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ON THE DIFFERENTIAL APPROXIMATION

IN RADIATIVE TRANSFER

by

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ON THE DIFFERENTIAL APPROXIMATION
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1. INTRODUCTION

Considerable emphasis has recently been placed upon use of the differential approximation in dealing with radiative transfer through media which absorb and emit thermal radiation. This method consists of replacing the integral formulation for the radiation flux vector by an approximate differential equation [1-5]²⁾. For the one-dimensional plane layer this is equivalent to both the Milne-Eddington approximation [1, 4, 5] and the exponential kernel approximation [2, 3, 5].

The applicability of the differential approximation to the one-dimensional plane layer is well known. The implied utility of the approximation, however, is to more general problems, since an exact expression for the radiation flux vector is not easily formulated for

1) This work was sponsored by the U. S. Air Force, Office of Aerospace Research, Aerospace Research Laboratories, under Contract AF 33(615)-3104.

2) Numbers in brackets refer to References.

multi-dimensional geometries. The purpose of the present note is to explore the applicability of the differential approximation to geometries other than the one-dimensional plane layer.

2. DIFFERENTIAL APPROXIMATION

For simplicity a nonscattering gray medium having an absorption coefficient that is independent of temperature will be assumed. From the differential approximation, one has [1-5]³⁾

$$\frac{\partial q_j}{\partial x_i \partial x_j} - 3\kappa^2 q_i = 4\sigma \frac{\partial T^4}{\partial x_i} \quad (1)$$

where κ , σ , T and q_i are, respectively, the volumetric absorption coefficient, Stefan-Boltzmann constant, absolute temperature, and radiation flux components. Equation (1) constitutes a differential equation for the radiation flux vector, and for a given problem this must be coupled with an equation expressing overall conservation of energy.

It remains to specify boundary conditions for Equation (1). This has been accomplished by Cheng [2], and letting x_1 be a coordinate normal to a given gray surface, the boundary condition at that surface

3) It is worth mentioning that this three-dimensional equation actually follows from Rosseland's original derivation [6], even though his final result is restricted to the optically thick limit.

is ⁴⁾

$$B(x_2, x_3) = \sigma T^4(0, x_2, x_3) - \frac{1}{4\kappa} \left(\frac{\partial \mathcal{Q}_j}{\partial x_j} \right)_{x_1} + \frac{1}{2} \mathcal{Q}_1(0, x_2, x_3) \quad (2)$$

where $B(x_2, x_3)$ is the total radiant energy (radiosity) leaving the surface. In Cheng's application a black surface was assumed, such that $B(x_2, x_3) = \sigma T_w^4(x_2, x_3)$, where T_w is the surface temperature. Equation (2) may, however, easily be applied to nonblack surfaces by noting that

$$B(x_2, x_3) = \epsilon_w \sigma T_w^4(x_2, x_3) - (1 - \epsilon_w) [B(x_2, x_3) - \mathcal{Q}_1(0, x_2, x_3)] \quad (3)$$

where ϵ_w is the surface emittance. Upon combining Equations (2) and (3), the applicable boundary condition becomes

$$\left(\frac{1}{\epsilon_w} - \frac{1}{2} \right) \mathcal{Q}_1(0, x_2, x_3) - \frac{1}{4\kappa} \left(\frac{\partial \mathcal{Q}_j}{\partial x_j} \right)_{x_1=0} = \sigma T_w^4(x_2, x_3) - \sigma T^4(0, x_2, x_3) \quad (4)$$

For purposes of comparison with other results, it will be convenient to recast Equations (1) and (4) in terms of the optical coordinates $\tau_i = \kappa x_i$, and Equations (1) and (6) become, respectively

4) An alternate and somewhat more direct means of obtaining Equation (2) is presented by Wang [7].

$$\frac{\partial^2 q_i}{\partial \tau_i \partial \tau_j} - 3q_i = 4\sigma \frac{\partial T^4}{\partial \tau_i} \quad (5)$$

$$\left(\frac{1}{\epsilon_w} - \frac{1}{2}\right) q_i(0, \tau_2, \tau_3) - \frac{1}{4} \left(\frac{\partial q_i}{\partial \tau_j}\right)_{\tau_i=0} = \sigma T_w^4(\tau_2, \tau_3) - \sigma T^4(0, \tau_2, \tau_3) \quad (6)$$

Furthermore, let L be a characteristic dimension of the system, and correspondingly the characteristic optical thickness is $\tau_0 = \kappa L$, while $\tau_i = 0(\tau_0)$.

When continuum molecular conduction is included in the formulation of a given problem, the requirement of continuity of temperature at a bounding surface results in the right side of Equation (6) being zero. Conversely, if conduction is neglected, the right side of Equation (6) represents the well-known temperature jump at a surface. From Equation (5) it is readily seen that, with respect to the right side of Equation (6), the terms on the left side of Equation (6) have the following orders of magnitude:

$$\left(\frac{1}{\epsilon_w} - \frac{1}{2}\right) q_i(0, \tau_2, \tau_3) = O\left(\frac{1}{\tau_0}\right)$$

$$\frac{1}{4} \left(\frac{\partial q_i}{\partial \tau_j}\right)_{\tau_i=0} = O\left(\frac{1}{\tau_0^2}\right)$$

while the term $\partial^2 q_i / \partial \tau_i \partial \tau_j$ appearing in Equation (5) is of $O(1/\tau_0^2)$ compared to q_i .

It is apparent that for $\tau_0 \rightarrow \infty$ Equation (5) reduces to the

Rosseland equation

$$q_i = -\frac{4\sigma}{3} \frac{\partial T^4}{\partial \tau_i} \quad (7)$$

while Equation (6) requires continuity of temperature at the boundaries.

This is the conventional optically thick limit. On the other hand, if terms of $O(1/\tau_0)$ are retained, Equation (5) again reduces to Equation (7) while Equation (6) becomes

$$\sigma T_w^4(\tau_2, \tau_3) - \sigma T^4(0, \tau_2, \tau_3) = \left(\frac{1}{\epsilon_w} - \frac{1}{2}\right) q_i(0, \tau_2, \tau_3) \quad (8)$$

Equations (7) and (8) constitute the Rosseland equation with jump boundary conditions. Such a case has previously been considered by Deissler [8], except that terms of $O(1/\tau_0^2)$ were included in the formulation of the boundary conditions. If only first-order terms are retained in Deissler's equation, it coincides exactly with Equation (8).

When terms of $O(1/\tau_0^2)$ are retained, Equations (5) and (6) remain as is. These differ from Deissler's second-order formulation in two ways. First, Deissler's method utilizes Equation (7) rather than Equation (5); and second, the term of $O(1/\tau_0^2)$ in Deissler's boundary condition is

$$-\frac{3}{16} \left(\frac{\partial q_i}{\partial \tau_i} \right)_{\tau_i=0} - \frac{3}{16} \left(\frac{\partial q_j}{\partial \tau_j} \right)_{\tau_j=0}$$

instead of

$$-\frac{1}{4} \left(\frac{\partial \mathcal{E}_j}{\partial \tau_j} \right)_{\tau_j=0}$$

Since Equations (5) and (6) include terms of $O(1/\tau_0^2)$, it would appear that the differential approximation constitutes a second-order departure from optically thick radiation. However, it is often stated (somewhat incorrectly) that Equations (5) and (6) reduce to the correct optically thin limit, and if both thick and thin limits are achieved, then the differential approximation is possibly applicable for all optical thicknesses. Actually, the applicability of Equation (5) to the optically thin limit is quite restricted, and this will be illustrated in the next section.

3. OPTICALLY THIN LIMIT

For optically thin conditions ($\tau_0 \ll 1$), Equation (5) reduces to

$$\frac{\partial^2 \mathcal{E}_j}{\partial \tau_j \partial \tau_j} = 4\sigma \frac{\partial T^4}{\partial \tau_j} \quad (9)$$

and it follows that

$$\frac{\partial \mathcal{E}_j}{\partial \tau_j} = 4\sigma T^4 - G \quad (10)$$

where G is an integration constant. One may note that Equation (10) is of the same form as the general equation representing conservation of radiant energy (see for example Equation (114) of [5]).

It remains to satisfy either Equation (2) or Equation (6) at each surface, and substitution of Equation (10) into Equation (2) yields

$$G = 4B(\tau_2, \tau_3) - 2q_1(0, \tau_2, \tau_3) \quad (11)$$

Recall further that for optically thin conditions the radiation heat flux at each surface can be evaluated as if the medium were completely transparent (see for example Section 7-5 of [9]). Thus $B(\tau_2, \tau_3)$ and $q_1(0, \tau_2, \tau_3)$ for each surface may be determined from conventional enclosure theory [9].

For an optically thin medium bounded by N surfaces, Equation (11) yields N equations for the value of G at each surface. However, in order for Equation (10) to be the optically thin form of the differential approximation, G must be constant throughout the medium. This of course imposes a specific restriction upon the applicability of the differential approximation to optically thin conditions; that is, Equation (11) must yield the same value of G at each surface. In addition, inspection of Equations (3) and (11) illustrate that the surfaces must be isothermal and that the radiosity be uniform over each surface.

As pointed out by Vincenti and Kruger [4], there is one case for

which the differential approximation does reduce to the optically thin limit for all geometrical configurations. This is the emission-dominated limit, for which $G = 0$. The physical requirement is that the temperatures of all bounding surfaces be much smaller than the temperature of the medium. For problems that are not emission dominated, the foregoing restriction concerning Equation (11) must be imposed. This restriction will now be illustrated for the three geometries shown in Fig. 1, and for each of these the surfaces are taken to be isothermal.

4. ILLUSTRATIVE EXAMPLES

The first example is the plane layer. Recall again that in the optically thin limit the radiation heat flux at each surface is evaluated as if the medium were completely transparent. Thus

$$q_{w1} = -q_{w2} = B_1 - B_2$$

where the subscripts 1 and 2 now refer to the specific surfaces.

Upon employing Equation (11) at each surface, one has

$$G_{w1} = G_{w2} = 2B_1 + 2B_2$$

The requirement on G is thus satisfied, such that the differential

approximation reduces to the correct optically thin limit. This is evidently the reason why the differential approximation consistently gives good results for the plane layer geometry.

The second case consists of an absorbing-emitting medium contained within an infinitely long circular cylinder. Since there is a single isothermal surface, the restriction pertaining to Equation (11) is obviously satisfied. This conclusion, in fact, applies to any enclosure for which the entire enclosure surface is isothermal.

To give a specific example involving the cylindrical geometry, consider that there is uniform heat generation per unit volume Q within the medium and that the surface is black. It is easily shown that the differential approximation yields the result

$$\frac{q_w}{\sigma(T_w^4 - T_c^4)} = \frac{1}{3\tau_0/16 + 1/2 - 1/\tau_0} \quad (12)$$

where T_c is the temperature at the centerline and $\tau_0 = 2\kappa r_0 = \kappa D$. It is interesting to note that this differs from Deissler's slip solution only in that the last term in the denominator is $1/\tau_0$ rather than Deissler's $9/8\tau_0$. A comparison of Equation (12) with Howell and Perlmutter's Monte Carlo results for the same problem [10] is shown in Fig. 2, and this is quite comparable to Deissler's similar comparison [8]. The maximum departure from the Monte Carlo results is approximately six percent, and this lends support to the premise that when the differential approximation reduces to the correct

optically thin limit, it is in turn a useful approximation for all optical thicknesses.

The final example concerns two infinitely long concentric cylinders. In the transparent limit one finds from enclosure theory that

$$q_{w1} = B_1 - B_2 \quad , \quad q_{w2} = \frac{r_1}{r_2} (B_2 - B_1)$$

and from Equation (11)

$$G_{w1} = 2B_1 + 2B_2$$

$$G_{w2} = 2\left(\frac{r_1}{r_2}\right)B_1 + 2\left(2 - \frac{r_1}{r_2}\right)B_2$$

With the exception of equal surface temperatures⁵⁾, the requirement that G have the same value at each surface is not satisfied, and this is an example for which the differential approximation does not yield the optically thin limit. Furthermore, it is apparent that the difference between G_{w1} and G_{w2} increases with increasing values of r_2/r_1 .

To give a specific illustration, consider that the surfaces are black and that radiation is the sole mode of energy transfer within the medium. For this case the differential approximation yields the result

5) For $T_{w1} = T_{w2}$, one has in the transparent limit an isothermal enclosure, such that $B_1 = B_2 = \sigma T_w^4$.

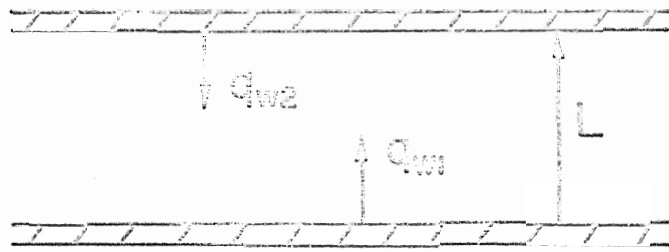
$$\frac{\sigma(T_1^4 - T_2^4)}{q_w} = \frac{3\tau_0}{4(r_2/r_1 - 1)} \ln\left(\frac{r_2}{r_1}\right) + \frac{1}{2}\left(\frac{r_1}{r_2} + 1\right) \quad (13)$$

This coincides exactly with Deissler's first-order slip solution.

A comparison of Equation (13) with other results is shown in Fig. 3. The limit $r_2/r_1 \rightarrow 1$ corresponds to the plane layer, and the excellent agreement between the differential approximation and the exact solution of Heaslet and Warming [11] further illustrates the applicability of the differential approximation to the plane layer geometry. The other comparisons are with the Monte Carlo results of Howell and Perlmutter [10]. As would be expected, the accuracy of the differential approximation diminishes with decreasing optical thickness, and this is more pronounced for large r_2/r_1 values. It is worth mentioning that Deissler's second-order slip solution gives better agreement with the Monte Carlo results than does the differential approximation (see Fig. 5 of [8]).

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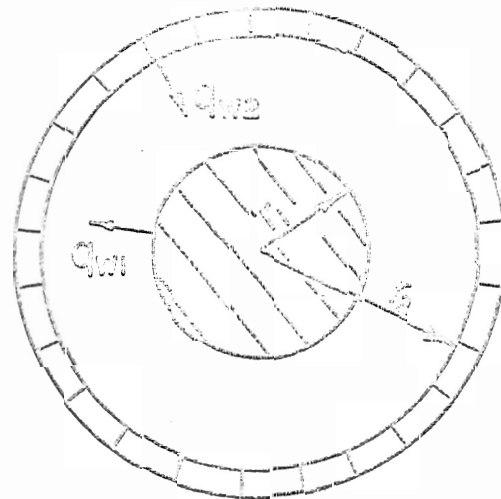
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(a) Plane Layer



(b) Circular Cylinder



(c) Concentric Cylinders

Fig. 1 Illustrative Geometries

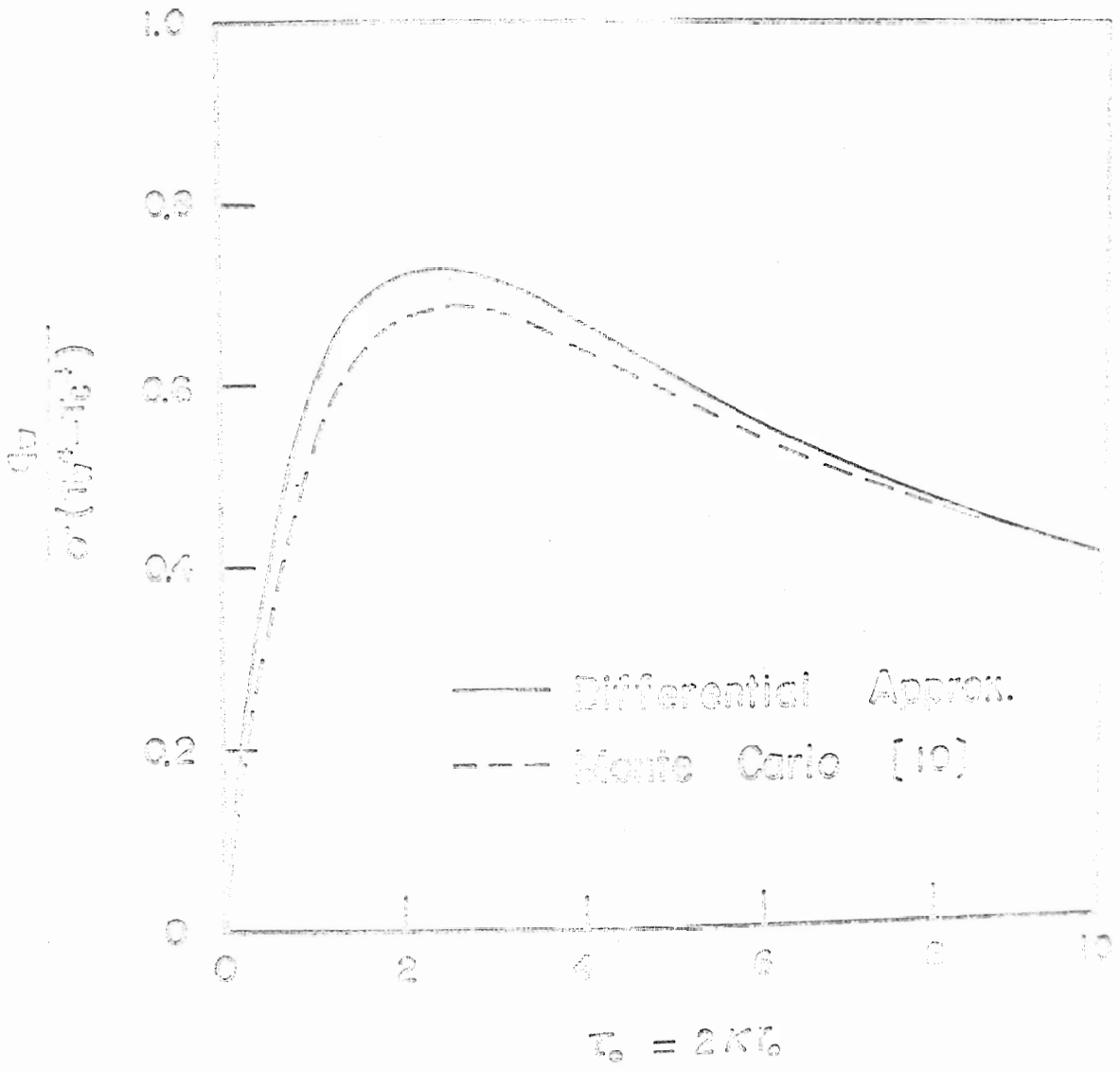


Fig. 2 Comparison of results for a circular cylinder with uniform internal heat generation.

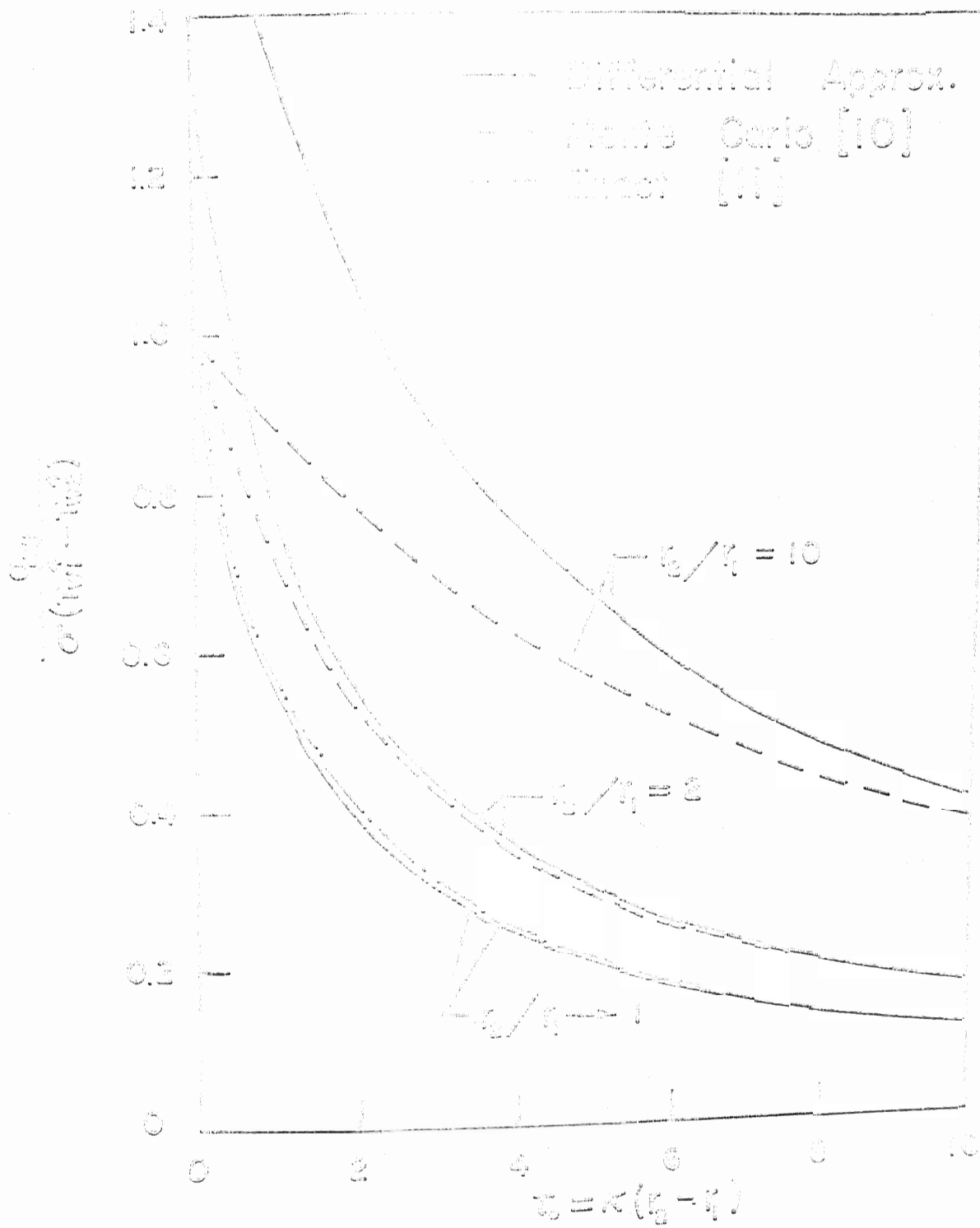


Fig. 3 Comparison of results for two concentric cylinders.