of the top surface due to its distortion increases the mean flux across such solid strip of length $(2\pi/m)$. This increase is represented by the first term in the curly bracket of equation (10). The contribution of this term to the solution for temperature distribution $T(x, y, t)$ in the wave-carrying fluid, is therefore recognized as the result of the geometrical distortion of the layer rather than a convective effect.

The possibilities of producing an overall x-directed heat (or mass) transfer will now be considered, bearing in mind that each of the two modes of transfer do not interact. Since the impressed gradient is essentially vertical the mean axial conduction is expected to vanish. Indeed, the time integral of $(\partial T / \partial x)_{x = \text{const.}}$ over a period $2\pi/mc$ is zero. The mean convective transfer similarly vanishes, because fluid particles retain their temperature while tracing closed contours around points which are fixed in space. Conversely convection is bound to be observed with wave motion in which the mean positions of the isothermal particles transverse axial distances. Such, so called, drift is associated with the infinite train of waves under consideration, but is of an order that is neglected within the framework of this analysis.^[4,5] It is shown in the next paragraph that drift and hence convection can be produced by wave-system

of $O(\epsilon)$.

Consider the following generalization of the expressions for φ and η

$$\varphi = c a \int_{-\infty}^{\infty} f(m) \frac{\cosh[m(y+h)]}{\sinh(mh)} \sin[m(x-ct)] dm + O(\epsilon^2), \quad (11)$$

$$\eta = a \int_{-\infty}^{\infty} f(m) \cos[m(x-ct)] dm + O(\epsilon^2) \quad (12)$$

Here (am) is of $O(\epsilon)$ and for the sake of brevity and simplicity terms of $O(\epsilon^2)$ are neglected. At time $t=0$ the following holds

$$\eta(x,0) = a F(x) = a \int_{-\infty}^{\infty} f(m) \cos(mx) dm \quad (13)$$

XXX This relationship could (but need not necessarily) be one of the initial conditions. In any case it is assumed here that $F(x)$ is continuous, even in x and decreases more rapidly than $|x|^{-1}$ for large $|x|$. Under some circumstances (which will be discussed later) c is independent of m so that at any time $t > 0$ η is $aF(x-ct)$. The deflection of the free surface and the associated velocity field therefore vanish far from the moving section $x = ct$. In the event that $F(x)$ is everywhere positive, i.e. when the travelling wave is associated with a localized 'swell' of the otherwise uniform layer, the propagating disturbance produces a drift. For hydrodynamic wave of the form (11) the temperature distribution is

$$T = T_- + (T_+ - T_-) \left\{ \frac{y+h}{h} - \frac{a}{h} \int_{-\infty}^{\infty} \rho(m) \frac{\sinh[m(y+h)]}{\sinh(mh)} \cos[m(x-ct)] dm \right\}. \quad (14)$$

It is important to note that the expressions in equations (11), (12) and (14) are linear superpositions of solutions of the form (7), (8) and (9). Therefore in the generalized wave system too fluid particles are isothermal and the arguments about the effect of this property on the convection hold. Thus with $T = \theta_0$ as a reference temperature the heat stored in a wave of breadth l is

$$Q = \frac{\delta l}{2\sigma} (T_+ - T_-) \frac{a}{h} \int_0^{\infty} \int_0^{\infty} [F(\xi) + F(-\xi)] \ln \left\{ \tanh\left(\frac{\sigma}{2h}|\xi-x|\right) \tanh\left(\frac{\sigma}{2h}|\xi+x|\right) \right\} d\xi dx, \quad (15)$$

where δ is the specific heat. This is obtained by setting in equation (14) $t=0$ and integrating the time-dependent term with respect to y , m and x . In view of the assumed properties of $F(x)$ Q is finite and non-zero. The form of the solution (14) indicates that the time-dependent component of T and the associated heat Q , propagates with the velocity c . Therefore though Q is proportional to (a/h) , which is small, the rate of heat flow cQ could under some circumstances be significant.

This section is concluded with some remarks about the validity and applicability of the foregoing arguments and results. The velocity of surface waves is related to the gravitational acceleration g by

$$c^2 = g m^{-1} \tanh(mh). \quad (16)$$

Therefore, when m is very small c is m -independent, as assumed, and is equal to the maximum attainable value $(gh)^{1/2}$. Therefore when the causing disturbance is such that the n^{th} derivatives of $F(x)$, $F^{(n)}(x)$ are of $O(h^{-n})$, $f(m)$ is very small beyond the range $-\epsilon h^{-1} < m < \epsilon h^{-1}$ where $\epsilon \ll 1$, and the resulting (hydrodynamic and) heat wave propagates with that velocity. However, if the surface wave is produced by a more abrupt or sharp disturbance, $F(x)$ is not smooth, and the major contributions to time dependent components of equations ((11), (12) and) (13) are due to integration with respect to m beyond the above-mentioned range. The (hydrodynamical and) heat wave gets dispersed, and a significant part of the heat Q propagates with a velocity slower than $(gh)^{1/2}$. Consequently the resulting rate of heat flow is lower than $(gh)^{1/2} Q$. A more exact expression for it can be obtained by multiplying the integrand of equation (13) by $c(m)$, setting $t=0$ and carrying out the integration with respect to y, m and x as before.

Heat Carried by Tidal Waves

In this case the depth of the layer h is taken to be the characteristic length L . The parameter ϵ is the ratio of the amplitude of the free surface deflection to h . The non dimensional components of velocity and the deflection are [1]

$$U_1 = - [G(x-ct) - K(x+ct)] ,$$

$$V_1 = (y+h)[G'(x-ct) - K'(x+ct)] ,$$

$$E_1 = - [G(x-ct) + K(x+ct)] .$$

The functions G and K are solutions of the one-dimensional wave equation which represent incoming and outgoing waves. The expressions for U_1 , V_1 and E_1 are inserted in equations (1₁), (2₁) and (3₁). The non homogeneous part of these turn out to be functions of y , $(x+ct)$ and $(x-ct)$. This leads one to seek a solution which is expressible in terms of the three variables. Since, as will be shown, such solution exists, it may be concluded that like surface waves, incoming and outgoing tidal waves are accompanied by heat waves.

The solution is assumed to have the following form:

$$\theta_1 = (T_+ - T_-) \left\{ \sum_{j=0}^{\infty} g_j(y) G^{(j)}(x-ct) + \sum_{j=0}^{\infty} k_j(y) K^{(j)}(x+ct) \right\} \quad (17)$$

In this expansion g_j and k_j are functions of y only.

With this, the governing equation (1₁) and boundary conditions (2₁) and (3₁) reduce to

$$\left. \begin{aligned} - \sum_{j=0}^{\infty} G^{(j)}(c g_{j-1} + K(g_{j-2} + g_j)) + \\ + \sum_{j=0}^{\infty} K^{(j)}(c k_{j-1} - K(k_{j-2} + k_j)) = c(y+h)(-G^{(0)} + K^{(0)}) \end{aligned} \right\} \quad (18)$$

$$g_j(-h) = k_j(-h) = 0 \quad (19)$$

$$g_j(0) = k_j(0) = \zeta_{j_0} \quad (20)$$

Since G and K are finite and continuous but otherwise arbitrary their derivatives are, in general, mutually independent. Equation (18) is therefore satisfied by requiring that the coefficients of $G^{(j)}$ and $K^{(j)}$ for $j = 0, 1, 2, \dots$ should vanish. For $j=0$ this requirement yields the following differential equations

$$g_0'' = k_0'' = 0$$

By utilizing the boundary conditions (19) and (20) these can be readily integrated. For $j > 0$ this requirement yields non-homogeneous second order equations governing g_j and k_j , in which the non homogeneous parts contain 'g's and 'k's of lower indices. By using the boundary conditions the coefficients in equation (17) can therefore be evaluated consecutively for $j = 0, 1, 2, \dots$. The resulting solution for is found to be

$$\begin{aligned} \theta = (T_+ - T_-) \left\{ \frac{y+h}{h} (G + K) - \frac{(y+h)^3 - h^3(y+h)}{6h} (G'' + K'') \right. \\ \left. + \left(\frac{ch}{k} \right) \frac{1}{360h^2} [3(y+h)^5 - 10h^2(y+h)^3 + 7h^4(y+h)] (G^{(5)} - K^{(5)}) \right\} \quad (21) \end{aligned}$$

Like the time-dependent term of equation (14) the solution (21) represents progressing heat waves. However, unlike surface wave tidal and associated heat waves do not get dispersed. Therefore when $G(x)$ or $K(x)$ are

(like $F(x)$) everywhere positive and vanish at infinity the tidal waves carry non-zero amount of heat, with the velocity of propagation (which happens to be $(gh)^{1/2}$ again). Since the form of solution (21) indicates that the two modes of transfer interact, it is impossible to recognize one of these as the sole cause of convection. It is nevertheless noted that the assumed positiveness of G and K imply the existence of 'swell' and therefore drift. It is assumed that in the case of tidal waves too, this drift plays a role in inducing convection.

Concluding Remarks

The diffusion equation is solved for a layer of fluid which is bounded on the top by a time dependent free surface. Through the layer either surface or tidal waves propagate. Availability of non-trivial solution is taken here to imply that heat waves, which have certain properties, can exist. It is, however, born in mind that in solving the problems initial conditions were altogether ignored. The resulting temperature (or concentration) distribution $T(x, y, t)$ can be regarded as a solution of a well-posed problem with $T(x, y, 0)$ as the initial condition. It is found that such condition is rather artificial. Hence though the result about the existence of heat waves is theoretically conclusive, the manner in which such waves could be generated has yet to be studied.

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