STATE UNIVERSITY OF NEW YORK AT STONY BROOK

COLLEGE OF ENGINEERING

Report No. 52

ON THE RELATIVE TIME OF ADAPTIVE PROCESSES

by

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September, 1964

Reprinted from IEEE TRANSACTIONS
ON AUTOMATIC CONTROL
Volume AC-10, Number 1, January, 1965
Pp. 11-17

On the Relative Time of Adaptive Processes

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***Abstract*—Adaptive control systems can be classified according to the response time of the adaptive loop T_a relative to that of the main servo loop T_m. The response time T_a cannot be smaller than the time needed for making the required measurement on which the adjustments are based.**

In a slow adaptive system, $T_a \gg T_m$, and the adjustments are made according to the estimated situation (statistical parameters of the inputs changing plant parameters, etc.) or the estimated performance (error, cost, intermediate parameters etc.). If $T_a \ll T_m$, usually a reference model is chosen and the system is forced to conform to the reference model. Sometimes the reference model is dimensionless so that the fastest response can be obtained.

In both cases, $T_a \gg T_m$ and $T_a \ll T_m$, the system can be analyzed by introducing suitable approximations. The condition of stability for the adaptive loop is derived in general terms.

If $T_a \approx T_m$, both analysis and synthesis become difficult. The concept of "dual control" is introduced, but not developed in the paper.

Introduction

THIS PAPER is an attempt to provide a systematic viewpoint or understanding of the various methods of adaptive control advanced in the literature. A key factor which can be used as a coordinate on which to attach the various methods seems to be the response time of the adaptive loop T_a in relation to the response time of the main servo loop T_m. Methods are distinctively different for the three cases with (I) $T_a > T_m$, (II) $T_a \approx T_m$, and (III) $T_a < T_m$. Cases (I) and (III) will be referred to as slow and fast systems, respectively. The classification is applicable to systems with continuous or discrete main servo loop and continuous or discrete adaptive loop.

Following previous literature, the term *situation parameters* will be used to denote the changing parameters governing the plant performance, and the statistical natures of the various inputs, desirable or undesirable. The term *adjustable parameters* will be used to denote the adjustment which can be made on the controller. The term *performance parameters* will be used to characterize the *closed-loop* response of the main servo loop. The situation, adjustable, and performance parameters will be denoted by ξ, ζ, α respectively. These are generally vectors, as there are usually more than one parameter of each kind.

In general T_a is determined by the time required to make fairly good measurements on the situation parameter ξ or the performance parameter α. Once the measured quantities become known or fairly well estimated, the controller is then adjusted in a more favorable direction. Therefore T_a is determined mainly by the time required for making measurements. Another factor which affects T_a is the time T_ξ in which the situation parameter ξ undergoes considerable change. Obviously T_a can be increased if T_ξ is large. On the other hand, if $T_\xi < T_a$, an adaptive system is not effective.

As the purpose of the present paper is to develop a theoretical framework for the different types of adaptive systems, the latter will not be described in any detail. Only the literature references of the individual systems cited as examples will be given.

General Formulation of the Adaptive Problem

The block diagram of an adaptive control system is illustrated in Fig. 1. The controlled plant is described by

$$\dot{x} = f(x, u, v, \xi_p) \tag{1}$$

where x is the state vector, u is the control vector, v is a random disturbance and ξ_p is a vector representing the unknown or changing parameters.

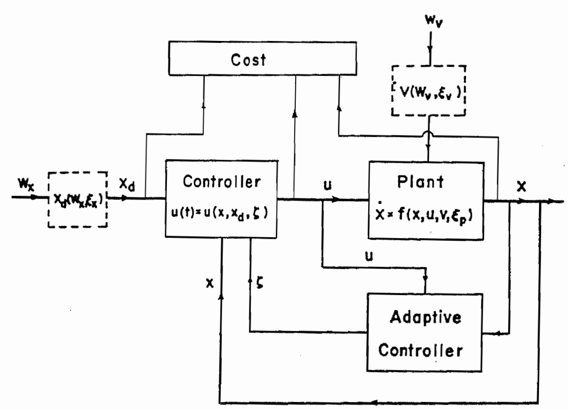

Fig. 1. Block diagram of an adaptive control system.

Sometimes the disturbance v and desired output x_d contain stochastic or random components which are nonwhite. Then v and x_d can be expressed as functions of parameters ξ_v, and ξ_x and white stochastic signals w_v and w_x;

$$x_d = x_d(w_x, \xi_x) \tag{2}$$

$$v = v(w_v, \xi_v) \tag{3}$$

The set of vectors ξ_p, ξ_x, and ξ_v is denoted by ξ.

Manuscript received July 9, 1964; revised September 9, 1964. The work upon which this paper was based was sponsored by the Office of Scientific Research and Development Command, Washington, D. C., under Grant AF-AFOSR-542-64.
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The control vector u is a function of the state variable x, the desired state x_d, and some adjustable parameter ζ;

$$u = u(x, x_d, \zeta) \tag{4}$$

The merit of the control system is determined by the value of a cost integral;

$$c \equiv \int_{t_1}^{t_2} h(x, x_d, u)dt \tag{5}$$

In designing an adaptive control system, the functions f and h are known. The function u, and the method of determining ζ from previous measurements are to be selected so that c is a minimum.

The Extreme Case of a Slow Adaptive System

In the extreme case of $T_a \gg T_m$, the variables ξ and ζ are very slow-varying and can be regarded as constants in calculating the performance of the main control loop. Equations (1) and (4) can be solved for u, and x

$$u = \phi(x_d, v; \xi_p, \zeta) \tag{6}$$

$$x = \chi(x_d, v; \xi_p, \zeta) \tag{7}$$

The integrand of the cost function c in (5) becomes a function of x_d, v, ξ_p and ζ;

$$h(x, x_d, u) = \psi(x_d, v; \xi_p, \zeta) \tag{8}$$

As x_d and v are random variables, the only meaningful value of h is its expected value;

$$\langle h(x, x_d, u)\rangle = \langle \psi(x_d, v; \xi_p, \zeta)\rangle = H(\xi, \zeta) \tag{9}$$

The last equality sign is interpreted as follows: The expected value of h is a function of ξ_p and ζ and the parameters ξ_x and ξ_v which govern the generating processes of x_d and v. Consequently it can be written simply as a function of ξ and ζ.

There are three ways of adjusting ζ.

1) Optimizing controller with estimated value of ζ: The parameters ξ are estimated from measured value of x and u. Let η denote the estimated value of ξ. In general, η is a functional of the present and past values of x and u.

$$\eta(t) = g\{x(t' \le t), u(t' \le t)\} \tag{10}$$

The curly brackets are used to represent functionals.

In adjusting ζ, the function $\zeta(\eta)$ is prespecified, and gives the optimum setting of ζ for each η. Then ζ is adjusted accordingly.

Schematically, this method is represented by the solid line of Fig. 2. The actual setting of ζ is determined by η_1. But since η is the estimated ξ and is generally different from the actual ξ, ζ_1 is different from the true optimum ζ_{01}. The adaptive control is open-loop and the question of stability does not arise. An example of such a system is one proposed by Kalman [1].

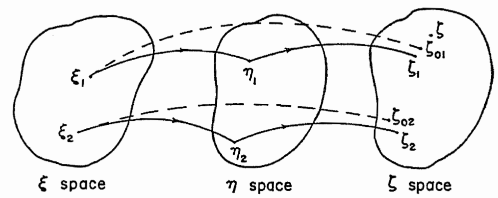

Fig. 2. Mapping from ξ space to η and ζ spaces of different types of adaptive control systems.

2) Peak-seeking systems. In peak-seeking systems, no attempt is made to estimate ξ. The adjustable parameter for peak performance is sought experimentally by the steepest descent method. To minimize H, $\dot{\zeta}$ or $\zeta(n+1)-\zeta(n)$ is made proportional to the measured value of

$$\frac{\partial H}{\partial \zeta} = \left(\frac{\partial H}{\partial \zeta_1}, \frac{\partial H}{\partial \zeta_2} \cdots \frac{\partial H}{\partial \zeta_m}\right)$$

in magnitude and antiparallel in direction;

$$\dot{\zeta}_i = -k\frac{\partial H}{\partial \zeta_i} \quad \text{(continuous)} \tag{11}$$

$$\zeta_i(n+1) - \zeta_i(n) = -k\frac{\partial H(\xi(n), \zeta(n))}{\partial \zeta_i(n)} \quad \text{(discrete)} \tag{12}$$

where $k > 0$.

As $\partial H/\partial \zeta$ depends on ζ, the adaptive control loop is closed. However, the stability of such a system when ζ is nearing $\zeta_0(\xi)$ can be readily proved in general terms under the assumption that $H(\xi, \zeta)$ is analytic in ζ in the vicinity of ζ_0. Since $H(\xi, \zeta_0)$ is at its minimum Taylor's series expansion gives

$$H(\xi, \zeta) = H(\xi, \zeta_0) + \frac{1}{2}\sum_i \sum_j \left(\frac{\partial^2 H}{\partial \zeta_i \partial \zeta_j}\right)_{\zeta=\zeta_0} (\zeta_{0i} - \zeta_i)(\zeta_{0j} - \zeta_j) \tag{13}$$

Let

$$A_{ij} \equiv \left(\frac{\partial^2 H}{\partial \zeta_i \partial \zeta_j}\right)_{\zeta=\zeta_0}$$

and A denote the matrix having A_{ij} as its elements. The condition that $H(\xi, \zeta_0)$ is at its minimum implies that none of the eigenvalues of A is negative. For simplicity, the rather remote possibility that one or more eigenvalues of A are exactly zero are excluded, and *A is assumed to be positive-definite.*

$$\frac{\partial H}{\partial \zeta_i} = -\sum_j A_{ij}(\zeta_{0j} - \zeta_j) = -(A(\zeta_0 - \zeta))_i \tag{14}$$

The expression on the right hand side of (14) is in matrix form; $\zeta_0 - \zeta$ is a column vector, A is a matrix and $(A(\zeta_0 - \zeta))_i$ means the ith component of the column vector $A(\zeta_0 - \zeta)$. Substituting (14) into (11) and (12) gives the following matrix equations

$$\dot{\zeta} = kA(\zeta_0 - \zeta) \quad (15)$$

$$\zeta(n+1) = \zeta(n) + kA(\zeta_0(n) - \zeta(n)) = (\Pi - kA)\zeta(n) + kA(\zeta_0(n)) \quad (16)$$

n (16), Π is the unit matrix with all diagonal elements 1 nd off-diagonal elements zero.

Since A is positive definite, the system represented ›y (15) is stable and ζ converges to ζ_0. The eigenvalues ıf the matrix $\Pi - kA$ are $1 - k\lambda_i$, where $\lambda_i > 0$ are eigenvalues of A. For $0 < k < 2/\lambda_m$ where λ_m is the maximum value of λ, $|1 - k\lambda_i| < 1$ and the system represented by 16) is stable.

Therefore we come to the following conclusion: A peak-seeking system using the control law (11) and (12) with $T_a \gg T_m$ is always stable if the adaptive loop s continuous or if the adaptive loop is discrete and the proportionality constant k is sufficiently small.

The derivative sensing system of Draper and Li is of this type [2].

3) Systems with Intermediate Parameters: Sometimes optimum or approximately optimum performance can be obtained by keeping a set of key performance parameters at certain preassigned values. In the final analysis, these performance parameters do not have much significance per se, and the ultimate goal is to minimize some cost function c. But c is minimized or nearly so if these parameters are kept at the preassigned values. These key parameters are sometimes referred to as intermediate parameters. The mapping from ξ space to ζ space is illustrated in Fig. 3. Let the intermediate parameters be denoted α, and their assigned values be denoted α_0. Then

$$\alpha_i - \alpha_{0i} = \sum_j \left(\frac{\partial \alpha_i}{\partial \zeta_j}\right)_{\zeta=\zeta_0} (\zeta_j - \zeta_{0j}) \quad (17)$$

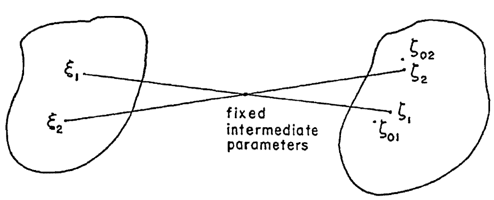

Fig. 3. Mapping from ξ space to ζ space of systems with intermediate parameters.

Let

$$B_{ij} \equiv \left(\frac{\partial \alpha_i}{\partial \zeta_j}\right)_{\zeta=\zeta_0} \quad (18)$$

and B denote the matrix with B_{ij} as its elements. In matrix notation (17) becomes

$$\alpha - \alpha_0 = B(\zeta - \zeta_0) \quad (19)$$

If B is a square matrix

$$\zeta - \zeta_0 = B^{-1}(\alpha - \alpha_0) \quad (20)$$

Therefore one way to make ζ approach ζ_0 is to synthesis the adaptive loop such that

$$\dot{\zeta} = -kB^{-1}(\alpha - \alpha_0) \quad (21)$$

Since the value of B varies with ζ_0, and the value of ζ_0 varies with ξ, B is not a constant matrix. In synthesizing the system it is convenient to assign some fixed or nominal value of B. Let it be denoted as B_0. The actual equation for the adaptive loop is not (21) but

$$\dot{\zeta} = -kB_0^{-1}(\alpha - \alpha_0) \quad (22)$$

where k and B_0 are a constant and a constant matrix respectively.

Consider the case in which ξ and consequently ζ_0 have just undergone a step change; $\zeta \neq \zeta_0$, $\dot{\zeta}_0 = 0$ and $\dot{B} = 0$. Differentiating (19) gives

$$\dot{\alpha} = B(\dot{\zeta} - \dot{\zeta}_0) = -kBB_0^{-1}(\alpha - \alpha_0) \quad (23)$$

Equation (23) holds not only at the nominal operating point but at all operating points. The adaptive loop is stable if BB_0^{-1} is positive-definite. It is noted that at the nominal operating point $B = B_0$ and BB_0^{-1} is the unit matrix. There is a neighborhood of the nominal operating point in which BB_0^{-1} is positive-definite.

The above analysis is not limited to the special case of α and ζ having equal number of components. Generally the number of adjustable parameters is at least as large as the number of intermediate parameters; otherwise it is generally not possible to adjust α to α_0. Therefore B has more columns than rows. The matrix B_0^{-1} in (22) can be replaced by a matrix C with m rows and n columns, $m > n$, which satisfies

$$B_0C = 1 \quad (24)$$

Equation (24) does not determine C uniquely. To see this, one may add $m - n$ rows to B_0 to make it into a $m \times m$ matrix D. The elements of the new rows are completely arbitrary except for the restriction that D must be nonsingular. Then D^{-1} can be found and C is the part of D^{-1} as illustrated in Fig. 4.

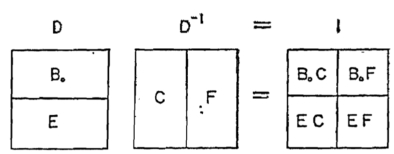

Fig. 4. One way of determining matrix C when $m > n$.

For a discrete adaptive loop, (22) and (23) can be replaced by

$$\zeta(n+1) - \zeta(n) = -kB_0^{-1}(\alpha(n) - \alpha_0) \quad (25)$$

$$\alpha(n+1) - \alpha_0 = -kBB_0^{-1}(\alpha(n) - \alpha_0) \quad (26)$$

The system is stable if all the eigenvalues of $\Pi - kBB_0^{-1}$ are less than one in magnitude.

An example of an intermediate parameter is the area ratio proposed by Anderson, Aseltine, et al [3] and Kuntsevich and Chugunnaya [4].

Each of the above-mentioned systems has its relative advantages and disadvantages. The function $\zeta(\eta)$ of a type 1) system is difficult to realize. Because the adaptive loop is open, this function must be accurately simulated. It usually requires considerably more knowledge on the controlled plant and the statistical natures of x_d and v than what is available in practice. For systems of type 2) explorative variations of ζ must be used to determine the gradient $\partial H/\partial\zeta$. Consequently ζ must be different from ζ_0 at times and the so-called "hunting loss" is introduced. While systems of type 3) do not have the above mentioned handicaps, one prerequisite is that the intermediate parameters α must exist and must be properly selected.

In the above examples, the adjustment of ζ is based entirely on the estimated situation or estimated performance of the system. These methods are slow because it usually takes a measuring interval many times T_m to obtain a good estimate. To illustrate this point, consider a system in which some plant parameter ξ_p or over-all performance parameter α is to be estimated. Usually there is load disturbance or some other source of fluctuation which cannot be measured directly. In order to obtain a fairly good estimate on ξ_p or α, the effect or error introduced by these disturbances must be in some way neutralized, and the only effective way appears to be correlating the input and output over an interval large compared to T_m. Sometimes the adjustment is based on estimated input parameters ξ_x and ξ_v, and these parameters are defined and measurable only over an interval of time long enough to contain many fluctuations of x_d and v. As the servo bandwidth must be larger than that of x_d, and is approximately equal to the pertinent bandwidth of v, the required measuring interval for ξ_x and ξ_v is again many times T_m.

Relatively Slow Systems [5], [6]

When the variation of ξ is relatively fast, the measuring interval needs to be shortened. Three factors which have not been considered in the previous section become important under the present case:

a) The finite time for measurement,

b) The effect of ζ in previous measuring intervals on the system especially on H and α of the present measuring interval

c) The statistical nature of the variations of ξ.

Mathematically, a) requires that the adaptive loop be a discrete system. Also, since the value of ζ at the present interval is determined by the measurements made at previous intervals, there is a time delay of one interval. The condition b) can be expressed as follows: Nearing a minimum value of H

$$H(\xi, \zeta, n) = H_0(n) + \sum_{k,k'=0}^{k,k'=\infty} [\zeta(n-k) - \zeta_0(n-k)]'a(k,k')[\zeta(n-k') - \zeta_0(n-k')] \quad (27)$$

$$\alpha(\xi, \zeta, n) = \alpha_0 + \sum_{k=0}^{k=\infty} b(k)[\zeta(n-k) - \zeta_0(n-k)] \quad (28)$$

There are many ways to account for the statistical nature of the variations of ξ. One most convenient way is to note that for each ξ, there is an optimum value ζ_0. When ξ is random, ζ_0 is also random. What really counts in designing the adaptive loop is the statistical nature of ζ_0. Without loss of generality, we may redefine ζ_0 as the variation from its mean value. Then the newly defined ζ_0 has zero mean, and its statistical property is represented by a spectral density $\Phi_{\zeta_0\zeta_0}(j\omega)$. It is noted that as ζ is a vector of m components, $\Phi_{\zeta_0\zeta_0}(j\omega)$ is a matrix of $m\times m$ elements.

Because there is a random statistical error, the accuracy of the measurement improves with the measuring interval. But if the measuring interval is made too long, the delay of one interval as mentioned under (a) becomes too much of a handicap. A compromise has to be made between the accuracy and timeliness of the measured results. Because of this inaccuracy, better results can generally be obtained by making an optimum design based on present as well as all previous measurements. For systems of type 1), the adjustments may be made in two ways:

a)
$$\eta_f(n+1) = \sum_{k=0}^{k=\infty} f(k)\eta(n-k) \quad (29)$$

$$\zeta = \zeta(\eta_f) \quad (30)$$

b) Alternatively ζ is computed first, and the actual adjustments are made according to a filtered ζ_f:

$$\zeta(n) = \zeta(\eta(n)) \quad (31)$$

$$\zeta_f(n+1) = \sum_{k=0}^{k=\infty} f(k)\zeta(n-k) \quad (32)$$

The two cases are represented in the block diagrams of Fig. 5(a) and (b).

For systems of types 2) and 3), the adjustments are based on all previously measured $\partial H/\partial\zeta$ and $\alpha-\alpha_0$. Instead of (12) and (25) the adjustment equations are

$$\zeta(n+1) - \zeta(n) = -\sum_{k=0}^{k=\infty} f(k)\left(\frac{\partial H}{\partial\zeta}\right)_{n-k} \quad (33)$$

$$\zeta(n+1) - \zeta(n) = -\sum_{k=0}^{k=\infty} f(k)(\alpha(n-k) - \alpha_0) \quad (34)$$

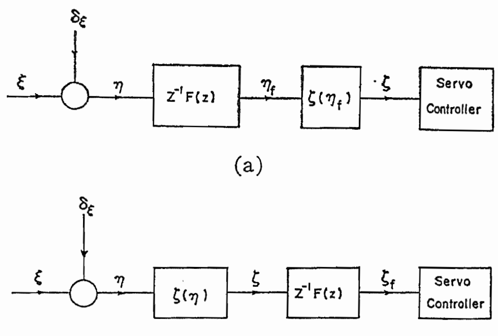

Fig. 5. Block diagrams of open adaptive loop control based on best estimates.

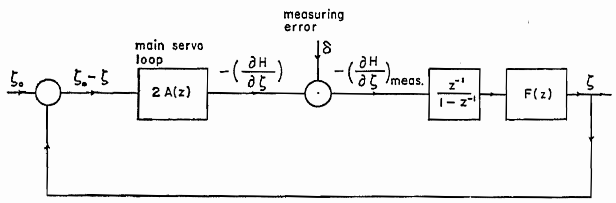

Fig. 6. Block diagram of peak-seeking system.

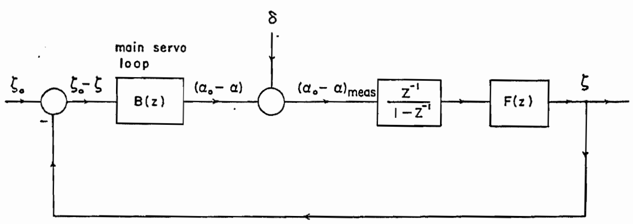

Fig. 7. Block diagram of adaptive system with intermediate parameters.

where $(\partial H/\partial \zeta)$ represents the gradient of H at the $(n-k)$th interval.

In terms of z transform, $F(z)$, $\zeta(z)$, and $n(z)$ are defined as

$$F(z) = \sum_{k=0}^{k=\infty} f(k) z^{-k}$$

$$\zeta(z) = \sum_{k=0}^{k=\infty} \zeta(k) z^{-k}$$

$$\eta(z) = \sum_{k=0}^{k=\infty} \eta(k) z^{-k}$$

Multiplying (29) by $z^{-(n+1)}$ and summing over n give

$$\eta_f(z) = z^{-1}F(z)\eta(z) \tag{29a}$$

Similarly (32) becomes

$$\zeta_f(z) = z^{-1}F(z)\zeta(z) \tag{32a}$$

Differentiating (27) with respect to $\zeta(n)$ gives

$$\frac{\partial H}{\partial \zeta(n)} = 2\sum_{k=0}^{k=\infty} a(0, k)[\zeta(n-k) - \zeta_0(n-k)] \tag{35}$$

Equation (35) can be rewritten in terms of z transform,

$$\frac{\partial H}{\partial \zeta}(z) = 2A(z)[\zeta(z) - \zeta_0(z)] \tag{36}$$

where

$$A(z) = \sum_{k=0}^{k=\infty} z^{-k} a(0, k) \tag{37}$$

Multiplying (33) by $z^{-(n+1)}$ and summing over n give

$$(1 - z^{-1})\zeta(z) = -z^{-1}F(z)\frac{\partial H}{\partial \zeta}(z)$$

$$\zeta(z) = -\left(\frac{z^{-1}}{1 - z^{-1}}\right)F(z)\frac{\partial H}{\partial \zeta}(z) \tag{38}$$

Similarly (28) and (34) give

$$(\alpha - \alpha_0)(z) = B(z)[\zeta(z) - \zeta_0(z)] \tag{39}$$

$$\zeta(z) = -\left(\frac{z^{-1}}{1 - z^{-1}}\right)F(z)(\alpha - \alpha_0)(z) \tag{40}$$

where

$$(\alpha - \alpha_0)(z) = \sum_{k=0}^{k=\infty} (\alpha(k) - \alpha_0) z^{-k} \tag{41}$$

It is to be noted that for actual systems, the measured values of $\partial H/\partial \zeta$ and $\alpha - \alpha_0$ must be used in (33) and (34). They differ from the true values by the respective measuring errors.

Summarizing the above these equations are represented by the block diagrams of Fig. 6 and Fig. 7, respectively.

In designing the adaptive loop, the function to be selected is $F(z)$ in all cases. The problem is reduced to one of minimizing mean square error with given data on input spectral densities. As the solution is well-known, it will not be discussed here. The following is a table of required input data and the quantities to be minimized for each case:

System	Input Data, Spectral Densities	Error to be Minimized
Fig. 5a	ξ, δ_ξ	$\overline{(\eta_f - \eta)^2}$
Fig. 5b	ζ_0, $\zeta - \zeta_0$	$\overline{(\zeta_f - \zeta_0)^2}$
Fig. 6	ζ_0, δ	$\overline{(\zeta_0 - \zeta)^2}$
Fig. 7	ζ_0, δ	$\overline{(\zeta_0 - \zeta)^2}$

Sometimes it is possible to make continuous measurements on certain intermediate parameters [7]. The system can be considered as a special case with the sampling interval approaching zero, and z transform approaching Laplace transform [8].

Fast Adaptive Systems $T_m \gg T_a$

In some control systems, the time required for measurement of ξ accurately is very short, and x and x_d do not change much in such a short interval. Consequently, ξ is known most of the time. The system is still different from a time-varying system because $\xi(t)$ is random and its variations cannot be predicted beforehand. For simplicity, it is assumed that $\xi(t)$ is completely known at t_1 for $t \leq t_1$.

There are two different situations.

1) ξ varies appreciably within T_m, and in fact may go through many fluctuations in T_m: A theory of optimum control of systems of this type has not been worked out. However, there is a widely adapted practice, the model reference scheme [9]. The model is usually so selected that its response can be duplicated by some choice of u most of the time. Then the system is forced to duplicate the response of the model. Figuratively, let it be assumed that the attainable ranges of $f(x, u, v, \xi_p)$ for different values of ξ_p are the ellipse, the rectangle, and the triangle as shown in Fig. 8. Then the common shaded area is selected as the range of $f(x, u, v, \xi_p)$ of the reference model, and an optimum controller is designed accordingly.

There are many ways of implementing the model reference idea. Two examples are illustrated in Fig. 9(a) and (b). In Fig. 9(a) the input x_d is applied to both the model and the control system. The outputs are compared, and the difference or error is fed back to the control system to force x to as close to x_m as possible. In Fig. 9(b) the model response is linear and can be represented by the transfer function $H_m(s)$. Then $1/H_m(s)$ is introduced in the feedback loop. The success of these methods depends on the loop gains of the two loops illustrated by the heavy lines. Many methods are devised to push the loop gain to the limit [10].

2) ξ varies very slowly and is approximately constant within T_m: Theoretically one can regard ξ as constant and determine u for optimum performance. In an adaptive system the optimum control law varies continuously as ξ is varied. While the concept is simplicity itself, it is difficult to implement with practical hardware.

Alexandro [11] has shown that under certain conditions two systems can be governed by the same dimensionless control law, and one need only to change the proportionality constants between measured state variables and the dimensionless state variables used in the control law. The conditions are as follows: Both systems can be described by the same transfer function

$$\frac{Y(s)}{U(s)} = \frac{\sum_{k=0}^{k=m'} a_k s^{m'-k}}{s^n \prod_{i=1}^{i=m} (1 + T_i s)} \tag{42}$$

$$| u(t) | < u_m \tag{43}$$

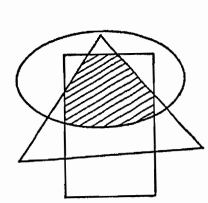

Fig. 8. Control range of a model reference system.

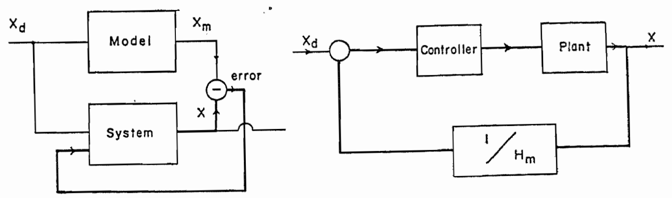

Fig. 9. Two ways of implementing a model reference system.

where Y is the output, and $m' < m+n$. The two systems may have completely different u_m, m' and a_k, $k=0, 1, \cdots, m'$. But m and n must be the same, and the constants T_i must vary proportionately.

The proof is as follows: Let a dimensionless time τ and time constants λ_i be defined as

$$\tau = t/T_1$$

$$\lambda_i = T_i/T_1 \qquad i = 1, 2 \cdots m$$

The Laplace transform variable s is replaced by the new variable $\sigma \equiv T_1 s$. Equation (42) becomes

$$\frac{Y(\sigma)}{U(\sigma)} = \frac{T_1^{n-m'} \sum_{k=0}^{k-m'} a_k T_1^k \sigma^{m'-k}}{\sigma^n \prod_{i=1}^{i=m} (1 + \lambda_i \sigma)} \tag{44}$$

By partial-fractioning (44), it is readily shown

$$y(\tau) = \sum_{k=1}^{k=n+m} b_k x_k(\tau) \tag{45}$$

where

$$\begin{aligned} \frac{dx_k(\tau)}{d\tau} &= x_{k+1}(\tau) \quad k - 1, 2 \cdots n - 1 \\ \frac{dx_n(\tau)}{d\tau} &= \frac{u}{u_m} \\ \lambda_i \frac{dx_{n+i}}{d\tau} &= -x_{n+i} + \frac{u}{u_m} \quad i = 1, 2, \cdots m \end{aligned} \tag{46}$$

While the b_k's are different for the two systems, the λ_i's are identical. The same set of dimensionless state variables x_k obeying the same set of differential equations (46) applies to both systems. Consequently the same switching boundaries for reducing (x_1, xi, $\cdots$, x_{n+m}) to zero in minimum time can be used for both systems.

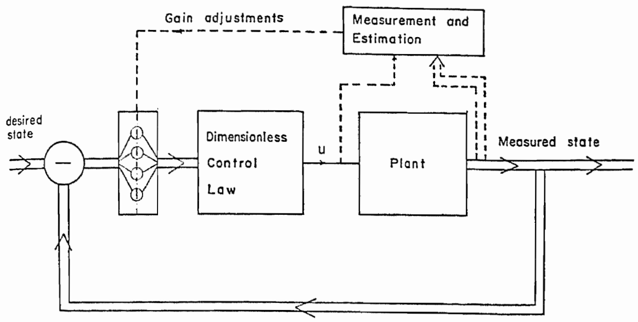

Fig. 10. Block diagram of an instantaneously optimum adaptive system.

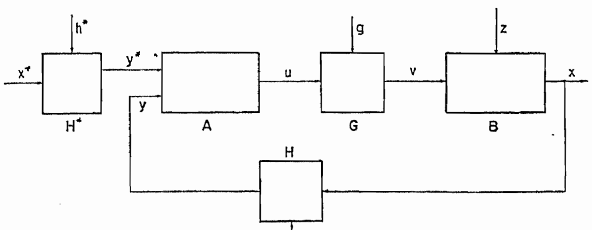

Fig. 11. An adaptive system illustrating dual-control theory.

Obviously the above discussion applies to the situation where a_k, u_m and T_1 vary at random and T_i's vary proportionately with T_1. The switching surfaces in x space remain unchanged, but the point in x space representing the state of the system drifts as the plant parameters a_k, u_m and T_1 are varied. The adaptive system is illustrated in Fig. 10. The system obeys the optimum control law at every instant, and may be referred to as an instantaneously optimum system. It is not as widely applicable as the model reference system. But wherever it can be applied, it is a faster system than the latter.

Another way of looking at Alexandro's system is that the controlled system is made to conform to a dimensionless model rather than an absolute model. Since the unit of time is now adjustable it is always set at the shortest practicable value to obtain fast response.

The Case With $T_m \approx T_a$

In case T_m is of the same order of magnitude as T_a, very little can be done in the way of approximations or physical intuition. It seems that the only recourse left is a brute force formulation of the problem and leave the question of a solution to wherever it may be. In Feldbaum's dual control theory [12], the author pointed out that each input to the plant serves two functions; control of the plant to yield the desired output, and testing of the plant so that the knowledge gained about the plant can be used to reduce future error.

Figure 11 illustrates the system studied by Feldbaum. The desired output is x^* and the actual output is x. A and B represent the controller and controlled plant respectively. H, H^* and G represent the contamination of the output, input and control signals during transmission, and h, h^*, and g are random noises in the links. The Markov variable z may be a vector and represents both the load disturbance and the changing plant parameters,

$$x = F(v, z) \tag{47}$$

For instance one may write

$$F(v, z) = z_1 v + z_2$$

then z_1 is the variable gain and z_2 is the load disturbance. The problem is to determine A for least expected value of the square error $<(x^* - x)^2>$ or some other cost criterion. The difficulty lies in the fact that each different output u gives a different estimate of the Markov states x^* and z are in. The only general result obtained by Feldbaum is that if B is memoryless, that is the nth sample of x is a function of the nth samples of v and z only, then A is a deterministic controller in the sense that the best u to use is completely determined by the present and past samples of y^* and y. But the best u as a functional of y^* and y is very difficult to calculate except for a few simple cases.

Conclusion

Many different types of adaptive control systems can be classified according to the ratio of two time constants, T_a of the adaptive loop, and T_m of the main servo loop. If $T_m < T_a$, the system can be analyzed by taking sample or time average over the input and load disturbances first. Peak-seeking and estimating, adjusting systems belong to this type. If $T_m \gg T_a$, the changing parameters of the plant are practically known. Model reference systems belong to this type. The most difficult case is the one with $T_m \approx T_a$, because no general approximation can be made. Feldbaum's dual control theory applies to this case but it has yielded very little practical results.

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