STATE UNIVERSITY OF NEW YORK AT

STONY BROOK

CEAS Technical Report 530

TRANSFINITE GRAPHS AND ELECTRICAL NETWORKS : II

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This work was supported by the National Science Foundation

under Grant DMS - 8521824

December 15, 1988

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Abstract

All prior theories of infinite electrical networks assume that such networks are finitely connected, that is, between any two nodes of the network there is a finite path. This work establishes a theory for transfinite electrical networks wherein some nodes are not connected by finite paths but are connected by transfinite paths. The main difficulty to surmount for this extension is the construction of an appropriate generalization of the concept of connectedness. This is accomplished by means of an unorthodox definition of ordinary graphs, which is amenable for generalization to transfinite graphs. The construction appears to be novel. An existence and uniqueness theorem is then established for transfinite resistive electrical networks based upon Tellegen's equation.

1. INTRODUCTION

Infinite electrical networks have appeared intermittently in both the mathematical and electrical engineering literature for most of this century, but the earlier works were restricted to networks having graphs with regular repetitive patterns, such as ladders and grids. It has been only during the past two decades that networks with arbitrary graphs have been examined. The seminal work in this area was by Flanders [5] and appeared in 1971. It established an existence and uniqueness theorem for the voltage-current regime on a locally finite, linear, resistive network having only a finite number of sources and open circuits everywhere at infinity. This was followed by a series of papers that generalized the theory in various ways; see [1], [2], [10]-[14], and the references therein. Actually, infinite electrical networks arise in quite a different context as well, namely, in the theory of random walks on infinite graphs [3], [4], [7]-[9]. All the infinite electrical networks considered up to now have been finitely connected, that is, between every two nodes there exists a finite path. Nonetheless, infinite networks having some pairs of nodes connected by infinite paths but not by finite ones is an idea worth pursuing.

This paper was inspired by the following question. What kind of connections can be made between the "extremities" of an infinite network? That short circuits as well as pure voltage or current sources can be so connected was established in [10] and [13], but resistances between extremities remained an open problem, which this paper now resolves. Moreover, if resistances can be connected out at infinity, so too can other infinite networks, and we are thereby led naturally to a transfinitely connected infinite networks.

Much of this paper is devoted to a variety of definitions that generalize the idea of connectedness, allow the flow of currents from here to infinity and beyond, and as a result establish transfinite electrical networks. Those definitions set up a fairly elaborate structure of ranked extremities, but a simpler way of generalizing connectedness does not seem to be at hand if one wishes to make connections not only between extremities at infinity but also between such an extremity and a node or between extremities of different ranks. In fact, a transfinite network can be partitioned into sections whereby nodes in different sections are transfinitely distant, that is, the natural numbers do not suffice to number consecutively the branches in any path connecting two such nodes. The sections form a hierarchy ranked by the ordinal numbers; in effect, in order to define sections of higher and still higher ranks, one must keep expanding the concept of transfinite connectedness.

Finally, an existence and uniqueness theorem is established for the voltage-current regime in such a network. Under certain assumptions on the branch resistances, each section affects the other sections, and thus the network cannot be disconnected into a collection of finitely connected networks. The fundamental principle upon which the existence and uniqueness theorem is based is Tellegen's equation, which implies Kirchhoff's laws. Actually, Kirchhoff's laws need not hold everywhere in the network, but his current law will hold at every "restraining node" and his voltage law will hold around every "perceptible loop".

The idea of transfinite networks in the special cases of ladders and grids occurs in [15] and [16], but those works use the regular structure of their graphs in essential ways. The arbitrariness of the graphs in this paper requires a much different analysis.

Just a word or two about terminology: When we say that x is a subset of y, we allow $x = y$. As usual, a partition $\{x_m\}$ of a nonvoid set y means $y = \cup x_m$ and $x_m \cap x_n$ is void for $m \neq n$. A singleton is a set with exactly one member.

2. AN EXAMPLE

Consider the infinite ladder network of Figure 1 having the indicated resistance values. We take it that, if the upper nodes are indexed consecutively from left to right, then all the natural numbers are needed and suffice for this purpose. In analogy to a finite ladder network, we might suppose that output terminals exist at the end of this infinite ladder network, as indicated by the small circles. If so, a load resistance R_L might be connected thereto. Let us suppose still further that the monotonicity principle for resistance functions continues to hold for this ladder network. Consequently, the driving-point resistance R_D should be less than the value it becomes when all the shunting resistance values are replaced by ∞. So, when $R_L = 0$, this results in an infinite series circuit and $R_D < .111\cdots = 1/9$. On the other hand, R_D should be larger than the value it becomes when all the series resistances are replaced by 0. Now, for $R_L = \infty$, we obtain an infinite parallel circuit and $R_D > 1/.111\cdots = 9$. We could then conclude that R_D changes when R_L changes, and so we might infer that, in order for a voltage-current regime to be determined when a source is impressed at the input to the ladder, we must specify what the connection at infinity is - at least for this particular network. In short, infinity is perceptible to an observer at the input. This heuristic argument for ladder networks has a completely rigorous justification [15].

The point here is that we now have a network with a connection "at infinity". Moreover, R_L may be replaced by the input to another infinite ladder network to get a network that extends "beyond infinity". To put this another way, if we index the upper nodes consecutively from left to right starting with 0, the natural numbers will carry through the first ladder, but the upper circled node will have to be indexed by ω, the least transfinite ordinal; moreover, the upper nodes of the next ladder will have the indices $\omega+1, \omega+2, \cdots$. This is a transfinite network characterized by the fact that any node of the first ladder is transfinitely - but not finitely - connected to any node of the second ladder.

Transfinite networks with arbitrary graphs may similarly be conceptualized. What is needed is a fundamental theory for their voltage-current regimes, and the basic problem is the generalization of the idea of connectedness to allow currents to flow from local regions to regions beyond infinity.

3. 0-GRAPHS

In order to conform with our subsequent definitions, we shall define an ordinary (finite or infinite) graph in an unusual way. Let $\underset{\approx}{T}$ be a finite set with an even number of elements or a denumerably infinite set. Call each element of $\underset{\approx}{T}$ an *elementary tip* or just *tip* for short. Partition $\underset{\approx}{T}$ into subsets of two elementary tips each and call each subset a *branch*. Thus, no two branches have a tip in common, and every tip appears in a branch. $\underset{\approx}{B}$ denotes the set of all branches; it is countable.

Also, partition $\underset{\approx}{T}$ in an arbitrary way: $\underset{\approx}{T} = \cup x_m$ where $x_m \cap x_n$ is void if $m \neq n$. The subsets x_m will be called *0-extremities* or *extremities of rank* 0 or simply *nodes*. If x_m has two or more members, it is also called a *0-connection*; this may be interpreted as the shorting together of the tips in the 0-connection. All the other nodes are singletons. As a terminology we shall need later on, we shall say that a node *embraces* itself. Also, a branch is said to be *incident* to any node that contains one or both of its tips. Furthermore, two nodes are called *adjacent* if there is a branch that is incident to both nodes. A *finite* (or *infinite*) node is a node with a finite (or, respectively, infinite) number of incident branches.

An *ordinary graph* or a *graph of rank* 0 or a *0-graph* is defined as the pair $\underset{\approx}{G} = (\underset{\approx}{B}, \underset{\approx}{C}_0)$, where $\underset{\approx}{C}_0 = \{c_m\}$ is the set of all 0-connections. $\underset{\approx}{C}_0$ may be void, in which case all 0-extremities are singletons, and no tip is shorted to any other tip. Note that we do not display $\underset{\approx}{G}$ as the triplet $(\underset{\approx}{T}, \underset{\approx}{B}, \underset{\approx}{C}_0)$ simply because $\underset{\approx}{T}$ can be recovered as the union of all the sets in $\underset{\approx}{B}$. In words of the customary electrical-network terminology, this definition of a graph allows branches that are self-loops in themselves, as well as parallel branches. However, there are no isolated nodes; every node has an incident branch. Our definition of a graph is equivalent to the customary definition of a graph with these properties.

Let $\underset{\approx}{B}^*$ be a subset of $\underset{\approx}{B}$. Consider any $c \in \underset{\approx}{C}_0$. Any tip in c belonging to a branch in $\underset{\approx}{B}^*$ is taken to be a member of a set c^*, which perforce is a subset of c. If under this rule c^* has two or more members, it is taken to be a member of $\underset{\approx}{C}_0^*$, the set of ω^0-connections for the ordinary graph $\underset{\approx}{G}^* = (\underset{\approx}{B}^*, \underset{\approx}{C}_0^*)$. $\underset{\approx}{G}^*$ is called the ***subgraph of $\underset{\approx}{G}$ induced by $\underset{\approx}{B}^*$***. Also, $\underset{\approx}{C}_0^*$ is called the ***restriction of $\underset{\approx}{C}_0$ to $\underset{\approx}{B}^*$***.

A 0-***path*** or synonymously an ***ordinary path*** is an alternating sequence of nodes x_m and branches B_m

$$\{ \cdots , x_m , b_m , x_{m+1}, b_{m+1}, \cdots \} \tag{3.1}$$

that satisfies the following conditions.

Conditions Π^0:

(i) The sequence is either finite, one-way infinite, or two-way infinite; if it is one-way infinite (or two-way infinite), it is required that the natural numbers (or, respectively, the rational integers) suffice to index all the elements consecutively as indicated.

(ii) If the sequence terminates in either direction, it terminates at a node.

(iii) Each x_m that is not a terminal node is a 0-connection.

(iv) Each b_m is a branch that is incident to the two nodes immediately preceding and succeeding it in the sequence.

(v) No node appears more than once in the sequence. (Consequently, the branches are all different from each other.)

The construction of an infinite sequence such as (3.1) may require an infinity of selections from an infinity of sets, namely, the selection of an incident branch at each node; this is sanctioned by the axiom of choice.

The 0-path (3.1) is said to *meet* or *embrace* the nodes in (3.1). A 0-path is called *nontrivial* if it has at least three elements, *finite* if it has two terminal elements, *one-ended* if it has exactly one terminal element, and *endless* if it has no terminal element. We allow the special case of a finite path that is a singleton containing just one node and no branch; this is called the *trivial* 0-path. Two 0-paths are called *totally disjoint* if there is no node that is embraced by both paths. A 0-*loop* is defined exactly as is a finite 0-path except that the two terminal nodes are required to be identical.

Two nodes x_a and x_b are said to be 0-*connected* or *finitely-connected* if there exists a finite 0-path with x_a and x_b as its terminal elements. Two branches b_a and b_b are called 0-*connected* if b_a is incident to a node x_a, b_b is incident to a node x_b, and x_a and x_b are 0-connected. Also, $\underset{\approx}{G}$ is called 0-*connected* if every two nodes are 0-connected.

A 0-*section* or *nodal section* of the graph $\underset{\approx}{G}$ is the subgraph induced by a maximal set of branches that are pairwise 0-connected. $\underset{\approx}{G}$ may be a 0-section by itself or it may have more than one 0-section. At this stage of our definitions, a 0-section is simply a component of $\underset{\approx}{G}$, but we will shortly generalize the idea of connectedness and thereby render a 0-section into something other than a component.

4. 1-GRAPHS

Two one-ended 0-paths in a 0-graph $\underset{\approx}{G}$ are taken to be equivalent if they are identical except for a finite number of nodes and branches. This equivalence relationship partitions the set of all one-ended 0-paths in $\underset{\approx}{G}$ into equivalence classes, which will be called the 0-*pathlike tips* of $\underset{\approx}{G}$. A *representative* of a 0-pathlike tip is any one-ended path in that equivalence class. A one-ended or endless 0-path that contains a representative of a 0-pathlike tip is said to *meet* or *have* that tip. This idea of 0-pathlike tips is fundamental to our discussion, for it is to them that "connections at infinity" will be made.

Although the branch set $\underset{\approx}{B}$ is countable, $\underset{\approx}{G}$ may have a noncountably infinite set of pathlike tips. For example, this is the case when $\underset{\approx}{G}$ is the infinite binary tree.

A 1-*connection* or synonymously a *connection of rank* 1 is a finite or infinite set of the form

$$c^1 = \{x_0, t_1^0, t_2^0, t_3^0, \cdots\}$$

where the t_m^0 are 0-pathlike tips and x_0 is a node, which may not be present in the set. We say that c^1 *embraces* its elements as well as itself. (For notational convenience, we have indexed the pathlike tips in c^1 with the natural numbers. However, c^1 is allowed to be a noncountably infinite set of pathlike tips, in which case another indexing system should be used.)

The following conditions are required.

Conditions Γ^1:

(i) Each 1-connection has at least two members.

(ii) All elements of a 1-connection are 0-pathlike tips t_m^0 except possibly for one of them; that one x_0 is a node - if it is present.

(iii) No two 1-connections have an element in common.

It follows that each 1-connection has at least one 0-pathlike tip. Also, x_0 is called the *exceptional element of* c^1.

A physical interpretation of a 1-connection is that of short circuits connected to the 0-pathlike tips and possibly to one node as well. This will allow the flow of current along a path out to infinity, through a short circuit at infinity, and then along another infinite path. Alternatively, the current may jump along a short circuit from a node (the exceptional element) out to infinity and then continue along an infinite path.

A 1-*extremity* or an *extremity of rank* 1 is either an 1-connection or a singleton whose member is a 0-pathlike tip that does not appear in any 1-connection. In other words, we get the 1-extremities by first partitioning the set of 0-pathlike tips and then adding nodes to none, some, or all of the sets of the partition, at most one node to each set, all nodes different. It follows that two 1-extremities are either identical or have a void intersection. Again we say that a 1-extremity *embraces* itself and all its elements too.

Let x_0 be a node in a 1-connection c^1. A branch that is incident to x_0 is said to *meet* c^1. A 0-path P^0 that contains x_0 is said to *meet* c^1 *at* x_0, and, if x_0 is a terminal node of P^0, P^0 is said to *terminate at* c^1 *with* x_0. Similarly, a one-ended or endless 0-path that contains a representative of a 0-pathlike tip t^0 in a 1-extremity x^1 is said to *meet* x^1 *with* t^0.

A 1-*graph* or synonymously a *graph of rank* 1 is a triplet $\underset{\approx}{G} = (\underset{\approx}{B}, \underset{\approx}{C}_0, \underset{\approx}{C}_1)$, where $\underset{\approx}{B}$ and $\underset{\approx}{C}_0$ are as before and $\underset{\approx}{C}_1$ is a set of 1-connections. $\underset{\approx}{C}_1$ may be void. So too may $\underset{\approx}{C}_0$, or $\underset{\approx}{C}_0$ may have only a finite number of members; in this case $\underset{\approx}{C}_1$ is perforce void because $\underset{\approx}{G}$ will have no one-ended ω^0-paths.

As an example, consider Figure 2, which shows an infinite lattice cascade $\underset{\approx}{G}_1$ and an infinite ladder $\underset{\approx}{G}_2$ that are "connected at infinity" in such a fashion that their infinite extensions reach toward each other. The line segments indicate branches, the heavy dots indicate nodes, and the two small circles indicate 1-connections, which will be specified momentarily. We can identify any 0-path just by listing its branches in sequence. For instance, we have the following 0-paths.

$$P_a = \{a_1, a_2, a_3, \cdots\}$$
$$P_b = \{b_1, b_2, b_3, \cdots\}$$
$$P_{abc} = \{a_1, c_2, c_3, a_4, b_5, b_6, a_7, \cdots\}$$
$$P_d = \{d_2, d_3, d_4, \cdots\}$$
$$P_e = \{e_1, e_2, e_3, \cdots\}$$

$\underset{\approx}{G}_1$ has an infinity of 0-pathlike tips. Representatives of three of them are P_a, P_b, and P_{abc} respectively. Even though P_a, P_b, and P_{abc} have an infinity of nodes in common, they represent distinct 0-pathlike tips, which need not be declared to be "connected at infinity", that is, members of the same 1-connection.

Let t_a, t_d, and t_e be the 0-pathlike tips with the representatives P_a, P_d, and P_e respectively. Also, let x_f be the infinite node of the ladder network $\underset{\approx}{G}_2$. We might take for our 1-connections the two sets $c_1^1 = \{t_a, t_e\}$ and $c_2^1 = \{x_f, t_d\}$. Thus, $\underset{\approx}{C}_1 = \{c_1^1, c_2^1\}$. This is what is intended in Figure 2. According to some of our forthcoming definitions, this allows the flow of a "1-loop current" along P_a, through c_1^1, along P_e in the reverse direction, along branch f_1, through c_2^1, along P_d in the reverse direction, and finally along branch b_1. Moreover, we have at hand the 1-graph $(\underset{\approx}{B}, \underset{\approx}{C}_0, \underset{\approx}{C}_1)$ where $\underset{\approx}{B}$ and $\underset{\approx}{C}_0$ are implicitly specified by $\underset{\approx}{G}_1$ and $\underset{\approx}{G}_2$ together.

Alternatively, we could construct another 1-graph by changing $\underset{\approx}{C}_1$. In particular, we could let $\underset{\approx}{C}_1$ be the singleton $\{c_3^1\}$ where $c_3^1 = \{t_a, t_b\}$ and t_b is the 0-pathlike tip with P_b as a representative. This would disconnect $\underset{\approx}{G}_1$ from $\underset{\approx}{G}_2$. On the other hand, it would allow the flow of current along P_a, through c_3^1, and backwards along P_b. This flow would not be a 1-loop current according to our upcoming definition because P_a and

P_b share nodes.

Let $\underset{\approx}{G} = (\underset{\approx}{B}, \underset{\approx}{C}_0, \underset{\approx}{C}_1)$ be a given 1-graph. Given a subset $\underset{\approx}{B}^*$ of $\underset{\approx}{B}$, let $\underset{\approx}{C}_0^*$ be the restriction of $\underset{\approx}{C}_0$ to $\underset{\approx}{B}^*$. Next, let $c^1 \in \underset{\approx}{C}_1$. If a node x_0 exists in c^1, then the set of all elementary tips in x_0 belonging to branches in $\underset{\approx}{B}^*$ is taken to be a node x_0^* in a set c^* so long as x_0^* is not void. Similarly, if any 0-pathlike tip $t^0 \in c^1$ has a representative with all its branches in $\underset{\approx}{B}^*$, then the subclass of t^0 consisting of all such representatives is also taken to be a member of c^*. If under these circumstances c^* has two or more nonvoid members, then c^* is taken to be a member of a set $\underset{\approx}{C}_1^*$. It follows that the members of $\underset{\approx}{C}_1^*$ satisfy Conditions Γ^1, and thus $\underset{\approx}{C}_1^*$ is a set of 1-connections. We say that $\underset{\approx}{C}_1^*$ is the *restriction of* $\underset{\approx}{C}_1$ *to* $\underset{\approx}{B}^*$. In this way, the 1-graph $(\underset{\approx}{B}^*, \underset{\approx}{C}_0^*, \underset{\approx}{C}_1^*)$ is defined as the *subgraph of* $\underset{\approx}{G}$ *induced by* $\underset{\approx}{B}^*$. For instance, if $\underset{\approx}{B}^*$ consists of the branches in $\underset{\approx}{G}_1$ in Figure 2 and if $\underset{\approx}{C}_1 = \{c_1^1, c_2^1\}$ as before, then the subgraph induced by $\underset{\approx}{B}^*$ is in effect $\underset{\approx}{G}_1$ and $\underset{\approx}{C}_1^*$ is void because c_1^* and c_2^* are singletons.

The idea of 0-connectedness applies to any 1-graph $(\underset{\approx}{B}, \underset{\approx}{C}_0, \underset{\approx}{C}_1)$ since 0-paths are defined in terms of $\underset{\approx}{B}$ and $\underset{\approx}{C}_0$. For instance, the subgraph $\underset{\approx}{G}_1$ of Figure 2 is 0-connected (in fact, is a 0-section) but the entire graph $\underset{\approx}{G}$ is not because there is no finite 0-path that connects a node of $\underset{\approx}{G}_1$ to a node of $\underset{\approx}{G}_2$. However, by generalizing the idea of a path, we can say that $\underset{\approx}{G}$ is connected in a wider sense.

We need some more definitions. Let x be an extremity of either rank 0 or rank 1, and similarly for y. Also, let P^0 be a 0-path. x and y are said to be *totally disjoint* if they do not embrace a common element. Also, x and P^0 are called *totally disjoint* if P^0 does not meet x. Furthermore, x and P^0 are said to be *terminally connected* if P^0 meets x either with a terminal node or with a 0-pathlike tip. In this case, x and P^0 are said to be *terminally connected but otherwise totally disjoint* if they do not meet at any other node or with any other 0-pathlike tip or node; thus, P_0 meets x with only one (not both) of its ends or tips.

Now consider the alternating sequence

$$\{ \cdots, c_m^1, P_m^0, c_{m+1}^1, P_{m+1}^0, \cdots \} \tag{4.1}$$

which may or may not have terminal elements x_a or x_b to the left or to the right respectively. This sequence is called *nontrivial* if it has at least three terms. We require the following.

Conditions Π^1:

(i) No more than the rational integers are needed to index the elements consecutively as indicated.

(ii) If a terminal element x_a or x_b exists, it is an extremity of rank 0 or 1.

(iii) Each c_m^1 that is not a terminal element is a 1-connection.

(iv) Each P_m^0 is a nontrivial 0-path (finite, one-ended, or endless) that is terminally connected to the two extremities immediately preceding and succeeding P_m^0 in the sequence but is otherwise totally disjoint from those extremities.

(v) Any extremity and 0-path that are not adjacent in (4.1) are totally disjoint.

(vi) The extremities in (4.1) - including the terminal elements - are pairwise totally disjoint.

(vii) The 0-paths in (4.1) are pairwise totally disjoint.

Under these conditions, (4.1) is called an *1-path* or a *path of rank* 1. The adjectives, finite, one-ended, and endless, are defined for 1-paths as they are for 0-paths. A 1-*loop* is a finite 1-path except that the following is required: One of the two terminal elements embraces the other.

Two 1-paths P_1^1 and P_2^1 are said to be *totally disjoint* if every extremity or 0-path in P_1^1 is totally disjoint from every extremity and every 0-path in P_2^1.

For some examples refer to Figure 2 again. We use the same notation as before. Also, the x's denote nodes as indicated. Then,

$$P_{ef} = \{ \cdots, e_3, x_3, e_2, x_2, e_1, x_1, f_1, x_f \}$$

is a one-ended 0-path, and

$$P_{aef}^1 = \{x_a, P_a, c_1^1, P_{ef}, c_2^1\}$$

is a finite 1-path. The 0-path P_d is terminally connected to c_2^1 but is otherwise totally disjoint from c_2^1. Also, $\{x_a, P_a, c_1^1\}$ and $\{x_d, P_d, c_2^1\}$ are totally disjoint 1-paths, but $\{x_d, P_d, c_2^1\}$ and $\{c_1^1, P_{ef}, x_f\}$ are not because c_2^1 embraces x_f.

Let x_a be a node or a 1-extremity, and similarly for x_b. x_a and x_b are said to be 1-*connected* if there exists a finite 1-path with x_a and x_b as its terminal elements. (This meaning for "1-connected" is different from the customary one. If this is displeasing, one might say instead "transfinitely 1-connected" to mark the difference.) Two branches are said to be 1-*connected* if x_a and x_b are 1-connected nodes with one branch incident to x_a and the other branch incident to x_b. An 1-graph is said to be 1-*connected* if every two extremities of rank 0 or 1 are 1-connected. It follows directly from our definitions that, if two nodes x_a and x_b are 0-connected, then they are also 1-connected. Indeed, let P^0 be a finite 0-path with x_a and x_b as its terminal nodes. Then, $\{x_a, P^0, x_b\}$ is a finite 1-path.

A 1-*section* of a 1-graph $\underset{\approx}{G}$ is a subgraph of $\underset{\approx}{G}$ induced by a maximal set of branches that are pairwise 1-connected. For instance, the graph $\underset{\approx}{G}$ of Figure 2 is 1-connected but not 0-connected and is a 1-section by itself. Moreover, the subgraphs $\underset{\approx}{G}_1$ and $\underset{\approx}{G}_2$ are 0-sections but, under our generalized concept of connectedness, are not components (i.e., disconnected parts) of $\underset{\approx}{G}$. Neither $\underset{\approx}{G}_1$ nor $\underset{\approx}{G}_2$ is a 1-section because it is not maximal with respect to 1-connectedness. A 1-path cannot proceed from one 0-section to another 0-section without passing through an 1-connection. It may enter or leave a 0-section either through a node or through a 0-pathlike tip, and its sojourn within that section may be either a finite, one-ended, or endless 0-path.

5. 2-GRAPHS, HEURISTICALLY

The idea of connectedness can be extended still further by applying strong recursion on the definitions used so far. However, before considering the general case, it may be helpful to sketch out the next level of generalization.

We start by partitioning the set of all one-ended 1-paths in a given 1-graph into equivalence classes by treating two one-ended 1-paths as being equivalent if their sequences (4.1) differ on no more than a finite number of terms. Each such equivalence class is called a 1-pathlike tip and denoted by t^1. A 2-connection is a finite or infinite set

$$c^2 = \{x_0, t_1^1, t_2^1, t_3^1, \cdots\}$$

having at least two members, no more than one of them being an extremity x_0 of rank 0 or 1. A 2-extremity is either a 2-connection or a singleton containing a 1-pathlike tip. It is also required that no two 2-extremities "embrace" an element in common; thus, the exceptional element in one 2-extremity is allowed to embrace neither the exceptional element in the other nor the exceptional element in that other exceptional element, if it exists.

A 2-graph is a quadruplet $\underset{\approx}{G} = (\underset{\approx}{B}, \underset{\approx}{C}_0, \underset{\approx}{C}_1, \underset{\approx}{C}_2)$ where $\underset{\approx}{C}_2$ is the set of 2-connections specified for the given 1-graph $(\underset{\approx}{B}, \underset{\approx}{C}_0, \underset{\approx}{C}_1)$.

A 2-path is a (finite, one-ended, or endless) alternating sequence of the form

$$\{\cdots, c_m^2, P_m^1, c_{m+1}^2, P_{m+1}^1, \cdots\} \tag{5.1}$$

where the indices m traverse no more than the rational integers, the P_m^1 are nontrivial 1-paths, the c_m^2 are 2-connections, a terminal element - if it exists - is an extremity of rank 0, 1, or 2, and all the terms are pairwise "totally disjoint" except for adjacent terms which are "terminally connected but otherwise totally disjoint".

Now, a 2-section of a 2-graph is a subgraph induced by a maximal set of branches that are pairwise "2-connected", that is, for every two branches of the subgraph there is a finite 2-path connecting those branches. Thus, within every 2-graph there are 0-sections, which are encompassed within 1-sections, which in turn, are encompassed within 2-sections. A 2-path cannot proceed from one 1-section to another 1-section without passing through a 2-connection. Moreover, it is possible for a 2-path to "touch down", so to speak, in an 0-section, by passing through just one branch in that 0-section, and to "jump off" at both ends of that branch into different 2-sections that do not contain the 0-section or even the 1-section in which the branch occurs. This can happen when in (5.1) the 1-path P_m^1 contains only one 0-path which in turn consists of only the said branch, whose nodes x_a and x_b are the exceptional elements in c_m^2 and c_{m+1}^2, and when in addition P_{m-1}^1 and P_{m+1}^1 meet c_m^2 and c_{m+1}^2 respectively with 1-pathlike tips.

6. p-GRAPHS

We now apply strong recursion to the definitions given in Sections 3 and 4. Let p be a natural number greater than 1. Assume that for each $q = 0, 1, \ldots, p-1$ the q-graphs $(\underset{\approx}{B}, \underset{\approx}{C}_0, \ldots, \underset{\approx}{C}_q)$ have been defined for a given branch set $\underset{\approx}{B}$ and specified sets $\underset{\approx}{C}_q$ of q-connections c^q, and also defined are the q-extremities x^q, q-paths P^q, q-connectedness, and q-sections, along with the terminology pertaining to these ideas. This has explicitly been done for $q = 0$ and $q = 1$.

Consider the $(p-1)$-graph $(\underset{\approx}{B}, \underset{\approx}{C}_0, \ldots, \underset{\approx}{C}_{p-1})$. Two one-ended $(p-1)$-paths are called *equivalent* if they differ at most by a finite number of $(p-1)$-connections and

$(p-2)$-paths. Thus, all the one-ended $(p-1)$-paths in that graph are partitioned into equivalence classes, called *$(p-1)$-pathlike tips*, and a *representative* of any such class is any one of its members. A *p-connection* or synonymously a *connection of rank p* is a finite or infinite set of the form

$$c^p = \{x_0, t_1^{p-1}, t_2^{p-1}, t_3^{p-1}, \cdots\} \tag{6.1}$$

where the t_m^{p-1} are $(p-1)$-pathlike tips, x_0 is a q-extremity ($q<p$) which need not be present, and the following conditions are satisfied.

Conditions Γ^p :

(i) Each p-connection has at least two members.

(ii) All elements of a p-connection are $(p-1)$-pathlike tips t_m^{p-1}, except possibly for one of them; that one x_0, if it exists, is a q-extremity with $q<p$ and does not appear as an element of any other n-connection where $q<n\leq p$. (x_0 is called the *exceptional element* of c^p.)

(iii) No two p-connections have an element in common.

A *p-extremity* or an *extremity of rank p* is either a p-connection or a singleton whose element is a $(p-1)$-pathlike tip that is not an element of any p-connection. A p-extremity x^p is said to *embrace* itself as well as all its elements, and, if it has an exceptional element x_0, it is also said to *embrace* all the elements of x_0, and all the elements of the exceptional element in x_0 if that exists, and so forth. Thus, we take it that a p-extremity does *not* embrace any other p-extremity. Now, let x_1^q and x_2^n be extremities of rank q and n respectively, where $n\leq q\leq p$. We shall say that x_1^q and x_2^n are *totally disjoint* if their sets of embraced elements have a void intersection. On the other hand, x_1^q is said to *embrace* x_2^n if x_1^q embraces all the elements embraced by x_2^n including x_2^n itself.

Proposition 6.1. Let q and p be natural numbers with $0 \leq q \leq p$. If x_0 and y_0 are respectively a q-extremity and a p-extremity and if x_0 and y_0 embrace a common extremity, then y_0 embraces x_0. If in addition $p = q$, then $x_0 = y_0$.

Proof. Let z denote an extremity that is embraced by both x_0 and y_0. If z's rank is q, then, since by definition x_0 does not embrace another extremity of the same rank q but does embrace itself, we must have that $z = x_0$, and so y_0 embraces $z = x_0$.

Now assume assume that the rank of the embraced common extremity z is less than q. Let x_{-1} be the unique exceptional extremity in x_0, and in general let x_{-k} be the unique exceptional extremity in x_{-k+1} for $k = 1, 2, \ldots$ .Thus, we have a sequence of extremities $x_0, x_{-1}, x_{-2}, \ldots$ of strictly decreasing ranks, one of which is the extremity z. Similarly, let $y_0, y_{-1}, y_{-2}, \ldots$ comprise the sequence of extremities such that y_{-k} is the unique exceptional extremity in y_{-k+1} for $k = 1, 2, 3, \ldots$; z is also one of those extremities.

Suppose that y_0 does not embrace x_0. It follows that there will be an extremity $w = x_{-i} = y_{-j}$ appearing in both sequences such that its predecessors x_{-i+1} and y_{-j+1} $(i, j \geq 1)$ are not the same. This violates Γ^p (ii). We can conclude that y_0 embraces x_0.

If $p = q$, we must have that $x_0 = y_0$ because again a p-extremity cannot embrace another p-extremity. QED

A p*-graph* or synonymously a *graph of rank* p is a $(p+2)$-tuplet

$$\underset{\approx}{G} = (\underset{\approx}{B}, \underset{\approx}{C}_0, \cdots, \underset{\approx}{C}_p) \tag{6.2}$$

where $\underset{\approx}{B}$ is a set of branches and each $\underset{\approx}{C}_q$ for $q = 0, \ldots, p$ is a set of q-connections. As the above construction indicates, each $\underset{\approx}{C}_q$ can be specified only after $\underset{\approx}{B}$ and all the $\underset{\approx}{C}_m$ for $m = 0, \ldots, q-1$ have been specified.

Let $\underset{\approx}{B}^*$ be a subset of $\underset{\approx}{B}$. For each $q = 0, \ldots, p-1$ let $\underset{\approx}{C}_q^*$ be the restriction of $\underset{\approx}{C}_q$ to $\underset{\approx}{B}^*$. Consider any $c \in \underset{\approx}{C}_p$. If an exceptional element x_0 exists in c and is in fact a node, then all the elementary tips in x_0 belonging to branches in $\underset{\approx}{B}^*$ are taken to comprise a node x_0^* in a set c^* so long as x_0^* is not void. More generally, by following the sequential construction of c from elementary tips to pathlike tips of higher and higher order, one can build another set c^* by restricting the construction to the branches in $\underset{\approx}{B}^*$. If this yields a c^* with two or more nonvoid members, then c^* is taken to be a member of a set $\underset{\approx}{C}_p^*$. $\underset{\approx}{C}_p^*$ is called the *restriction of* $\underset{\approx}{C}_p$ *to* $\underset{\approx}{B}^*$. Since the members of $\underset{\approx}{C}_p$ satisfy Conditions Γ^p, so too will the members of $\underset{\approx}{C}_p^*$ except that $\underset{\approx}{C}_p^*$ may be void; thus, $\underset{\approx}{C}_p^*$ will be a (possibly void) set of p-connections. The p-graph ($\underset{\approx}{B}, \underset{\approx}{C}_0^*, \ldots, \underset{\approx}{C}_p^*$) is called the *subgraph of* $\underset{\approx}{G}$ *induced by* $\underset{\approx}{B}^*$.

Let c^p be given by (6.1). If in (6.1) x_0 is a node and if a branch b is incident to x_0, we say that b *meets* c^p *at* x_0. More generally, consider the $(p-1)$-path

$$P^{p-1} = \{\cdots, c_m^{p-1}, P_m^{p-2}, c_{m+1}^{p-1}, P_{m+1}^{p-2}, \cdots\} \tag{6.3}$$

which is an alternating sequence of $(p-1)$-connections c_m^{p-1}, $(p-2)$-paths P_m^{p-2}, and possibly a terminal element to the left and/or to the right. (This was defined for $p-1 = 1$ in Section 4. It will be defined for every natural number $p-1$ once we finish stating our recursive definitions.) Each P_m^{p-2} can be expanded into another alternating sequence of extremities and paths of still lower rank, and so forth repeatedly. Any extremity embraced by the extremities of P^{p-1} or embraced by the extremities arising in this repeated expansion of paths is said to be *embraced* by P^{p-1}. Similarly, P^{p-1} is said to *embrace* itself as well as any path of rank $p-2$ or lower arising in this repeated expansion of paths. Also, P_{p-1} is called *nontrivial* if it has at least three elements.

As is indicated in (6.1), x_0 is the exceptional element in c^p and is therefore a q-extremity ($q < p$). If P^{p-1} embraces an extremity y that embraces or is embraced by x_0

(that is, if P^{p-1} and x_0 embrace an extremity in common), then P^{p-1} is said to *meet* x_0; we also say that P^{p-1} *meets* c^p *at* x_0 or alternatively P^{p-1} *meets* c^p *with* y. If in addition y is a terminal element of P^{p-1}, we say that P^{p-1} *terminates at* x_0 (or *at* c^p) *with* y. Similarly, a one-ended or endless $(p-1)$-path P^{p-1} that contains as a subsequence a representative of a $(p-1)$-pathlike tip t^{p-1} in a p-extremity x^p is said to *meet* x^p *with* t^{p-1}. For the last situation, it should be noted that, even though P^{p-1} contains a representative of t^{p-1} as a subsequence, t^{p-1} is not an element of P^{p-1}, and any extremity containing t^{p-1} is not embraced by P^{p-1}.

Furthermore, x^q and P^{p-1} are called *totally disjoint* if P^{p-1} does not meet x^q. Also, x^q and P^{p-1} are said to be *terminally connected* if P^{p-1} terminates at x^q or if, for $q=p$, P^{p-1} meets x^p with a $(p-1)$-pathlike tip in x^p. Moreover, x^q and P^{p-1} are called *terminally connected but otherwise totally disjoint* if they are terminally connected and P^{p-1} does not meet x^q with any other extremity embraced by P^{p-1} or with any other $(p-1)$-pathlike tip.

Two $(p-1)$-paths P_1^{p-1} and P_2^{p-1} are called *totally disjoint* if the set of all extremities embraced by P_1^{p-1} has a void intersection with the set of all extremities embraced by P_2^{p-1}.

We now complete our recursive definition of a path of higher rank by explicating the conditions that such a path must satisfy. Consider the alternating sequence of $(p-1)$-paths P_m^{p-1} interspersed with p-connections c_m^p:

$$\{ \cdots, c_m^p, P_m^{p-1}, c_{m+1}^p, P_{m+1}^{p-1}, \cdots \} \tag{6.4}$$

which may or may not terminate to the left with the terminal element x_a or to the right with the terminal element x_b. This sequence is called *nontrivial* if it has at least three elements. We require the following:

Conditions Π^p :

(i) No more than the rational integers are needed to index the elements consecutively as indicated.

(ii) If a terminal element x_a or x_b exists, it is a q-extremity, where $0 \leq q \leq p$.

(iii) Each c_m^p that is not a terminal element is a p-connection.

(iv) Each P_m^{p-1} is a nontrivial $(p-1)$-path (finite, one-ended, or endless) that is terminally connected to, but otherwise totally disjoint from, the p-connection or terminal extremity immediately preceding or succeeding P_m^{p-1} in the sequence.

(v) Any extremity and any $(p-1)$-path in (6.4) that are not adjacent are totally disjoint.

(vi) The p-connections and terminal extremities in (6.4) are pairwise totally disjoint.

(vii) The $(p-1)$-paths in (6.4) are pairwise totally disjoint.

Under Conditions Π^p, (6.4) is called a *p-path*. Finite, one-ended, and endless p-paths are defined as expected. Also, all the terminology used with (6.3) is carried over to (6.4). A *p-loop* is a finite p-path except for the following requirement: One of the terminal elements embraces the other one.

Proposition 6.2. Assume that the p-path (6.4) contains at least one p-connection, say, c_{m+1}^p that is not a terminal element. Then, at least one of the adjacent paths, P_m^{p-1} or P_{m+1}^{p-1}, meets c_{m+1}^p with an $(p-1)$-pathlike tip.

Proof. In view of Conditions Γ^p (ii) and Π^p (iv), the only way the conclusion can be negated is if both P_m^{p-1} and P_{m+1}^{p-1} terminate at the single exceptional element x_0 in c_{m+1}^p in such a way that P_m^{p-1} has a terminal element y_m and P_{m+1}^{p-1} has a terminal element y_{m+1}, each of which embraces or is embraced by x_0. Three cases arise:

1. y_m and y_{m+1} both embrace x_0. By Proposition 6.1, either y_m embraces y_{m+1} or y_{m+1} embraces y_m.

2. y_m embraces x_0 and x_0 embraces y_{m+1} (or conversely). By definition, y_m embraces all the elements embraced by x_0. Hence, y_m embraces y_{m+1}. (Conversely, y_{m+1} embraces y_m.)

3. x_0 embraces both y_m and y_{m+1}. We now invoke the fact that x_0 contains as an element of itself no more than one exceptional element w, and the rank of w is lower than the rank of x_0. Moreover, w contains no more than one exceptional element u, and u is of still lower rank. Continuing in this way, we find that x_0 and all its embraced exceptional elements form a sequence $\{x_0, w, u, \cdots\}$ whose elements have strictly decreasing ranks. So, y_m and y_{m+1} must appear in this sequence. This implies that y_m embraces y_{m+1}, or conversely.

In all three cases, we obtain a contradiction to the fact that P_m^{p-1} and P_{m+1}^{p-1} are totally disjoint according to Condition Π^p (vii). QED

If (6.4) terminates on the left (or right) at x_a, then the $(p-1)$-path P_m^{p-1} of lowest (of highest) index m will be called the *leftmost* (or *rightmost*) *subpath of rank* $p-1$ *embraced by* (6.4). Similarly, the $(p-2)$-path in that leftmost (rightmost) subpath of lowest (of highest) index, if it exists, will be called the *leftmost* (or *rightmost*) *subpath of rank* $p-2$ *embraced by* (6.4). This terminology is extended to subpaths of still lower rank.

Proposition 6.3. If a p-path P^p terminates on the left (right) at an extremity x_a of rank q where $q < p$, then P^p embraces leftmost (rightmost) subpaths of every rank n, where $n = q-1, \ldots, p-1$.

Proof. Since P^p terminates on, say, the left, it possesses a leftmost subpath P^{p-1}

of rank $p-1$. If P^{p-1} does not possess a leftmost subpath of rank $p-2$, then it can meet x_a only with a $(p-1)$-pathlike tip. Hence, x_a must be of rank p at least. Thus, if x_a's rank is less than p, P^{p-1} must possess a leftmost subpath of rank $p-2$. This argument can be continued inductively to obtain the proposition. QED

Note: Since P^p may terminate at x_a with an extremity of rank smaller than that of x_a, the conclusion may also hold for some values of n smaller than $q-1$.

Proposition 6.4. Let $\underset{\approx}{G}$ be the p-graph (6.2) and let $1 \leq q \leq p$. If $\underset{\approx}{C}_{q-1}$ is a finite set, then each $\underset{\approx}{C}_n$, where $q \leq n \leq p$, is void. (Thus, $\underset{\approx}{G}$ is effectively a $(q-1)$-graph.)

Proof. The finiteness of $\underset{\approx}{C}_{q-1}$ implies that $\underset{\approx}{G}$ has no one-ended $(q-1)$-path and thus no $(q-1)$-pathlike tip. By Condition Γ^p (i) and (ii), $\underset{\approx}{C}_q$ is void. Continuing this argument inductively, we obtain Proposition 6.4. QED.

To each p-path P^p there corresponds a set $\underset{\approx}{B}(P^p)$ of branches and a set $\underset{\approx}{N}(P^p)$ of nodes. Those nodes are all the 0-extremities embraced by P^p. The branches of all the 0-paths embraced by P^p comprise $\underset{\approx}{B}(P^p)$.

Proposition 6.5. If P_1^p and P_2^p are totally disjoint p-paths, then $\underset{\approx}{N}(P_1^p) \cap \underset{\approx}{N}(P_2^p)$ is void.

This proposition follows directly from our definition of totally disjoint paths.

In order for P_1^p and P_2^p to be totally disjoint, it is not sufficient that $\underset{\approx}{N}(P_1^p) \cap \underset{\approx}{N}(P_2^p)$ be void. For example, in Figure 2 the two 1-paths $\{x_a, P_a, c_1, P_{ef}, x_f\}$ and $\{x_d, P_d, c_2\}$ have nonintersecting node sets. However, they are not totally disjoint because c_2 embraces x_f.

Let x_a^n be an n-extremity and x_b^q be a q-extremity, where n and q are no larger than p. x_a^n and x_b^q are said to be *p-connected* if there exists a finite p-path with x_a^n and x_b^q as its terminal elements. Two branches are called *p-connected* if one branch is incident

to node x_a^0, the other branch is incident to node x_b^0, and x_a^0 and x_b^0 are p-connected. A graph is called p-connected if all its nodes are p-connected.

Proposition 6.6. If two extremities in a p-graph are q-connected, then they are n-connected for each $n = q+1, \ldots, p$.

Proof. Let

$$P^q = \{x_a^k, P_0^{q-1}, c_1^q, P_1^{q-1}, \cdots, P_i^{q-1}, x_b^m\}$$

be a finite q-path connecting the two extremities x_a^k and x_b^m. Thus, both k and m are no larger than q. Then, $P^{q+1} = \{x_a^k, P^q, x_b^m\}$ is a finite $(q+1)$-path, $P^{q+2} = \{x_a^k, P^{q+1}, x_b^m\}$ is a finite $(q+2)$-path, and so forth. QED

For any $0 \leq q \leq p$, a q-*section* of a p-graph $\underset{\approx}{G}$ is a subgraph of $\underset{\approx}{G}$ induced by a maximal set of branches that are pairwise q-connected.

Proposition 6.7. An n-path can pass from one q-section to another q-section only if $n > q$.

Proof. Suppose this is not so, that is, there is an n-path P^n with $n \leq q$ which terminates at both ends at nodes having incident branches b_a and b_b lying in different q-sections. By Condition Π^p (i) for $p=q$, since P^n terminates at both ends, a finite number of integers suffice to index the terms of P^n consecutively. Hence, P^n is finite. Thus, b_a and b_b are n-connected and, by Proposition 6.6, q-connected. By the maximality condition of q-sections, b_a and b_b lie in the same q-section, a contradiction. QED

The last proposition implies that, if $n \leq q$, any n-path or n-loop is confined to a single q-section. On the other hand, the condition $n > q$ is not in general sufficient for the existence of an n-path between two given q-sections because connections of rank larger than q may not be suitably located in $\underset{\approx}{G}$.

7. ω-GRAPHS

The next step in generalization occurs when p is replaced by the least transfinite ordinal ω; it requires some modifications in our constructions. We start with a graph that has p-connections for every natural number p. Consider one-ended paths of the form

$$\{x_0^{q_0},\ P_0^{p_0-1},\ c_1^{p_1},\ P_1^{p_1-1},\ c_2^{p_2},\ P_2^{p_2-1},\ \cdots\} \tag{7.1}$$

where the natural numbers suffice to index the elements consecutively as shown, $x_0^{p_0}$ is a p_0-extremity, $c_m^{p_m}$ is a p_m-connection, $P_m^{p_m-1}$ is a nontrivial (p_m-1)-path, $q_0 \le p_0$, the p_m are strictly increasing (i.e., $p_0 < p_1 < p_2 < \ldots$), and the members of (7.1) are pairwise totally disjoint except for adjacent members, which are terminally connected but otherwise totally disjoint. These conditions imply that $P_m^{p_m-1}$ meets $c_{m+1}^{p_{m+1}}$ at an extremity of rank p_m or less and therefore $P_{m+1}^{p_{m+1}-1}$ meets $c_{m+1}^{p_{m+1}}$ with a $(p_{m+1}-1)$-pathlike tip (see Proposition 6.2). Under these circumstances, we shall refer to (7.1) as a *one-ended $\vec{\omega}$-path.*

An equivalence class of all one-ended $\vec{\omega}$-paths that pairwise differ by no more than a finite number of terms is called an *$\vec{\omega}$-pathlike tip* $t^{\vec{\omega}}$. Then, an *ω-connection* or a *connection of rank* ω is a finite or infinite set of the form

$$c^{\omega} = \{x_0,\ t_1^{\vec{\omega}},\ t_2^{\vec{\omega}},\ t_3^{\vec{\omega}},\ \cdots\} \tag{7.2}$$

where x_0 is an extremity whose rank q is a natural number. We require that every ω-connection satisfy the *Conditions* Γ^{ω}, which read exactly as do the Conditions Γ^{p} except that p is replaced by ω and $p-1$ by $\vec{\omega}$. As before, an *ω-extremity* is either an ω-connection or a singleton having an $\vec{\omega}$-pathlike tip that does not appear in any ω-connection.

Let $\underset{\approx}{C}_\omega$ be a set of ω-connections. An *ω-graph* is the infinite set

$$\underset{\approx}{G} = (\underset{\approx}{B}, \underset{\approx}{C}_0, \underset{\approx}{C}_1, \cdots, \underset{\approx}{C}_\omega)$$

A *subgraph* G^* *induced by a subset* $\underset{\approx}{B}^*$ of $\underset{\approx}{B}$ is obtained by restricting the construction of each of $\underset{\approx}{C}_0, \ldots, \underset{\approx}{C}_\omega$ to $\underset{\approx}{B}^*$.

Another way of representing a one-ended $\vec{\omega}$-path is obtained by replacing every m by $-m$ in (7.1) and in the conditions imposed upon (7.1). Furthermore upon appending the result to the left of (7.1) (and striking out the extra $x_0^{q_0}$), we obtain an *endless $\vec{\omega}$-path.* All the terminology for p-paths extend to one-ended and endless $\vec{\omega}$-paths.

We can define an *ω-path* as an alternating sequence

$$\{\cdots, c_m^\omega, P_m^{\vec{\omega}}, c_{m+1}^\omega, P_{m+1}^{\vec{\omega}}, \cdots\} \tag{7.3}$$

that satisfies *Conditions* Π^ω, which read exactly as do the Conditions Π^p with p replaced by ω and $p-1$ replaced by $\vec{\omega}$ except for one more change. In Condition Π^ω (iv), the $\vec{\omega}$-paths must be one-ended or endless, not finite. An *ω-loop* is a finite ω-path, except that one of its two terminal extremities embraces the other one.

With these alterations, Propositions 6.1 through 6.5 and their proofs hold as before except for some obvious modifications. For example, in Proposition 6.4 we should require that $q < p = \omega$ because with $q = \omega$ there is no ordinal $q-1$. Similarly, in Proposition 6.3 the possible values of n are now $q-1, q, q+1, \ldots$, but not $p-1$.

As for connectedness, let x_a^n and x_b^q be extremities of ranks n and q respectively, where $0 \le n \le \omega$ and $0 \le q \le \omega$. These extremities are said to be *ω-connected* if there is a finite p-path, where $\max(n, q) \le p \le \omega$, having x_a^n and x_b^q as its two terminal elements. This definition may appear to be less demanding as compared to our prior definition of p-connectedness, but, by virtue of Proposition 6.6, it is quite analogous. Finally, an *ω-section* is defined as expected.

8. GRAPHS OF STILL HIGHER RANKS

With ω-graphs in hand, we can proceed as in Section 6 to obtain $(\omega+p)$-graphs for any natural number $p > 0$ by using $(\omega+p-1)$-pathlike tips to define $(\omega+p)$-connections. Then, the method of Section 7 provides $(\omega+\vec{\omega})$-pathlike tips from which $(\omega \cdot 2)$-connections and $(\omega \cdot 2)$-graphs can be obtained. This process can be continued to generate k-graphs where k is any countable ordinal that can be explicitly constructed as above from lesser ordinals. The procedure of Section 6 (Section 7) is used when k is a successor ordinal (respectively, limit ordinal). Thus, we have transfinite graphs of rank k for quite a range of ordinals k, and these graphs have their q-sections for every q from 0 to k.

9. (k, q)-PATHS AND TERMINAL BEHAVIOR AT EXTREMITIES

Again let p be a natural number larger than 0. Given the p-path (6.4), we can think of each P_m^{p-1} being explicitly written out as a $(p-1)$-path. This will yield an expanded display of (6.4) involving the p-connections c_m^p and the possible terminal elements of (6.4), as well as the $(p-1)$-connections, possibly other terminal elements, and $(p-2)$-paths arising from the expansions of all the P_m^{p-1} in (6.4). (For an example wherein $p = 4$, see the second line of Figure 3.) If a P_m^{p-1} terminates at an i-extremity d^i $(i \leq p-1)$ that is embraced by a p-connection, the notation d^i is deleted from the expanded version of (6.4). No such deletion is needed if P_m^{p-1} meets the p-connection with a $(p-1)$-pathlike tip. By virtue of Proposition 6.2, no more than one such deletion need be made at each p-connection. On the other hand, if P_m^{p-1} is a leftmost (rightmost) subpath, its terminal element on the left (right) is compared in rank with the terminal element on the left (right) in (6.4). If those ranks are the same, the two terminal elements will be identical, according to our definition of a terminating path and Proposition 6.1, and just one extremity notation is retained. If not, we discard the extremity notation with the lower rank. In this way, no two extremities appear as adjacent terms in the expansion of (6.4). (In the example of Figure 3, we have taken it that each of P_0^3, P_0^2, P_0^1 terminate on the left with x_0^2.)

The rational integers may no longer suffice to index consecutively all the terms of this expanded form of (6.4) Moreover, its terms, when ordered in accordance with this sequence of sequences, are totally ordered but may not be well-ordered. Well-ordering may be absent, for example, when one of the P_m^{p-1} is an endless path. We will refer to this expanded form of (6.4) as a $(p, p-1)$*-path* and will denote it by $P^{p,p-1}$; we may also refer to it as a ***transfinite path***, even though in special cases the rational integers may suffice for the stated purpose.

This process can be repeated, as is indicated in Figure 3. An expansion of all the $(p-2)$-paths in the transfinite $(p, p-1)$-paths yields a transfinite $(p, p-2)$*-path* $P^{p,p-2}$. Continuing in this way, we obtain for $q<p$ the transfinite (p, q)*-path* $P^{p,q}$ and finally a transfinite $(p, 0)$*-path* $P^{p,0}$, which is totally ordered but not necessarily well-ordered. The elements of $P^{p,0}$ will be branches interspersed with extremities of various ranks. Two adjacent branches will be separated by the node to which they are incident or by a connection embracing that node. The higher-order extremities will separate various finite or transfinite sequences, and the terminal elements of $P^{p,0}$, if they exist (they will exist if the original p-path had them), will be q-extremities where $0 \leq q \leq p$.

The (p, q)*-loops* are defined from the p-loops in just the same way and are called ***transfinite loops*** if they have more than a finite number of branches.

Assume P^p is a p-path that terminates on the left at a q-extremity x_0^q where $q<p$. Then, by Proposition 6.3, P^p embraces leftmost subpaths P_0^n of every rank n, where n varies from $p-1$ down to $q-1$ and perhaps lower. Let m be the smallest n for which P^p embraces a leftmost extremity of rank $n-1$. (This is illustrated in Figure 3

for $p=4$, $q=2$, and $m=2$.) In short, there will be a critical value of n, namely, m such that P^p embraces a leftmost subpath P_0^{n-1} of rank $n-1$ for every $n = m, \ldots, p$, but not for $n<m$. This means that $P_0^m, P_0^{m+1}, \ldots, P_0^p$ all terminate at x_0^q with an extremity, that P_0^{m-1} meets x_0^q with an $(m-1)$-pathlike tip, and that for $n = 0, \ldots, m-2$ there is no embraced n-path that terminates at or meets x_0^q.

A similar pattern will exist at all the p-connections of P^p at which $(p-1)$-subpaths terminate.

Now consider an ω-path such as (7.3). Each $P_m^{\vec{\omega}}$ in it has the form of (7.1) except that it may be endless. Each $P_m^{p_m-1}$ in (7.1) may be expanded into nested sequences of paths and extremities of lower ranks. Ultimately, we obtain a totally ordered (but not in general well-ordered) set of 0-paths interspersed with extremities whose ranks vary from 0 to ω; this will be called an $(\omega, 0)$-path. An $(\omega, 0)$-loop is an $(\omega, 0)$-path having however a least term and a largest term (with respect to the total ordering) such that one term embraces the other.

These ideas extend directly to k-paths, where k is any constructible countable ordinal as before. We obtain thereby $(k, 0)$-paths and $(k, 0)$-loops.

10. k-NETWORKS

Now that we have constructed transfinite graphs, we shall assign an analytical structure to every branch to obtain thereby transfinite electrical networks. First of all, note that, since $\underset{\approx}{B}$ is a countable set, the natural numbers suffice to index all the branches of a given k-graph $\underset{\approx}{G}$ in some fashion. (It is when we try to index the branches of a $(k, 0)$-path in the order of a tracing along that path that the natural numbers or even a well-ordered indexing system may not suffice.) Henceforth, we assume that every branch has a natural number j as an index, where $j = 0, 1, 2, \ldots$. Furthermore, we assume that every branch has an orientation, with respect to which the polarities of voltages and currents will be measured.

The jth branch's analytical structure is given by Thevenin's circuit, shown in Figure 4, where a pure voltage source of value e_j volts and a resistance of value r_j ohms are connected in series, the two ends of the series circuit being the tips from which the branch at hand was defined as an entity in the graph $\underset{\approx}{G}$. e_j is a real number, possibly zero, and r_j is a real positive number. g_j will always denote the branch conductance $1/r_j$. Ohm's law and Kirchhoff's laws dictate that

$$v_j = r_j i_j - e_j \tag{10.1}$$

where v_j is the value of the branch voltage and i_j is the value of the branch current, both being real quantities. (To simplify notation, we will use r_j to designate the resistor as well as its resistance value, and similarly for e_j, v_j, and i_j.) In accordance with this analytical structure, we will only examine purely resistive networks having no dependent sources. Moreover, every branch has a positive resistance and therefore any current source within a branch can be converted into a voltage source by a Norton-to-Thevenin transformation.

In this paper an *electrical network of rank k* or simply an *k-network* is taken to mean a k-graph every branch of which has the analytical representation shown in Figure 4 with its parameters satisfying (10.1).

Henceforth, the symbol Σ will denote a summation $\Sigma_{j=0}^{\infty}$ over all the branch indices j, that is over all the natural numbers, unless something else is explicitly indicated. We shall impose

Condition E.

All the branch voltage sources e_j satisfy the condition of *finite total isolated power*, namely, $\Sigma e_j^2 g_j < \infty$.

As we shall see below, the total power absorbed in all the resistors, which will equal the total power delivered by all the sources, is no larger than $\Sigma e_j^2 g_j$. Thus, Condition E implies that the k-network will be in a finite-power regime.

11. THE UNIQUE VOLTAGE-CURRENT REGIME

Boldface notation will denote one-way infinite vectors whose elements are indexed by the natural numbers; thus, $\mathbf{i} = (i_0, i_1, i_2, \ldots)$ is the vector of all branch currents, $\mathbf{v} = (v_0, v_1, v_2, \ldots)$ is the vector of all branch voltages, and $\mathbf{e} = (e_0, e_1, e_2, \ldots)$ is the vector of all branch voltage-source values. R will denote the operator that assigns to every branch-current vector $\mathbf{i}$ the vector $(r_0 i_0, r_1 i_1, r_2 i_2, \ldots)$ consisting of the voltages across the branch resistances (i.e., the voltage drops $r_j i_j$ measured in the direction of the branches' orientations).

$\underset{\approx}{I}$ denotes the space of all branch-current vectors $\mathbf{i}$ for which $\Sigma i_j^2 r_j < \infty$, that is, for which the total power dissipated in all the resistors is finite. The linear operations are defined componentwise on the vectors $\mathbf{i}$. Moreover, we assign the inner product $(\mathbf{i}, \mathbf{s})$ to two elements $\mathbf{i}, \mathbf{s} \in \underset{\approx}{I}$, where $(\mathbf{i}, \mathbf{s}) = \Sigma r_j i_j s_j$; $\|\cdot\|$ denotes the corresponding norm. A standard argument [6; p. 21] shows that $\underset{\approx}{I}$ is complete under this norm and is therefore a Hilbert space and that convergence in $\underset{\approx}{I}$ implies componentwise convergence.

The next step is to assign currents to various $(q, 0)$-loops where $q \le k$. Any $(q, 0)$-loop is confined to a q-section. A (0, 0)-loop is the same as a 0-loop and is in fact just a finite loop, which perforce is confined to an 0-section. We now assign an orientation to every $(q, 0)$-loop; it is one of the two possible ways of tracing around the loop. A $(q, 0)$-*loop current* or simply a q-*loop current* or just *loop current* is an assignment of branch currents such that the currents i_j in all branches are zero except for the branches in some given $(q, 0)$-loop L; in those latter branches the currents are

$i_j = \pm i$,

where i is a real constant and the plus (minus) sign is used if the jth branch's orientation agrees (respectively, disagrees) with the orientation of the loop L.

Kirchhoff's current law asserts that, given a node x_0,

$$\sum_{j \in N} \pm i_j = 0, \tag{11.1}$$

where N is the branch-index set for all the branches incident to x_0, i_j is the branch current in branch $j \in N$, and the plus (minus) sign is used if the branch j is oriented toward (away from) the node. A loop current will satisfy Kirchhoff's current law at every node except possibly when the node is embraced by a q-connection c^q where $q > 0$. In the latter case, Kirchhoff's current law will still be satisfied if the loop passes from one branch b_1 incident to the node to another branch incident to the node but will not do so if the loop passes from b_1 to a pathlike tip embraced by c^q.

A $(q, 0)$-loop L will be called *perceptible* if $\Sigma_{j \in \Lambda} r_j < \infty$ where Λ is the index set for all the branches in L. It follows immediately that a loop current will be a member of $\underset{\approx}{I}$ if and only if its corresponding loop is perceptible.

For a given k-network, $\underset{\approx}{K}^0$ will denote the span of all $(q, 0)$-loop currents $(0 \leq q \leq k)$ that are members of $\underset{\approx}{I}$. Thus, $\underset{\approx}{K}^0 \subset \underset{\approx}{I}$. $\underset{\approx}{K}$ will denote the closure of $\underset{\approx}{K}^0$ in $\underset{\approx}{I}$, and so $\underset{\approx}{K} \subset \underset{\approx}{I}$ as well. In fact, $\underset{\approx}{K}$ is a Hilbert space by itself when it is equipped with the inner product of $\underset{\approx}{I}$. Moreover, convergence in $\underset{\approx}{K}$ implies componentwise (i.e., branchwise) convergence.

In general, $\underset{\approx}{K}$ will contain transfinite loop currents that are not members of $\underset{\approx}{K}^0$; in fact, augmented connections may effectively be introduced when taking the closure of $\underset{\approx}{K}^0$. For example, refer to the 1-graph of Figure 5. Assume that the branch resistance values decay so rapidly as one proceeds to the right that $\Sigma r_j < \infty$. Figure 5(a) shows a

1-loop current consisting of a flow along the branches a_m, through a 1-connection $c^1 = \{x, t_a^0\}$, and back through a single return branch; x is a node, to which the return branch is incident, and t_a^0 is the 0-pathlike tip having as a representative the 0-path of a_m branches. Since this 1-loop is perceptible, its loop current can be taken to be a member of $\underset{\approx}{K}^0$. Also, all the 0-loop currents of part (b) can be taken to be members of $\underset{\approx}{K}^0$ too. Furthermore, we may assume that the pathlike tip t_b^0 corresponding to the b_m branches is not a member of any connection. However, if all the indicated loop currents of parts (a) and (b) have 1-ampere values, their superposition will be the 1-ampere 1-loop current shown in Figure 5(c) and will be a member of $\underset{\approx}{K}$ because that 1-loop is perceptible too. Thus, an enlarged 1-connection $d^1 = \{x, t_a^0, t_b^0\}$ has effectively been introduced even though d^1 was not declared to be a connection for this 1-graph.

We now turn to the voltage sources. Any branch voltage-source vector **e** defines a mapping (i.e., a functional) from $\underset{\approx}{K}$ into the real line R^1 according to

$$<\mathbf{e}, \mathbf{i}> = \Sigma e_j i_j , \quad \mathbf{i} \in \underset{\approx}{K} \tag{11.2}$$

whenever $\Sigma e_j i_j$ converges.

Lemma 11.1. If **e** satisfies Condition E, then **e** defines a continuous linear mapping of $\underset{\approx}{K}$ into R^1 according to (11.2).

Proof. We first show that $\Sigma e_j i_j$ converges absolutely. By Schwarz's inequality,

$$\Sigma \mid e_j i_j \mid = \Sigma \mid g_j^{1/2} e_j r_j^{1/2} i_j \mid \leq [\Sigma g_j e_j^2 \, \Sigma r_j i_j^2]^{1/2} \tag{11.3}$$

The right-hand side is finite by virtue of Condition E and the fact that $i \in \underset{\approx}{K}$.

Since absolutely convergent series can be rearranged, the functional defined by (11.2) is linear. Moreover, it is continuous because, according to (11.3) and the norm of $\underset{\approx}{K}$,

$$\mid <\mathbf{e}, \mathbf{i}> \mid \leq \Sigma \mid e_j i_j \mid \leq [\Sigma g_j e_j^2]^{1/2} \, || \mathbf{i} || .$$

QED

Here at last is the principal result of our paper.

Theorem 11.2. Given a k-network with a branch voltage-source vector $\mathbf{e}$ that satisfies Condition E, there exists a unique $\mathbf{i} \in \underset{\approx}{K}$ such that

$$\langle \mathbf{e} - R\,\mathbf{i}, \mathbf{s} \rangle = 0 \tag{11.4}$$

for every $\mathbf{s} \in \underset{\approx}{K}$. This equation implies the uniqueness of $\mathbf{i}$ in $\underset{\approx}{K}$ even when $\mathbf{s}$ is restricted to $\underset{\approx}{K}^0$.

Proof. Since $\mathbf{e}$ defines a continuous linear functional on $\underset{\approx}{K}$ according to Lemma 11.1, we can invoke the Riesz representation theorm to conclude that there is a unique $\mathbf{i} \in \underset{\approx}{K}$ such that $\langle \mathbf{e}, \mathbf{s} \rangle = (\mathbf{s}, \mathbf{i})$. On the other hand, $(\mathbf{s}, \mathbf{i}) = \Sigma r_j\, s_j\, i_j = \langle R\,\mathbf{i}, \mathbf{s} \rangle$. Thus, (11.4) holds for that unique $\mathbf{i}$. Moreover, $\mathbf{i}$ is uniquely determined as a member of $\underset{\approx}{K}$ by the values of $(\mathbf{s}, \mathbf{i})$ for all $\mathbf{s} \in \underset{\approx}{K}$, and in fact for just all the $\mathbf{s} \in \underset{\approx}{K}^0$ since $\underset{\approx}{K}^0$ is dense in $\underset{\approx}{K}$. QED

Equation (11.4) is known in the electrical engineering literature as Tellegen's equation. It, rather than Kirchhoff's laws, is the governing equation that determines the voltage-current regime for our k-network. Actually, the uniqueness of that regime arises from the conjunction of the finite-total-isolated-power condition (Condition E), the restriction of the allowable branch-current vectors to $\underset{\approx}{K}$, and Tellegen's equation (11.4). Nonetheless, as we shall see in the next section, Kirchhoff's laws do hold in certain circumstances, even though they have been relegated to a secondary role in this theory. Also, Ohm's law has been imposed upon every r_j by virtue of the term $R\,\mathbf{i}$ in (11.4).

Corollary 11.3. Under the hypothesis of Theorem 11.2, the total power $\Sigma i_j^2 r_j$ dissipated in all the resistors equals the total power $\Sigma e_j\, i_j$ supplied by all the voltage sources and is no larger than the finite total isolated power $\Sigma e_j^2 g_j$ available from all the voltage sources.

Proof. Set $\mathbf{s} = \mathbf{i}$ in (11.4) to get

$$\Sigma i_j^2 r_j = \langle R\,\mathbf{i}, \mathbf{i} \rangle = \langle \mathbf{e}, \mathbf{i} \rangle = \Sigma e_j i_j .$$

Upon combining this with (11.3), we obtain $\Sigma i_j^2 r_j \leq \Sigma e_j^2 g_j$. QED

12. KIRCHHOFF'S LAWS

A node in our given k-graph $\underset{\approx}{G}$ will be called *ordinary* if it is not embraced by any q-connection where $q > 0$. A node x_0 is called *restraining* if the sum of the conductances of all the branches incident to x_0 is finite, that is, if $\Sigma_{j \in N}\, g_j < \infty$ where N is the index set for all the branches incident to x_0. A finite node is restraining, but an infinite node may or may not be restraining.

Proposition 12.1. If x_0 is an ordinary restraining node, then, under the voltage-current regime dictated by Theorem 11.2, Kirchhoff's current law is satisfied at x_0 absolutely, that is, (11.1) holds where the series on the left-hand side converges absolutely.

Proof. Let Σ_N denote $\Sigma_{j \in N}$, and let $\mathbf{i} \in \underset{\approx}{K}$. Then,

$$\Sigma_N \mid i_j \mid = \Sigma_N r_j^{1/2} \mid i_j \mid g_j^{1/2} \leq [\Sigma_N r_j i_j^2 \,\Sigma_N g_j]^{1/2} \leq \; \| \mathbf{i} \| \, [\Sigma_N g_j]^{1/2} .$$

Since x_0 is restraining, the right-hand side is finite, which establishes the asserted absolute convergence.

Next, as was noted above, every loop current satisfies (11.1) at x_0. Consequently, so too does every member of $\underset{\approx}{K}^0$ since each such member is a (finite) linear combination of loop currents. Since $\underset{\approx}{K}^0$ is dense in $\underset{\approx}{K}$, we can choose a sequence $\{\mathbf{i}_m\}_{m=0}^{\infty}$ in $\underset{\approx}{K}^0$ which converges in $\underset{\approx}{K}$ to the unique $\mathbf{i} \in \underset{\approx}{K}$ specified in Theorem 11.2. Thus, with i_{mj} (or i_j) denoting the jth component of $\mathbf{i}_m$ (of $\mathbf{i}$), we may write $\Sigma_{j \in N} \pm i_{mj} = 0$ and

$$\begin{aligned} \mid \Sigma_{j \in N} \pm i_j \mid &= \mid \Sigma_N \pm i_j - \Sigma_N \pm i_{mj} \mid \leq \Sigma_N \mid i_j - i_{mj} \mid = \Sigma_N r_j^{1/2} \mid i_j - i_{mj} \mid g_j^{1/2} \\ &\leq [\Sigma_N r_j (i_j - i_{mj})^2 \,\Sigma_N g_j]^{1/2} \leq \| \mathbf{i} - \mathbf{i}_m \| \; (\Sigma_N g_j)^{1/2} \to 0 \end{aligned}$$

as $m \to \infty$. Thus, $\mathbf{i}$ satisfies Kirchhoff's current law at x_0. QED

Consider now Kirchhoff's voltage law. This asserts that

$$\sum_{j \in \Lambda} \pm v_j = 0 \tag{12.1}$$

where Λ is the branch-index set for all the branches in a given oriented $(q, 0)$-loop L, v_j is the branch voltage in branch $j \in \Lambda$, and the plus (minus) sign is used if the orientation of branch j agrees (disagrees) with the orientation of loop L. The loop L is called *perceptible* if $\Sigma_{j \in \Lambda} r_j < \infty$.

Proposition 12.2. If L is a perceptible $(q, 0)$-loop, then, under the voltage-current regime dictated by Theorem 11.2, Kirchhoff's voltage law (12.1) holds around L, and the series on the left-hand side of (12.1) converges absolutely.

Proof. Let $\mathbf{s}$ be the loop current corresponding to a unit current flow around L. Since $v_j = r_j i_j - e_j$ for each branch, the substitution of $\mathbf{s}$ into (11.4) yields (12.1).

Let us now show that the left-hand side of (12.1) converges absolutely. As before, Σ_Λ will denote $\Sigma_{j \in \Lambda}$. We may write

$$\Sigma_\Lambda \mid e_j \mid = \Sigma_\Lambda r_j^{1/2} \mid e_j \mid g_j^{1/2} \leq [\Sigma_\Lambda r_j \ \Sigma_\Lambda e_j^2 g_j]^{1/2} .$$

By Condition E and the perceptibility of L, the right-hand side is finite. Similarly,

$$\Sigma_\Lambda \mid r_j i_j \mid = \Sigma_\Lambda r_j^{1/2} \mid i_j \mid r_j^{1/2} \leq [\Sigma_\Lambda r_j i_j^2 \ \Sigma_\Lambda r_j]^{1/2} .$$

Since $\mathbf{i} \in \underset{\approx}{K}$ and L is perceptible, the last right-hand side is finite too. Since $v_j = r_j i_j - e_j$, we are done. QED

13. SOME FINAL REMARKS

1. *Pure sources.* We have assumed that every branch has a positive (not zero) branch resistance. However, we can allow some branches to have zero resistance, that is, to be pure voltage sources; in fact, pure current sources can also be allowed. Our theory can be so extended by using the technique of transferring pure sources into branches with positive resistances and adapting the arguments employed in Section VII,

VIII, and XII of [13].

2. *Reciprocity theorem.* The reciprocity theorem continues to hold for k-networks. That theorem states that the current in branch j due to a unit voltage source in branch m is equal to the current in branch m due to a unit voltage source in branch j. Flanders' proof of this fact (see Corollary 3 in [5]) extends directly to our k-networks.

3. *Other fundamental currents.* The fundamental currents upon which the space $\underset{\approx}{K}^0$ and thereby our existence and uniqueness theorem (Theorem 11.2) are based are the finite and transfinite loop currents. Still greater generality can be achieved by allowing other kinds of fundamental currents as well. An example of the latter are the "extremity currents" introduced in Section XI of [13]. This will expand $\underset{\approx}{K}^0$ and may lead to a different, but nonetheless unique, voltage-current regime for the k-network.

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FOOTNOTE

*This work was supported by the National Science Foundation under Grant DMS-8521824.

LEGENDS FOR FIGURES

Figure 1.

An infinite ladder network whose driving-point resistance R_D as seen from the input terminals on the left depends on the load resistance R_L connected on the right to the extremities of the network represented by two hypothetical nodes at infinity shown by the small circles. The numbers are resistance values in ohms, which continue in the indicated pattern infinitely to the right.

Figure 2.

$\underset{\approx}{G}_1$ is an infinite cascade of lattices connected at infinity to an infinite ladder $\underset{\approx}{G}_2$. Each labeled line segment denotes a branch. c_1 and c_2 are 1-connections; c_2 embraces the infinite node x_f of the ladder network.

Figure 3.

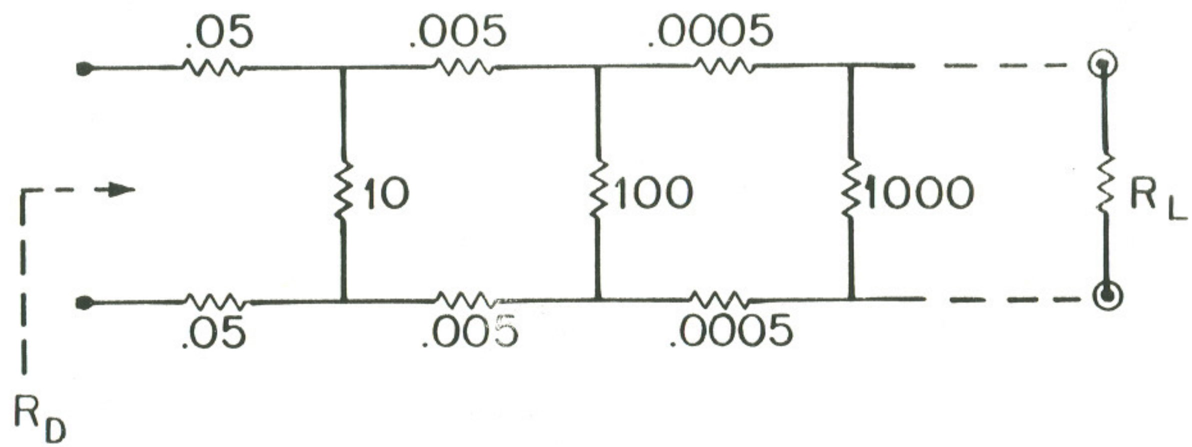
Illustration of the possible terminal behaviors of a 4-path and its corresponding $(4, q)$-paths $(q = 0, \ldots, 3)$. The 4-path is assumed to terminate on the left at a 2-extremity x_0^2. The c's, d's, e's, f's, and x's denote extremities, and the P's, Q's, R's, and S's denote paths of the indicated ranks. The b's are branches. The terminal element d_0^i $(i \leq 3)$ on the left-hand side of P_1^3 is deleted in the expansion of P_1^3 in the second line because it is embraced by c_1^4.

Figure 4.

Thevenin's circuit for the jth branch. The branch's orientation is taken to be in the direction of i_j and the polarities of v_j as a voltage drop and of e_j as a voltage rise.

Figure 5.

Assume that the sum of all resistance values is finite, that c^1 is a 1-connection, but that d^1 is not specified as a 1-connection. The 1-ampere loop currents shown in parts (a) and (b) can all be taken to be members of $\underset{\approx}{K}^0$. Then, $\underset{\approx}{K}$ will contain the loop current shown in part (c), which means that a new 1-connection d^1, an enlargement of c^1, has been introduced in effect just by taking the closure of $\underset{\approx}{K}^0$.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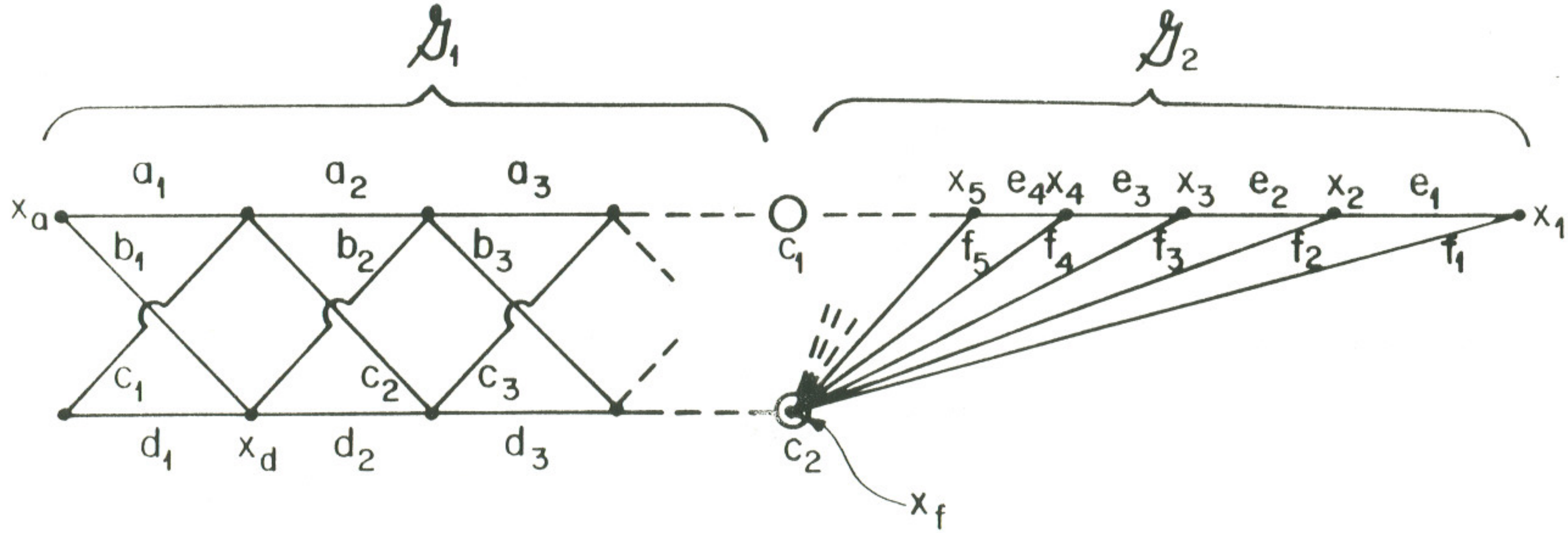
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C2
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x5
e4
x4
e3
x3
e2
x2
e1
x1
f5
f4
f3
f2
f1

An ω^4-path: $x_0^2, P_0^3, c_1^4, P_1^3, c_2^4, P_2^3, \cdots$

$\longleftarrow P_0^3 \longrightarrow \quad \longleftarrow P_1^3 \longrightarrow$

Its $\omega^{4,3}$-path: $x_0^2, P_0^2, c_1^3, P_1^2, c_2^3, \cdots, c_1^4,$ ~~d_0^1~~ $, Q_0^2, d_1^3, Q_1^2, \cdots$

$\longleftarrow P_0^2 \longrightarrow \quad \longleftarrow P_1^2 \longrightarrow$

Its $\omega^{4,2}$-path: $x_0^2, P_0^1, c_1^2, P_1^1, c_2^2, \cdots, c_1^3, \cdots, R_{-1}^1, e_0^2, R_0^1, e_1^2, \cdots$

$\longleftarrow P_0^1 \longrightarrow \quad \longleftarrow P_1^1 \longrightarrow$

Its $\omega^{4,1}$-path: $x_0^2, \cdots, P_{-1}^0, c_0^1, P_0^0, c_1^1, \cdots, c_1^2, \cdots, S_{-1}^0, f_0^1, S_0^0, f_1^1, \cdots$

$\cdots \leftarrow P_{-1}^0 \rightarrow \quad \longleftarrow P_0^0 \longrightarrow \quad \leftarrow P_1^0 \rightarrow$

Its $\omega^{4,0}$-path: $x_0^2, \cdots\cdots\cdots, c_0^1, \cdots, c_{-1}^0, b_{-1}, c_0^0, b_0, \cdots, c_1^1, \cdots\cdots$

Fig 3

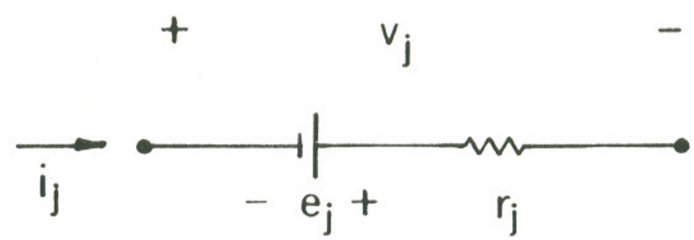
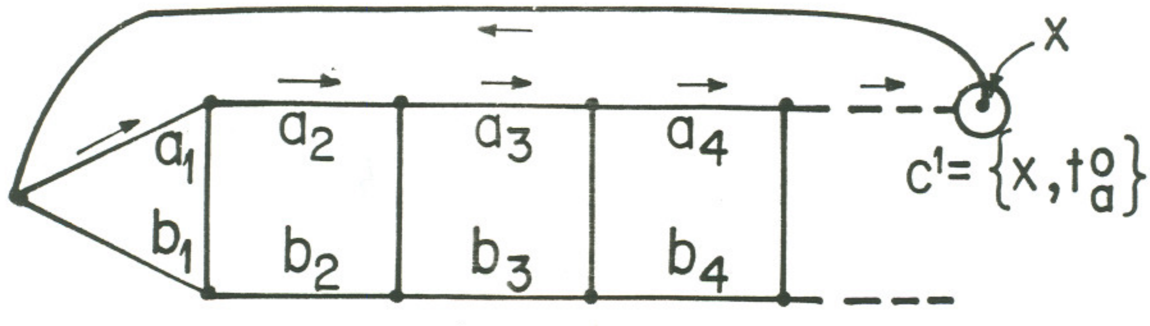
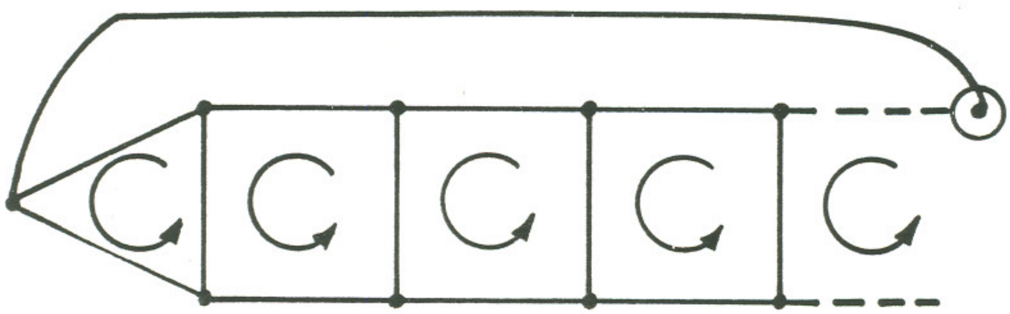


Fig 4



(a)



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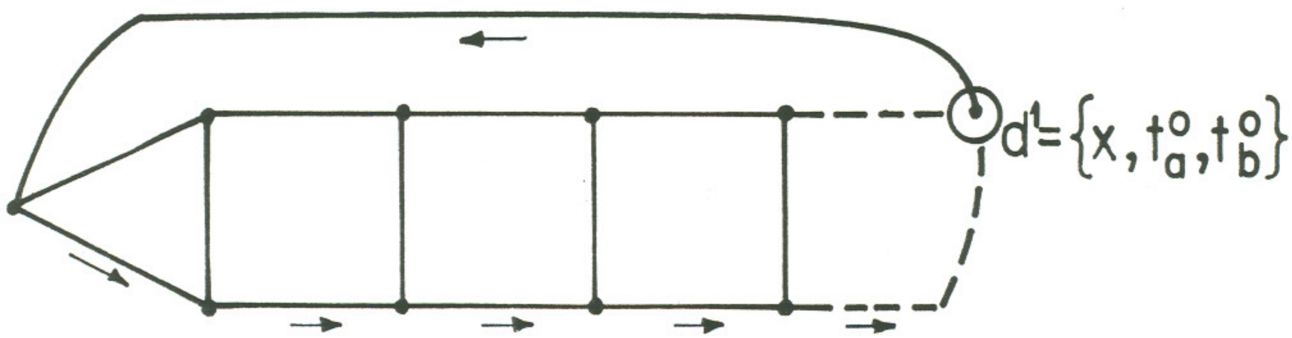

(c)

Figure 5

Zemanian