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ANALYSIS OF DISTRIBUTED PARAMETER NETWORKS -

A GENERAL METHOD

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Daniel Dicker

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ANALYSIS OF DISTRIBUTED PARAMETER NETWORKS - A GENERAL METHOD

DANIEL DICKER

State University of New York at Stony Brook
Stony Brook, New York

ABSTRACT

A general method is presented for obtaining approximate analytical solutions for distributed parameter networks where the parameters may be functions of the spatial variable. By means of this technique, an explicit approximate solution for the output of a rather general network is presented. For those cases where exact separable solutions exist the method yields the correct solutions.

INTRODUCTION

The distributed network considered in this paper is a fairly general one, and characterized by the "transmission line" equations [1] [2]

$$\begin{aligned} -\partial e(x,t)/\partial x &= r i(x,t) + l \partial i(x,t)/\partial t \\ -\partial i(x,t)/\partial x &= g e(x,t) + c \partial e(x,t)/\partial t \end{aligned} \quad (1)$$

For certain limiting cases general solutions of (1) are available, e.g. high-frequency lines and submarine cables, where the governing partial differential equations reduce to one of the standard types of applied mathematics - the wave equation or the heat equation. Examples of such solutions may be found in the references cited. For the full equations (1) and where the termination is neither open circuited nor short circuited, the boundary value problem is not separable, hence an analytical solution is generally not available. Furthermore, in the event the line parameters r , l , g , c are nonconstant the difficulty is compounded so that solutions for specific problems must be obtained numerically either by analog or digital computer.

A general method has been developed for obtaining approximate analytical solutions for (1) where the parameters may be dependent upon x but independent of time. By manipulating the differential equation it will be shown that the boundary-value problem can be related to a Sturm-Liouville system. The latter is then solved by means of the Bubnov-Galerkin method to yield the current and voltage as functions of x and t in terms of an infinite series, the coefficients of which are obtainable from a system of linear algebraic equations. By truncating the algebraic system approximations to any degree of precision may then be obtained. For those cases where exact solutions exist the algebraic system is easily solved in closed form.

REDUCTION OF BOUNDARY-VALUE PROBLEM TO STURM-LIOUVILLE SYSTEM

Consider the first equation of (1). Viewing it as a first-order nonhomogeneous differential equation on the current with time as the independent variable it is easily solved by standard techniques to yield

$$i(x,t) = \exp(-rt/l) \int_0^t \exp(r\tau/l) \partial e(x,\tau) / \partial x d\tau + C_1 \exp(-rt/l) \quad (2)$$

The coefficient C_1 may be a function of x . Although it is not necessary to assume initial quiescent conditions for this method, as a convenience in this study it will be assumed that $e(x,0)$ and $i(x,0)$ are zero. Under this assumption C_1 vanishes and the formal solution for the current becomes

$$i(x,t) = \exp(-r(x)t/l(x)) \int_0^t \exp(r(x)\tau/l(x)) \partial e(x,\tau) / \partial x d\tau \quad (3)$$

Next substituting for the current from (3) into the second equation of (1) yields

$$-\partial / \partial x \left[\exp(-r(x)t/l(x)) \int_0^t \exp(r(x)\tau/l(x)) \partial e(x,\tau) / \partial x d\tau \right] = g(x)e(x,t) + c(x) \partial e(x,t) / \partial t \quad (4)$$

which is an integro-partial-differential equation on the variable $e(x,t)$. Applying next the Laplace Transformation to (4) gives

$$\frac{d}{dx} \frac{dE(x;s)/dx}{s l(x) + r(x)} = [g(x) + sc(x)] E(x;s) \quad (5)$$

$$\text{where } \mathcal{L}[e(x,t)] \equiv \int_0^\infty \exp(-st) e(x,t) dt \equiv E(x;s) \quad [3]$$

The problem has thus been reduced to the solution of the self-adjoint differential equation (5). To prove that the original boundary value problem is equivalent to a Sturm-Liouville system requires a brief discussion of the boundary conditions, as follows.

Reduction of Boundary Conditions

Consider the terminal end of a transmission line consisting of R , L and C elements. The line voltage and current at the end of the line, $x = b$, are designated $e(b,t)$ and $i(b,t)$ respectively. Consider next an arbitrary path across the line in the termination circuit. It is easily shown that the following relationship is satisfied

$$e(b,t) = \sum_{k=1}^n [I_{jk}/C_k + R_k \dot{I}_{jk} + L_k \ddot{I}_{jk}] \quad (6)$$

where I_{jk} is the current in the k^{th} element of j^{th} path; and $j=1,2,\dots,m$, where m is the number of such paths. Hence there are " m " equations (6). Furthermore, summing currents at each junction supplies the requisite additional number of equations required to reduce the system to a single differential equation involving $e(b,t)$ and $i(b,t)$ only, viz.,

$$(\alpha_{n+1} d^{n+1}/dt^{n+1} + \alpha_n d^n/dt^n + \dots + \alpha_0) i(b,t) = (\beta_n d^n/dt^n + \beta_{n-1} d^{n-1}/dt^{n-1} + \dots + \beta_0) e(b,t) \quad (7)$$

Combining (7) above with the first equation of (1) evaluated at $x = b$ gives

$$-(\alpha_{n+1}D^{n+1} + \alpha_n D^n + \dots + \alpha_0) \partial e(b,t) / \partial x = (r(b) + \ell(b)D) (\beta_n D^n + \dots + \beta_0) e(b,t) \quad (8)$$

where $D \equiv d/dt$. Finally, applying the Laplace Transformation to (8) gives for the appropriate boundary condition at $x = b$

$$-h(s) dE(b;s) / dx = E(b;s) \quad (9)$$

$$\text{where } h(s) = [\alpha_{n+1} s^{n+1} + \alpha_n s^n + \dots + \alpha_0] / [\ell(b) \beta_n s^{n+1} + (r(b) \beta_n + \ell(b) \beta_{n-1}) s^n + \dots + r(b) \beta_0] \quad (10)$$

At the sending end the most usual boundary condition is the specification of the voltage,

$$e(0,t) = f(t) \quad (11)$$

the Laplace transformation of which is

$$E(0;s) = \bar{f}(s) \quad (12)$$

The boundary-value problem consisting of (5), (9) and (12) may be identified with a Sturm-Liouville eigenvalue problem ^{*}[4] and its solution, which can be shown to exist and to be unique, expressed in terms of an expansion of the eigenfunctions of the associated eigenvalue problem. However, there are relatively few solutions available in terms of tabulated functions, and these correspond to very special functions for the line parameters. ^{**} In the following section the Bubnov-Galerkin technique will be described, which will be employed to generate a convergent approximate eigenfunction expansion with arbitrary line parameters.

* The boundary-value problem is also of the Sturm-Liouville type when the boundary condition at $x = 0$ is similar to (9), i.e. $h'(s) dE(0;s) / dx = E(0;s)$, and, of course, when the boundary condition at $x = b$ is similar to (12), i.e. $E(b;s) = E(b;s) = \bar{f}'(s)$.

** One nontrivial example would be, for instance, $\ell(x) = \text{const.}$, $r(x) = \text{const.}$, $g(x) = \text{const.}$, $c(x) = \cos 2x$, the resulting solution for $E(x;s)$ being an expansion in Mathieu functions.

SOLUTION BY THE BUBNOV-GALERKIN METHOD

Bubnov-Galerkin Method

The following is a brief outline of the technique; for a detailed mathematical presentation the reader is referred to Mikhlín [5], and numerous examples in the study of differential equations and continuum mechanics may be found in Kantorovich and Krylov [6]. Consider a differential equation

$$Au = q \quad (13)$$

where A is a differential operator, u the dependent variable and q a nonhomogeneous term. The approximate solution is assumed to be of the form

$$u^{(n)} = \sum_{k=1}^n \rho_k \varphi_k \quad (14)$$

where the φ_k are elements of a complete set and ρ_k are coefficients. It is necessary that $u^{(n)}$ satisfy the prescribed boundary conditions. The Bubnov-Galerkin method consists in equating to zero the scalar products

$$\int_{\tau} (Au^{(n)} - q) \varphi_k d\tau, \quad k = 1, 2, \dots, n$$

where τ is the domain of definition of u , and solving the resulting linear algebraic simultaneous system for the coefficients, ρ_k .

Solution of the Network Problem

For the problem at hand, (5), (9), (12), it is convenient to introduce a new variable, u , such that

$$E = \lambda + u \quad (15)$$

where λ is a term which satisfies the given boundary conditions; u , therefore, satisfies homogeneous boundary conditions. After substituting (15) into (5), (9), (12), and in (5) carrying out the differentiations and multiplying by $(ls + r)^2$ there results as the problem on u

$$Au = -A\lambda \quad (16)$$

$$hdu/dx + u = 0 \quad \text{at } x = b \quad (17)$$

$$u = 0 \quad \text{at } x = 0 \quad (18)$$

where

$$A \equiv (ls+r)d^2/dx^2 - (l's+r')d/dx - (ls+r)^2(g+cs) \quad (19)$$

A simple choice for λ is

$$\lambda = \bar{f} [1 - x/(h - b)] \quad (20)$$

and a suitable collection of functions which are elements of a set complete in $(0, b)$ is

$$\varphi_1 = x + (h - b)x^2/b(2h - b), \quad \varphi_n = x(x - b)^n, \quad n > 1 \quad (21)$$

It is easily verified that (20) satisfies (9) and (12) and that each φ_k of (21) satisfies (17) and (18).

As is customary, define the scalar product of two functions, $v(x)$, $w(x)$ as

$$(v, w) \equiv \int_0^b v w dx \quad (22)$$

Similarly the energy product is

$$[v, w] \equiv \int_0^b w A v dx \quad (23)$$

where A is an operator. In terms of these definitions the simultaneous system of algebraic equations is given simply by

$$[u^{(n)} - \lambda, \varphi_k] = 0 \quad k = 1, 2, \dots, n \quad (24)$$

where

$$u^{(n)} = \rho_1 \varphi_1 + \rho_2 \varphi_2 + \dots + \rho_n \varphi_n \quad (25)$$

The approximate solution is obtained by selecting a value of n , solving for the coefficients ρ_k , and inverting $u^{(n)} + \lambda$ to obtain $e(x, t)$.

Of great importance in applications is the output, $e(b, t)$.^{*} An examination of the assumed solution for $E(x; s)$, (15), (25), (21), shows that the series terminates after the first term and that

$$E(b; s) = \lambda + \rho_1 \varphi_1(b; s) \quad (26)$$

which yields, for the output,

$$e(b, t) = \mathcal{L}^{-1}[\lambda + \rho_1 \varphi_1] \quad (27)$$

This last result is of importance as it is an explicit formula for the output, for a fairly arbitrary network, in terms of known functions λ and φ_1 as given by (20) and (21). Because of the nature of the termination circuit $h(s)$ is simply the ratio of two polynomials in s , the degree of each of which depends upon the complexity of the termination. Likewise, the coefficient ρ_1 is a ratio of polynomials in s . In general, the only nonpolynomial function of s which appears in (26) is the input $\bar{f}(s)$. Therefore the inverse Laplace transform is not difficult to obtain once the polynomials have been factored.

It is to be noted, however, that the coefficients, ρ_k , are so called "flexible" coefficients. The value of ρ obtained from a one term approximation will differ somewhat from ρ_1 obtained from an 'n' term approximation. Designate this latter coefficient $\rho_1^{(n)}$. It is known [5] that $\rho_1^{(n)}$ converges, i.e.

$$\rho_1^{(n)} \rightarrow \rho_1, \text{ and } \rho_k^{(n)} \rightarrow \rho_k \quad (28)$$

* When it is rather the current $i(b, t)$ that is desired, equation (3) together with the solution for $e(b, t)$ will provide the required result. Alternatively, the entire problem may be formulated in terms of $i(x, t)$.

The power of the Bubnov-Galerkin algorithm lies not only in (28) but also in that the solutions so constructed generally yield very close approximations to the exact solution, even when the number of terms included in the approximate solution (25) is small [7], [8].

Various approximations will be exhibited for the solution to (5), (9), (12). For a one term approximation

$$u^{(1)} = \rho_1^{(1)} \varphi_1 \quad (29)$$

where φ_1 is given by (21). Substituting in (24) gives

$$[u^{(1)}, \varphi_1] = [\lambda, \varphi_1]$$

and utilizing (29) there results

$$\rho_1^{(1)} = [\lambda, \varphi_1] / \|\varphi_1\|^2$$

where $\|\varphi\|$ is the so-called "energy norm" defined by $\|\varphi_k\| \equiv [\varphi_k, \varphi_k]$. Similarly the two term approximation, given by

$$u^{(2)} = \rho_1^{(2)} \varphi_1 + \rho_2^{(2)} \varphi_2$$

together with (24) gives the simultaneous system

$$[u^{(2)}, \varphi_1] = [\lambda, \varphi_1]$$

$$[u^{(2)}, \varphi_2] = [\lambda, \varphi_2]$$

from which it follows directly that

$$\rho_1^{(2)} = \frac{[\lambda, \varphi_1] \|\varphi_2\|^2 - [\lambda, \varphi_2] [\varphi_2, \varphi_1]}{\|\varphi_1\|^2 \|\varphi_2\|^2 - [\varphi_1, \varphi_2] [\varphi_2, \varphi_1]}, \quad \rho_2^{(2)} = \frac{\|\varphi_1\|^2 [\lambda, \varphi_2] - [\varphi_1, \varphi_2] [\lambda, \varphi_1]}{\|\varphi_1\|^2 \|\varphi_2\|^2 - [\varphi_1, \varphi_2] [\varphi_2, \varphi_1]}$$

In an entirely analogous manner the coefficients for the n^{th} approximation

$$u^{(n)} = \rho_1^{(n)} \varphi_1 + \rho_2^{(n)} \varphi_2 + \dots + \rho_k^{(n)} \varphi_k + \dots + \rho_n^{(n)} \varphi_n$$

are given by

$$\rho_k^{(n)} = \frac{\begin{vmatrix} \|\varphi_1\|^2 [\varphi_2, \varphi_1] & \dots & [\varphi_{k-1}, \varphi_1] [\lambda, \varphi_1] & [\varphi_{k+1}, \varphi_1] & \dots & [\varphi_n, \varphi_1] \\ [\varphi_1, \varphi_2] \|\varphi_2\|^2 & \dots & [\varphi_{k-1}, \varphi_2] [\lambda, \varphi_2] & [\varphi_{k+1}, \varphi_2] & \dots & [\varphi_n, \varphi_2] \\ [\varphi_1, \varphi_n] [\varphi_2, \varphi_n] & \dots & [\varphi_{k-1}, \varphi_n] [\lambda, \varphi_n] & [\varphi_{k+1}, \varphi_n] & \dots & \|\varphi_n\|^2 \end{vmatrix}}{\begin{vmatrix} \|\varphi_1\|^2 [\varphi_2, \varphi_1] & \dots & \dots & \dots & \dots & [\varphi_n, \varphi_1] \\ [\varphi_1, \varphi_2] \|\varphi_2\|^2 & \dots & \dots & \dots & \dots & [\varphi_n, \varphi_2] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ [\varphi_1, \varphi_n] [\varphi_2, \varphi_n] & \dots & \dots & \dots & \dots & \|\varphi_n\|^2 \end{vmatrix}} \quad (30)$$

It is clear that when it is possible to choose a complete set which is orthogonal in energy, (30) degenerates into a noncoupled system and the coefficients do not change from one approximation to the next, i.e.

$$\rho_k^{(n)} = \rho_k = [\lambda, \varphi_k] / [\varphi_k]^{*2}$$

The problems for which this is possible are the "separable" ones; however, these are the exceptions rather than the rule.

Finally, the solution may be written

$$e(x,t) = \mathcal{L}^{-1}[\lambda] + \sum_{k=1}^n \mathcal{L}^{-1}[\rho_k^{(n)} \varphi_k]$$

* These are, of course, the Fourier coefficients.

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