Report No. 90

HEAT TRANSFER TO LAMINAR FLOW OF AN ABSORBING-EMITTING GAS BETWEEN PARALLEL PLATES

by

R. D. Cess and S. N. Tiwari

May 1967

HEAT TRANSFER TO LAMINAR FLOW OF AN ABSORBING-EMITTING GAS BETWEEN PARALLEL PLATES[1]

R. D. Cess[2] and S. N. Tiwari[3]
State University of New York at Stony Brook

Prepared for Presentation
at the
Third All-Union Heat and Mass Transfer Conference
Minsk, U.S.S.R.
May 14-18, 1968

---

1 This work was supported by the National Science Foundation through Grant Number GK-345.

2 Professor of Engineering

3 Research Assistant, College of Engineering

ABSTRACT

Analyses are presented concerning the problem of combined convection, conduction, and radiation within a gas for laminar flow between parallel black plates. The thermal boundary condition at the plate surfaces is taken to be that of a uniform heat flux, and fully developed flow and heat transfer are assumed. Results are given for infrared radiative heat transfer and the specific case for which the absorbing-emitting gas is carbon monoxide, which has a single important vibration-rotation band. The methods of analysis, however, are easily extended to gases having multiple band spectra, such as carbon dioxide and water vapor.

Several methods of treating the radiative transfer process within the gas are considered. It is found that the gray gas approximation greatly overestimates radiative transfer within the gas, and it thus predicts bulk temperature differences which are too small. An optically thin analysis is also found to be generally inapplicable, since radiative transfer will usually be of importance only under optically non-thin conditions. A method of analysis is illustrated which incorporates the realities of the vibration-rotation band structure.

INTRODUCTION

The purpose of the present paper is to illustrate several analytical methods for treating nongray radiative transfer in gases when other modes of energy transfer simultaneously occur. The specific physical problem consists of fully developed laminar flow and heat transfer between parallel plates having a uniform wall heat flux, and results are presented for carbon monoxide. This work constitutes an extension of the analyses reported by the authors in Reference 1.

Only a limited number of analyses concerning duct flow of absorbing-emitting gases are available. Viskanta (Ref. 2) has treated laminar flow in a parallel plate channel under the assumption that the gas is gray. Turbulent flow of water vapor in the entrance region of an annular duct has been investigated by Nichols (Ref. 3). The nongray nature of the gas was accounted for through use of the statistical model for vibration-rotation bands. Another nongray analysis is that of deSoto and Edwards (Ref. 4), which treats laminar flow of carbon dioxide in the entrance region of a circular tube. In this case an exponential model was used to describe the spectral absorption coefficient for each vibration-rotation band, although the temperature profile was assumed to be independent of the radiative transfer process.

In the present investigation attention is restricted to carbon monoxide, which has a single fundamental band, and overtone bands

will be neglected. The methods of analysis, however, are easily extended to gases having multiple band spectra, such as carbon dioxide and water vapor.

BASIC EQUATIONS

The physical model and coordinate system are illustrated in Figure 1. Laminar flow of carbon monoxide between infinite parallel plates is considered under the condition of a uniform wall heat flux. The plates are taken to be black, and only fully developed flow and heat transfer are considered. Attention will additionally be restricted to small temperature differences, such that constant properties and linearized radiation may be assumed.

Since, for a uniform wall heat flux, the wall temperature varies in the axial direction, there will exist radiative transfer between wall elements located at different axial positions, and in general this would preclude the possibility of achieving fully developed heat transfer. For linearized radiation, however, it is easily shown that fully developed heat transfer can be attained, with the subsequent result that there will be no <u>net</u> radiative transfer between wall elements. In addition, it will be assumed that radiative transfer within the gas in the axial direction may be neglected.

Noting that for a uniform wall heat flux and fully developed heat

transfer[4]

$$\frac{\partial T}{\partial x} = \frac{2\alpha}{u_m L k}$$

then within the confines of the foregoing assumptions, conservation of energy is expressed by

$$\frac{12\alpha q}{Lk}\left[\frac{y}{L} - \left(\frac{y}{L}\right)^2\right] = \alpha \frac{d^2T}{dy^2} - \frac{1}{\rho c_p}\frac{dq_R}{dy} \qquad (1)$$

Upon integrating this equation, and noting that $dT/dy = 0$ and $q_R = 0$ at $y = L/2$, one has

$$\frac{dT}{dy} - \frac{2q}{k}\left(3\frac{y^2}{L^2} - 2\frac{y^3}{L^3}\right) + \frac{q}{k} = \frac{q_R}{k} \qquad (2)$$

Now, the total radiative heat flux, q_R, may be expressed as

$$q_R = \int_{\Delta\omega} q_{R\omega}\, d\omega \qquad (3)$$

where the wave number integration is taken over the single vibration-rotation band of carbon monoxide. In turn, from Reference 5 the spectral radiative flux may be expressed as

---

4 Nomenclature is listed at the end of the paper.

$$q_{R\omega} = 2\int_0^y \left[e_\omega(z) - e_{1\omega}\right] K_\omega E_2\left[K_\omega(y-z)\right] dz - 2\int_y^L \left[e_\omega(z) - e_{1\omega}\right] K_\omega E_2\left[K_\omega(z-y)\right] dz \tag{4}$$

In the following it will be convenient to employ the exponential kernel approximation (Ref. 5)

$$E_2(x) \simeq \frac{3}{4} e^{-3x/2}$$

such that Eq. (4) becomes

$$q_{R\omega} = \frac{3}{2}\int_0^y \left[e_\omega(z) - e_{1\omega}\right] K_\omega \exp\left[-\frac{3K_\omega}{2}(y-z)\right] dz - \frac{3}{2}\int_y^L \left[e_\omega(z) - e_{1\omega}\right] K_\omega \exp\left[-\frac{3K_\omega}{2}(z-y)\right] dz \tag{5}$$

The actual variation of κ_η with η throughout the vibration-rotation band is quite complicated. With the exception of high pressures, the band consists of a large number of individual rotational lines, and any attempt at incorporating this discrete line structure into the foregoing equations would prove to be a formidable task. An alternate approach, which has been employed by the authors in Reference 1, consists of expressing the total radiative heat flux in terms of the derivative of the total band absorptance. This procedure does account for the

discrete line structure of the band, and its use will be illustrated in the following section.

BAND ABSORPTANCE MODEL

Since a vibration-rotation band is very narrow compared with the black-body spectrum, one may assume that $e_{\omega}(T) \simeq e_{\omega_c}(T)$ within Eq. (5), where $e_{\omega_c}(T)$ denotes Planck's function evaluated at the band center, and for the fundamental band of carbon monoxide $\omega_c = 2,143\ \text{cm}^{-1}$. Furthermore, with the assumption of linearized radiation

$$e_{\omega_c}(T) - e_{1\omega_c} = \left(\frac{de_{\omega_c}}{dT}\right)_{T_1}(T - T_1) \tag{7}$$

Equation (5) may thus be rephrased as

$$q_{R\omega} = \frac{3}{2}\left(\frac{de_{\omega_c}}{dT}\right)_{T_1}\left\{\int_0^y \left[T(z) - T_1\right] K_\omega \exp\left[-\frac{3K_\omega}{2}(y - z)\right] dz - \int_y^L \left[T(z) - T_1\right] K_\omega \exp\left[-\frac{3K_\omega}{2}(z - y)\right] dz\right\} \tag{8}$$

The total band absorptance is defined as (Ref. 6)

$$A = \int_{\Delta\omega} \left[1 - \exp(-K_\omega y)\right] d\omega$$

and correspondingly

$$A' = \int_{\Delta\omega} K_\omega \exp\left(-K_\omega y\right) d\omega$$

where the prime denotes differentiation with respect to y. Thus combination of Eqs. (3) and (8) yields

$$q_R = \frac{3}{2}\left(\frac{de_{\omega_c}}{dT}\right)_{T_1}\left\{\int_0^y \left[T(z) - T_1\right] A'\left[\frac{3}{2}(y-z)\right] dz - \int_y^L \left[T(z) - T_1\right] A'\left[\frac{3}{2}(z-y)\right] dz\right\} \tag{9}$$

It will be convenient to recast this result in terms of the dimensionless band absorptance $\bar{A} = A/A_o$, where A_o is the band width parameter, and for the carbon monoxide fundamental band this is given by (Ref. 7)

$$A_o = 38\left(\frac{T}{300}\right)^{1/2}$$

Furthermore, it will be advantageous to employ the dimensionless quantities

$$u = C_o^2 P y \qquad u_o = C_o^2 P L$$

$$\Theta = \frac{T - T_1}{qL/k} \qquad N = \frac{k C_o^2 P}{A_o \left(de_{\omega_c}/dT\right)_{T_1}}$$

where, for carbon monoxide (Ref. 7)

$$C_o^2 = 6.24\left(\frac{300}{T}\right)^{3/2}$$

With the above definitions, Eqs. (7) and (9) combine to yield

$$u_o \frac{d\Theta}{du} - 2\left(3\frac{u^2}{u_o^2} - 2\frac{u^3}{u_o^3}\right) + 1 = \frac{3u_o}{2H}\left\{\int_o^u \Theta(u')\bar{A}'\left[\frac{3}{2}(u-u')\right]du' - \int_u^{u_o} \Theta(u')\bar{A}'\left[\frac{3}{2}(u'-u)\right]du'\right\} \tag{10}$$

where $\bar{A}'(u)$ denotes the derivative of $\bar{A}(u)$ with respect to u. This integrodifferential equation describes the dimensionless gas temperature $\theta(u)$, and the boundary condition for the equation follows to be

$$\Theta(0) = 0$$

The dimensionless bulk temperature is in turn given by Eq. (6) as

$$\Theta_b = \frac{T_b - T_i}{qL/K} = \frac{6}{u_o}\int_o^{u_o} \Theta(u)\left(\frac{u}{u_o} - \frac{u^2}{u_o^2}\right)du \tag{11}$$

It remains to specify the dimensionless band absorptance $\bar{A}(u)$, and in the present study the semi-emperical formulation of Tien and Lowder (Ref. 8) has been employed. This is of the form

$$\bar{A}(u) = \ln\left\{u f(\beta)\left[\frac{u+2}{u+2f(\beta)}\right] + 1\right\} \tag{12}$$

where

$$f(\beta) = 2.94\left[1 - \exp(-2.60\beta)\right]$$

and for the carbon monoxide fundamental band (Ref. 7)

$$\beta = 0.0855\, p\left(\frac{300}{T}\right)^{0.42}$$

It is important to realize that Eq. (12) does, at least in a semi-emperical sense, account for the line structure of the band. This line structure dependency is introduced through the parameter β, which is a measure of the ratio of line halfwidth to line spacing.

Equation (10) has been solved numerically, and the subsequent results for θ_b will be presented in the last section of the paper. Aside from these numerical results, however, Eq. (10) possesses two limiting solutions, and these will be described in the following two sections.

OPTICALLY THIN LIMIT

In the present notation, the optically thin limit corresponds to $u_o << 1$. There are basically two ways of obtaining this limiting solution. One is to employ Eq. (1) together with the optically thin formulation for dq_R/dy (see for example Ref. 1). The alternate method, which will be employed here, is to directly obtain the limiting

form of Eq. (10) for $u_o << 1$. Proceeding in this manner, one may note from Eq. (12) that for $u << 1$

$$\bar{A}(u) = u$$

and Eq. (10) correspondingly reduces to

$$u_o \frac{d\Theta}{du} - 2\left(3\frac{u^2}{u_o^2} - 2\frac{u^3}{u_o^3}\right) + 1 = \frac{3u_o}{2N}\left\{\int_o^u \Theta(u')\,du' - \int_u^{u_o} \Theta(u')\,du'\right\} \tag{13}$$

Upon differentiating this equation, and letting $\xi = y/L$, one has

$$\frac{d^2\Theta}{d\xi^2} - \frac{3u_o^2}{N}\Theta = 12(\xi - \xi^2) \tag{14}$$

with the boundary conditions

$$\Theta = 0 \quad ; \quad \xi = 0$$

$$\frac{d\Theta}{d\xi} = 0 \quad ; \quad \xi = \frac{1}{2}$$

Note that pressure appears in Eq. (14) through the grouping $u_o^2/N \sim P$.

Equation (14) possesses an elementary solution, and from Eq. (11) the dimensionless bulk temperature is found to be

form of Eq. (10) for $u_o << 1$. Proceeding in this manner, one may note from Eq. (12) that for $u << 1$

$$\bar{A}(u) = u$$

and Eq. (10) correspondingly reduces to

$$u_o \frac{d\Theta}{du} - 2\left(3\frac{u^2}{u_o^2} - 2\frac{u^3}{u_o^3}\right) + 1 = \frac{3u_o}{2N}\left\{\int_o^u \Theta(u')\,du' - \int_u^{u_o} \Theta(u')\,du'\right\} \tag{13}$$

Upon differentiating this equation, and letting $\xi = y/L$, one has

$$\frac{d^2\Theta}{d\xi^2} - \frac{3u_o^2}{N}\Theta = 12(\xi - \xi^2) \tag{14}$$

with the boundary conditions

$$\Theta = 0 \quad ; \quad \xi = 0$$

$$\frac{d\Theta}{d\xi} = 0 \quad ; \quad \xi = \frac{1}{2}$$

Note that pressure appears in Eq. (14) through the grouping $u_o^2/N \sim P$.

Equation (14) possesses an elementary solution, and from Eq. (11) the dimensionless bulk temperature is found to be

$$\Theta_b = \frac{576}{M_1^{7/2}}\left(\frac{1-e^{-\sqrt{M_1}}}{1+e^{-\sqrt{M_1}}}\right) - \frac{288}{M_1^3} + \frac{24}{M_1^2} - \frac{12}{5M_1} \qquad (15)$$

where

$$M_1 = \frac{3u_o^2}{N}$$

It should be noted that this form of the optically thin limit is based upon the exponential kernel approximation; i.e., it employs the optically thin form of Eq. (5) rather than Eq. (4). If, instead, one employed Eq. (4) and made use of the Planck mean and modified Planck mean coefficients (as in Ref. 1), the optically thin solution would be identical to Eq. (15), except that M_1 would be replaced by $M_1 = 4u_o^2/N$.

LARGE u_o LIMIT

The second limiting solution corresponds to $u_o >> 1$. Conventionally, this might be regarded as an optically thick limit, and one might attempt to employ the Rosseland equation for the total radiative flux. As pointed out in Reference 1, however, a useable Rosseland mean absorption coefficient does not appear to exist for vibration-rotation bands. The reason for this is that, in order for the Rosseland equation to apply, the gas must be optically thick for all values of wave number that contribute to the absorption-emission process.

With respect to a vibration-rotation band in the limit $u_o >> 1$, the central portion of the band will be opaque, and radiative transfer within the gas will be due solely to the wing regions of the band. Here the spectral absorption coefficient, κ_ω, decreases in an asymptotic manner towards zero. Consequently, regardless of the magnitude of the gas layer thickness L, the spectral optical thickness, $\kappa_\omega L$, will vary from optically thick to optically thin within the band wings, and this precludes the possibility of employing the Rosseland equation. A more complete discussion of this is given in Reference 1.

Even though the Rosseland equation is specifically inapplicable, Eq. (10) does possess a proper limiting form for $u_o >> 1$. To obtain this limit, note that for large u Eq. (12) may be written as

$$\bar{A}(u) = \ln u + \frac{1}{u} + \cdots \tag{16}$$

The first term in this expression denotes the dimensionless band absorptance, in the limit $u >> 1$, for the molecular model of a vibrating nonrigid rotator (Ref. 9).

Upon substituting Eq. (16) into Eq. (10), it is found that, in the limit $u_o >> 1$, only the first term in Eq. (16) contributes to the integrals in Eq. (10). Thus, with $\bar{A} = \ln u$, and again letting $\xi = y/L$, Eq. (10) reduces to

$$\frac{d\Theta}{d\xi} - 2\left(3\xi^2 - 2\xi^3\right) + 1 = \frac{u_o}{N}\int_o^1 \Theta(\xi')\,\frac{d\xi'}{\xi - \xi'} \tag{17}$$

$$q_R = 60\sigma K_p T_1^3 \int_0^y [T(z) - T_1] \exp\left[-\frac{3K_p}{2}(y-z)\right] dz$$

$$- 60\sigma K_p T_1^3 \int_y^L [T(z) - T_1] \exp\left[-\frac{3K_p}{2}(z-y)\right] dz \qquad (19)$$

Upon differentiating this equation twice, the integrals repeat themselves and may be eliminated, with the result

$$\frac{d^2 q_R}{dy^2} - \frac{9}{4} K_p^2 q_R = 120\sigma K_p T_1^3 \frac{dT}{dy} \qquad (20)$$

This is, in fact, simply one form of the well-known differential approximation (Ref. 5).

The simultaneous solution of Eqs. (1) and (20) is straightforward, and the final result for the dimensionless bulk temperature is

$$\Theta_b = C\left[24 - 12M_2 + M_2^2 + \left(M_2^3 - 12M_2 - 24\right)e^{-M_2}\right]$$

$$- \frac{12}{5}\frac{\gamma}{M_2^4} + \frac{17}{70}\frac{\gamma}{M_2^2} - \frac{17}{70}\Bigg]$$

where

$$C = \frac{\gamma}{M_2^8}\left[\frac{48 - 3\tau_o M_2^2 + 36\tau_o}{3\tau_o(1-e^{-M_2}) + 2M_2(1+e^{-M_2})}\right]$$

$$M_2^2 = 3\tau_o^2\left(\frac{3}{4} + \frac{1}{\bar{N}}\right) \qquad \gamma = \frac{3\tau_o^2}{\bar{N}}$$

$$\bar{N} = \frac{k K_p}{4\sigma T_1^3} \qquad \tau_o = K_p L$$

COMPARISON OF RESULTS

Before turning to the numerical results which have been obtained from the previous analyses, it will be of interest to discuss the parameters which enter into the analyses. Consider first the gray gas solution. The governing parameters are $\bar{N}$ and τ_o, and these are the conventional parameters which describe combined conduction and radiation. The relative importance of conduction versus radiation within the gas is denoted respectively by $\bar{N}$ and $\bar{N}/\tau_o^2$ in the optically thick and optically thin limits (Ref. 5).

Considering next the band absorptance model, Eqs. (10) and (12) illustrate that the governing parameters are N, u_o, and β. The first two constitute the nongray counterparts of $\bar{N}$ and τ_o, while β is the line structure parameter and does not arise in a gray gas analysis.

It may further be noted from Eq. (15) that the optically thin limit involves the single parameter N/u_o^2, which is analogous to the gray gas parameter $\bar{N}/\tau_o^2$. On the other hand, in the large u_o limit Eq. (17) yields the parameter N/u_o, which is not analogous to the gray gas counterpart. This again illustrates that the large u_o limit for a vibration-rotation band differs from the conventional optically thick limit. The absence of the line structure parameter β in both the optically thin and large u_o limits is discussed in Reference 1.

For the purpose of comparing solutions, it will be advantageous to present results in terms of the dimensional quantities L and P. The band absorptance results are illustrated in Figures 2 and 3 for wall temperatures of 500°K and 1000°K respectively. The limiting value of $\theta_b = -0.243$ corresponds to negligible radiation, and the effect of radiation increases with increasing plate spacing. As would be expected, radiative transfer is more pronounced for the higher wall temperature.

Also illustrated in Figures 2 and 3 is the limiting solution for large u_o, and recall that this solution is independent of pressure. Since $u_o = C_o^2 PL$, it is evident that, for a given wall temperature, the large u_o limit can be obtained either by going to large values of L or to high pressures. This is evident from Figures 2 and 3, although for moderate pressures unrealistically large values of L are necessary.

The gray gas and optically thin solutions are compared in Figure 4 for a wall temperature of 1000°K and a pressure of one atmosphere. The optically thin limit greatly overestimates the influence of radiation,

and it is evident that when radiation is of importance the radiative transfer process is in turn not optically thin.

Note that the gray gas solution also overestimates the importance of radiation. The reason for this is that the optical thickness of the gas based upon the Planck mean coefficient can be several orders of magnitude less than that based upon the maximum absorption coefficient within the vibration-rotation band (Ref. 1). Correspondingly, the gray gas assumption may predict optically thin radiation under conditions for which the real process is not optically thin.

REFERENCES

1. R. D. Cess, P. Mighdoll and S. N. Tiwari, Infrared Radiative Heat Transfer in Nongray Gases, Submitted for publication to the International Journal of Heat and Mass Transfer.

2. R. Viskanta, Interaction of Heat Transfer by Conduction, Convection and Radiation in a Radiating Fluid, Journal of Heat Transfer, vol. C85, pp. 318-328 (1963).

3. L. D. Nichols, Temperature Profile in the Entrance Region of an Annular Passage Considering the Effects of Turbulent Convection and Radiation, International Journal of Heat and Mass Transfer, vol. 8, pp. 589-608 (1965).

4. S. deSoto and D. K. Edwards, Radiative Emission and Absorption in Nonisothermal Nongray Gases in Tubes, Proceedings of the 1965 Heat Transfer and Fluid Mechanics Institute, Stanford University Press, Palo Alto, California, 1965, pp. 358-372.

5. E. M. Sparrow and R. D. Cess, Radiation Heat Transfer, Brooks/Cole Publishing Co., Belmont, California, 1966.

6. R. M. Goody, Atmospheric Radiation, Oxford University Press, 1964.

7. M. M. Abu-Romia and C. L. Tien, Measurements and Correlations of Infrared Radiation of Carbon Monoxide at Elevated Temperatures, Journal of Quantitative Spectroscopy and Radiative Transfer, vol. 6, pp. 143-167 (1966).

8. C. L. Tien and J. E. Lowder, A Correlation for Total Band Absorptance of Radiating Gases, International Journal of Heat and Mass Transfer, vol. 9, pp. 698-701 (1966).

9. D. K. Edwards and W. A. Menard, Comparison of Models for Total Band Absorption, Applied Optics, vol. 3, pp. 621-625 (1964).

NOMENCLATURE

A	total band absorptance, cm^{-1}
A_o	band width parameter, cm^{-1}
c_p	specific heat at constant pressure, cal/g/ ^oK
C_o^2	correlation parameter, $atm^{-1} cm^{-1}$
e_ω	Planck's function, watts/cm^2/cm^{-1}
$e_{1\omega}$	Planck's function evaluated at temperature T_1
e_{ω_c}	Planck's function evaluated at wave number ω_c
$E_n(x)$	exponential integral
k	thermal conductivity, watts/cm^2/^oK
L	distance between plates, cm
P	pressure, atm
q	wall heat flux, watts/cm^2
q_R	total radiation heat flux, watts/cm^2
$q_{R\omega}$	spectral radiation heat flux, watts/cm^2/cm^{-1}
T	temperature, ^oK
T_1	wall temperature, ^oK
T_b	bulk temperature, ^oK
u_m	mean velocity, cm/sec
u	dimensionless coordinate, $C_o^2 Py$
u_o	dimensionless path length, $C_o^2 PL$
x	axial coordinate, cm
y	transverse coordinate, cm

α	thermal diffusivity, cm^2/sec
κ_ω	spectral absorption coefficient, cm^{-1}
κ_P	Planck mean coefficient, cm^{-1}
ξ	dimensionless coordinate, y/L
ρ	density, g/cm^3
σ	Stefan-Boltzmann constant
τ_o	optical thickness, $\kappa_P L$
ω	wave number, cm^{-1}
ω_c	band center, cm^{-1}

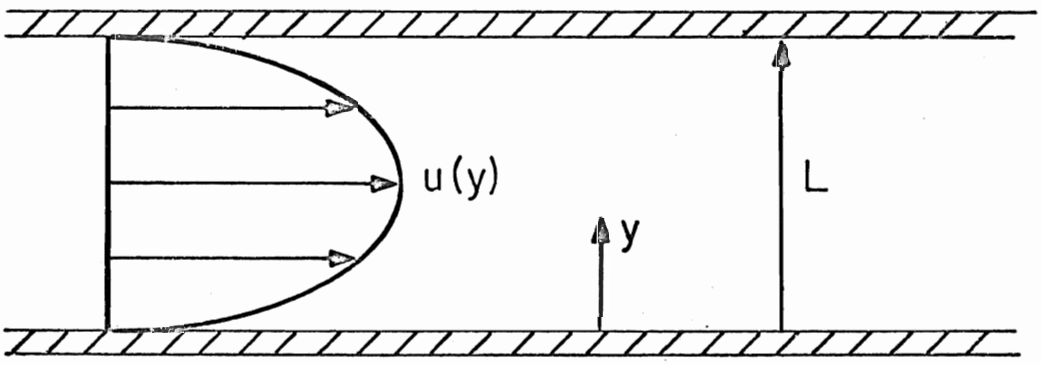

Fig. 1 Physical model and coordinate system.

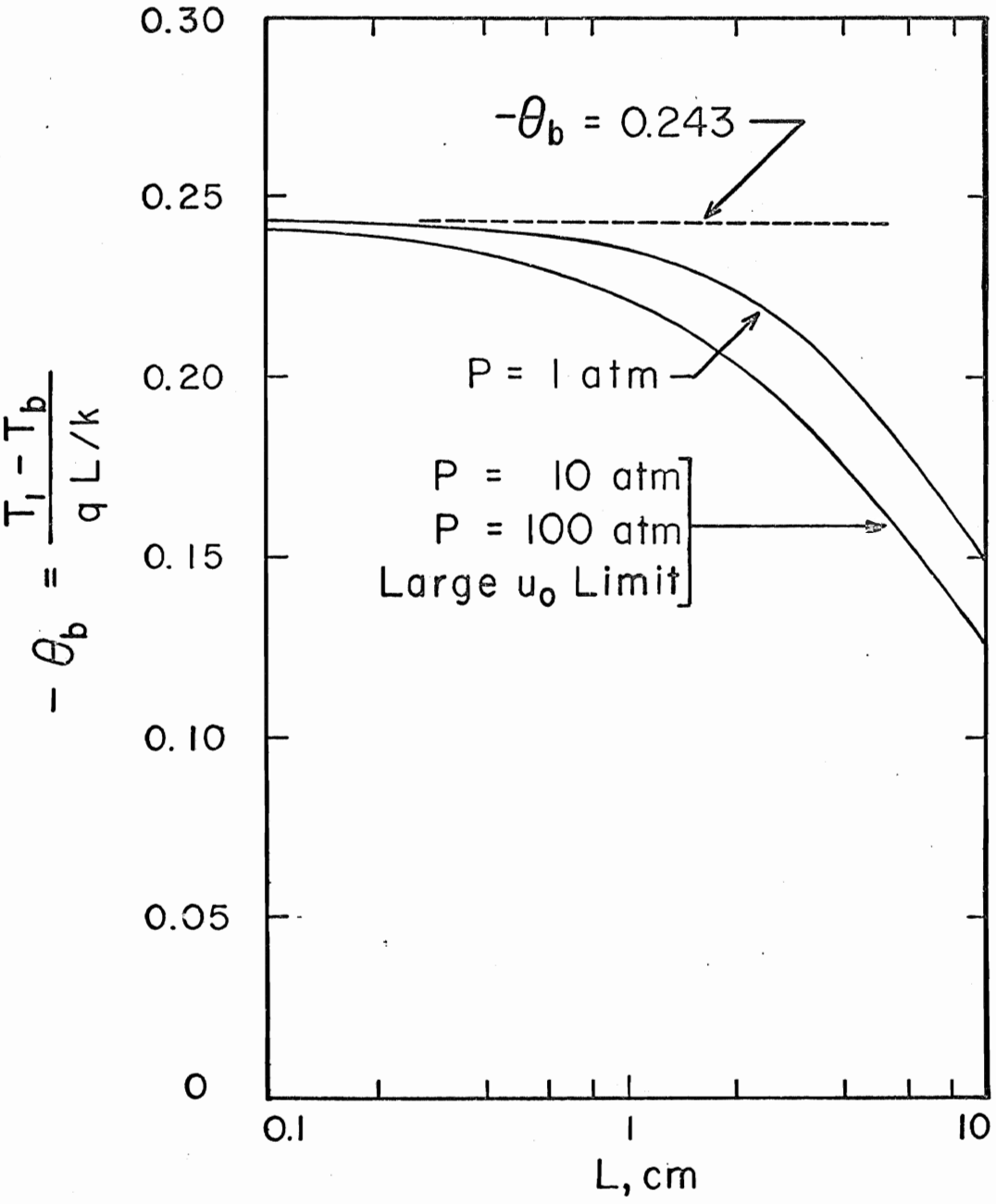

Fig. 2 Band absorptance results for $T_1 = 500^{\circ}K$.

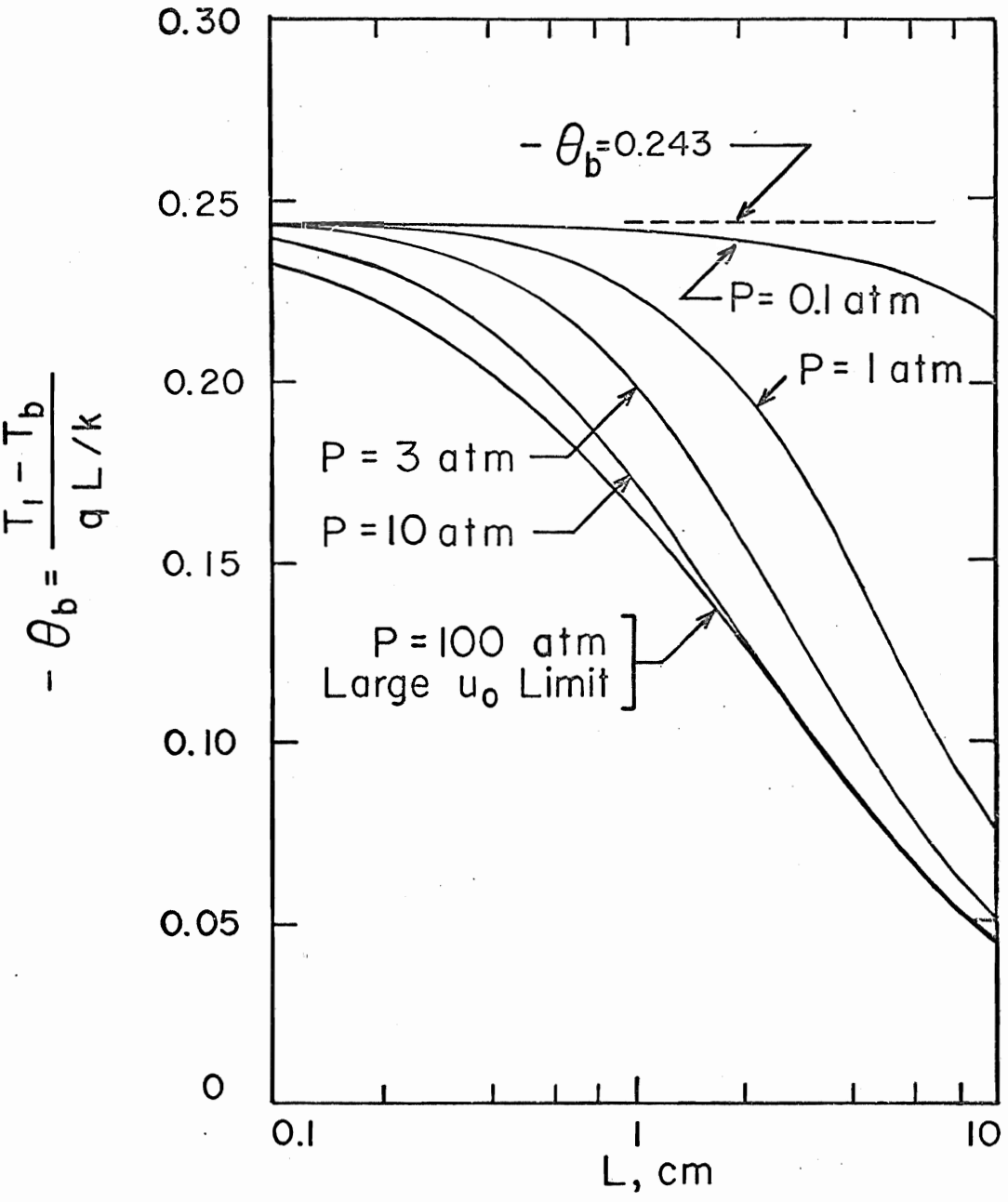

Fig. 3 Band absorptance results for $T_1 = 1000^{\circ}K$.

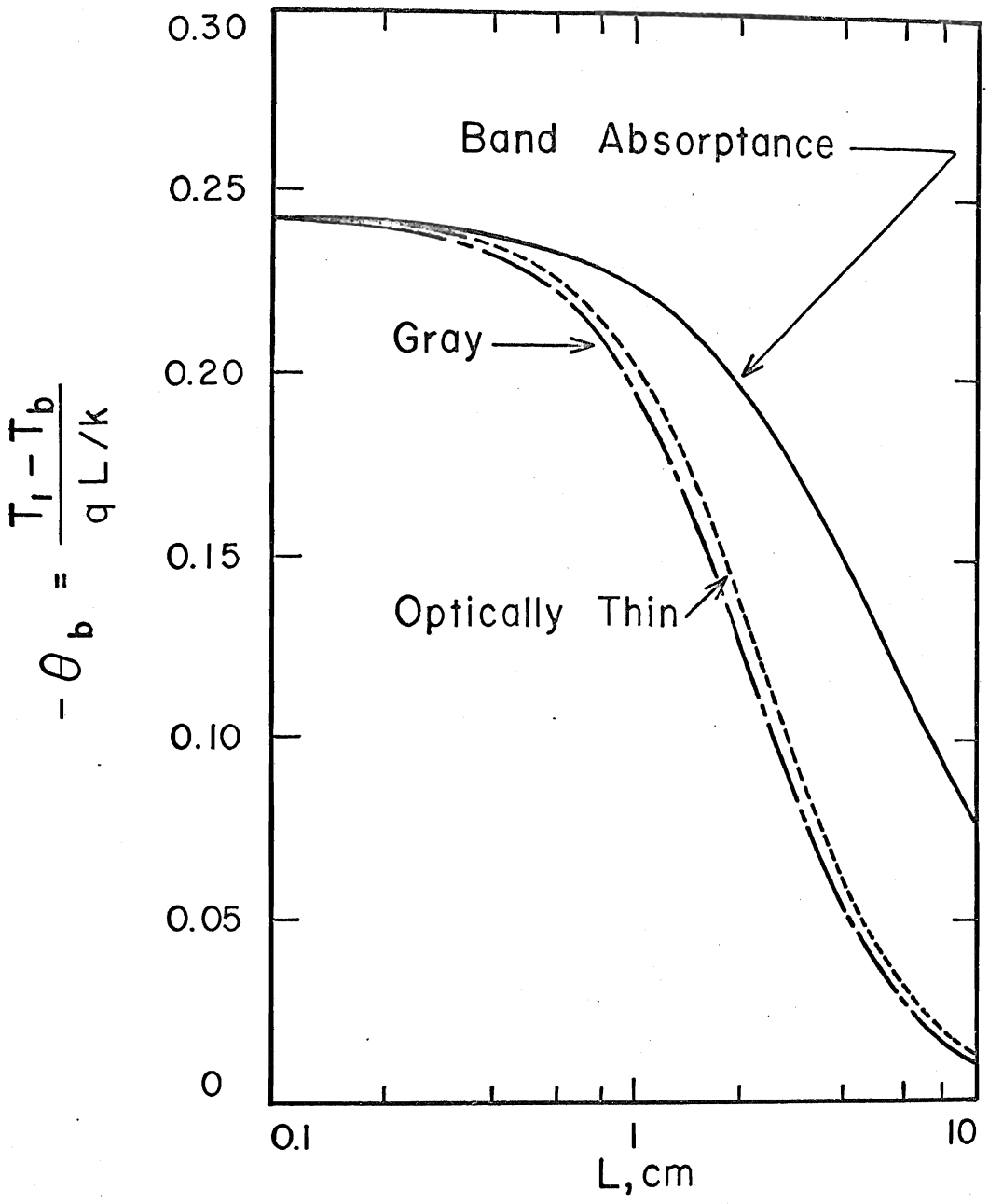

Fig. 4 Comparison of results for $T_1 = 1000^{\circ}K$ and $P = 1$ atm.