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TRANSMISSION OF OCEAN WAVES INTO BEACHES

by

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Introduction.

Coastal engineering - or near-shore oceanography - is concerned with, among many other scientific topics, the interaction of waves with real boundaries. Two groups of such problems, for example, which have received considerable attention within the last decade or so are diffraction of waves and breaking and run-up of waves on beaches. This paper treats a different type of problem, namely the interaction of ocean waves with the ground water level. Ground-water is generally fresh water, derived from rainfall, which may or may not flow, depending upon topography and the nature of the soil. In the vicinity of a shore there is a coupling of its motion with that of the ocean water if the common medium is a permeable soil, i.e., a sandy beach. Two important problems which come immediately to mind in this area are salt water intrusion via the "pumping" action of the ocean waves, and the influence of fluctuating ground-water levels on construction near shore, i.e., the possibility of deleterious buoyant forces on structures. In the problem studied the water is taken to be homogeneous throughout and the spatial and temporal behavior of the ground water elevation is sought as a function of the oncoming waves. This paper considers short-period, wind-induced waves, i.e. periods in the order of 5 minutes, and long-period waves, such as tides, of periods of one-half day or so. Tsunami's and other solitary waves are not included.

Statement of the Problem.

In this paper we consider a train of waves approaching a beach as shown in Fig. 1. The undisturbed water level in the soil is simply the extension of the mean water level of the sea in the vicinity of the shore.

The wave train may be represented by a function A sin $(\omega_0 t-\lambda x)$, where A is the amplitude, ω_0 the frequency, t the time and x the horizontal distance from source reference point. Upon breaking and striking the beach there is some run-up so that the maximum height, Fig. 2, may be different from the amplitude A, and in addition the height H is no longer a simple sinusoidal function, but a more complicated periodic function of time, Fig. 3. Nevertheless, assuming there is some regularity of the breaking wave on the beach, the height may be represented by a Fourier series expansion $H(t) = \Sigma\, a_n \sin \omega_n t$.

What is desired is the height of water above the undisturbed level in the soil, h(x,t), see Fig.2, caused by the imposed periodic motion at the boundary. Specifically, the questions to be answered are: is the fluctuating water level at the boundary H(t) of significant influence on the undisturbed water level for all, or any, distances x; and do the wave form and frequency, Fig. 3, of H(t), play an important role in this evaluation?

To answer these, it is first noted that the fundamental frequency of H(t) must be equal to the frequency of the wave train, ω_0. We therefore shall study the behavior of $\tilde{h}(x,t)$ with reference to a single frequency wave of the form

$$H = H_0 + a \sin \omega t \qquad (1)$$

where a is a constant. And while one may then obtain the response to an arbitrary H(t) simply by summing the responses to the individual components, $a_n \sin \omega_n t$, it will be shown that the dominant term in the solution for $\tilde{h}(x,t)$ comes from the zeroth term in the series, i.e. $a_0 \sin \omega_0 t$.

The field equations for flow through porous media are the Darcy equation

$$v_{x_i} = -k\frac{\partial\varphi}{\partial x_i} \tag{2}$$

where v_{x_i} is the velocity component in the x_i direction, φ a potential-like term equal to the pressure head plus the elevation above some datum, $\varphi = \frac{p}{\gamma} + z$, and k a constant for the soil, called the hydraulic conductivity and which is dependent upon the soil type, porosity and fluid. Equation (2) combined with the equation of continuity gives the Laplace equation for the law governing the flow. As is evident, this is a free-surface problem with all the attendant difficulties thereof. A greatly simplified boundary-value problem may be obtained, however, for the quasi-one-dimensional problem by considering one element in the soil, Fig. 4, for establishing continuity, and using an integrated form for (2). The new dependent variable then is $\tilde{h}(x,t)$ and the field equation becomes

$$k \frac{\partial}{\partial x}\left(\tilde{h}\frac{\partial \tilde{h}}{\partial x}\right) = n\frac{\partial \tilde{h}}{\partial t} \tag{3}$$

where n is the porosity of the soil. For the usual condition actually encountered the above may be easily linearized since the height H_o greatly exceeds the wave height. To accomplish this let $\tilde{h} = H_o + h$, Fig. 2. Substituting this into (3) and retaining the lowest order terms gives

$$k \frac{\partial}{\partial x}\left(H_o \frac{\partial h}{\partial x}\right) = n\frac{\partial h}{\partial t} . \tag{4}$$

h(x,t) is, of course, the vertical distance from the undisturbed water level to the free-surface. The boundary-condition, in terms of h(x,t) is, from (1)

$$h = a \sin \omega t \quad \text{at the sloped face} \tag{5}$$

and the initial condition is

$$h(x,0) = 0 \tag{6}$$

The problem represented by (4), (5) and (6), although well-posed in the mathematical sense, is awkwardly posed as regards its solution. This is so because the spatial variable and the solution function x and h are normal to each other while the boundary condition (5) is given on a surface sloped to these directions. In this problem, however, it is possible to introduce a nonorthogonal transformation which obviates this difficulty and at the same time leaves the form of the differential equation and initial condition unchanged.

Transformation.

Let α be the angle of slope of the beach. Consider the transformation

$$\begin{aligned} x &= \xi + \eta \cos \alpha \\ h &= \eta \sin \alpha \end{aligned} \tag{7}$$

which, for any point on the free surface (x,h) locates the same point, in terms of new coordinates, (ξ,η), parallel to the x axis and the sloping surface, Fig. 5. It is easily shown that the Jacobian of (7) vanishes only for the trivial case $\alpha = 0$ so the inversion, $\xi = \xi(x,h)$, $\eta = \eta(x,h)$ exists everywhere, i.e.

$$\begin{aligned} \xi &= x - h/\tan \alpha \\ \eta &= h/\sin \alpha \ . \end{aligned} \tag{8}$$

In terms of (7), (4), (5) and (6) become

$$\frac{\partial^2 \eta}{\partial \xi^2} = \frac{n}{kH_0} \frac{\partial \eta}{\partial t}, \quad \eta = \eta(\xi,t), \ \xi \geq 0, \ t \geq 0 \tag{9}$$

$$\eta(0,t) = \frac{a}{\sin \alpha} \sin \omega t \ , \qquad t \geq 0 \tag{10}$$

$$\eta(\xi,0) = 0 \ , \qquad \xi \geq 0 \tag{11}$$

The foregoing, (9), (10), (11) represent the final form of the boundary value problem which remains to be solved.

Solution of Boundary-Value Problem.

We start the solution by first taking the Fourier sine transform with respect to the variable ξ, i.e.

$$\mathfrak{F}_s[\eta] \equiv \int_0^\infty \eta(\xi,t) \sin p\xi \, d\xi \equiv \bar{\eta}(t;p) \tag{12}$$

with which (9) transforms to

$$a \sin \omega t/\sin \alpha - p^2\bar{\eta} = (n/kH_o)d\bar{\eta}/dt \tag{13}$$

where (10) has been utilized. The initial condition (11) transforms to

$$\bar{\eta}(0;p) = 0 \tag{14}$$

The solution to (13) and (14) is easily obtained by standard techniques to give

$$\bar{\eta}(t;p) = \frac{a}{\sin \alpha} \frac{\omega}{\varkappa} \frac{1}{p^4 + \omega^2/\varkappa^2} (e^{-p^2 \varkappa t} - \cos \omega t)$$

$$+ \frac{a}{\sin \alpha} \frac{p^2}{p^4 + \omega^2/\varkappa^2} \sin \omega t \tag{15}$$

where $\varkappa = kH_o/n$. Applying next the inversion integral to (15) gives

$$\eta(\xi,t) = \frac{2}{\pi} \int_0^\infty \frac{a}{\sin \alpha} \frac{\omega}{\varkappa} \frac{pe^{-p^2\varkappa t} \sin p\xi}{p^4 + \omega^2/\varkappa^2} dp$$

$$- \frac{2}{\pi} \int_0^\infty \frac{a}{\sin \alpha} \frac{\omega}{\varkappa} \frac{p \cos \omega t \sin p\xi}{p^4 + \omega^2/\varkappa^2} dp$$

$$+ \frac{2}{\pi} \int_0^\infty \frac{a}{\sin \alpha} \frac{p^3 \cdot \sin p\xi}{p^4 + \omega^2/\varkappa^2} \sin \omega t \, dp \tag{16}$$

$$= I_1 + I_2 + I_3 \tag{17}$$

I_1 cannot be evaluated in closed form. However, it is the periodic,or steady-state, response, given by I_2 and I_3 which is desired, and if it can be established that the transient response settles down to zero, i.e., $\lim_{t \to \infty} I_1 = 0$, then this term may be neglected or approximated. Consider,

therefore, the factor $p/(p^4 + \omega^2/\kappa^2)$ appearing in the denominator of I_1.
It is clear that as a function of p this is always positive and it is easily shown that it has a maximum at $p = (3)^{-\frac{1}{4}} (\omega/\kappa)^{\frac{1}{2}}$. Moreover, since $\sin p\xi$ is bounded by one, we have

$$I_1 \leq \frac{2}{\pi} \frac{a}{\sin\alpha} \frac{(3)^{3/4}}{4} (\omega/\kappa)^{-\frac{1}{2}} \int_0^\infty e^{-p^2 \kappa t} dp$$
$$= \frac{1}{\sqrt{\pi}} \frac{a}{\sin\alpha} \frac{(3)^{3/4}}{4} (\omega/\kappa)^{-\frac{1}{2}} \cdot \frac{1}{\sqrt{2\kappa t}} \qquad (18)$$

Hence I_1 does in fact vanish as $t \to \infty$. It should also be noted that if an approximation to I_1 is desired the fact that its integrand vanishes at the limits of integration suggests that the major contribution to the integral comes from its value in the neighborhood of $p = (3)^{-\frac{1}{4}} (\omega/\kappa)^{\frac{1}{2}}$. Therefore, if $p/[p^4 + (\omega^2/\kappa^2)]$ is expanded in a Taylor series and the first few terms are substituted into I_1, the result is integrable in closed form thus yielding the desired approximation.

For the purpose at hand it is sufficient to note that for large times I_1 is of negligible importance. To evaluate I_2 and I_3, which are known to be periodic, a slightly different method of attack is employed. Since all the terms involving time may be removed from under the integrals (which was not the case with I_1), we see that the solution sought must be of the form

$$\eta(\xi, t) = F_1(\xi) \cos\omega t + F_2(\xi) \sin\omega t. \qquad (19)$$

Substituting (19) into (9) and (10) and equating coefficients of like terms gives

$$d^2 F_1/d\xi^2 = (\omega/\kappa) F_2$$
$$d^2 F_2/d\xi^2 = -(\omega/\kappa) F_1 \qquad (20)$$

together with the boundary conditions

$$F_1(0) = 0, \quad F_2(0) = \frac{a}{\sin\alpha} \qquad (21)$$

Solving the simultaneous equations (20) by eliminating F_2 gives

$$d^4 F_1/d\xi^4 + (\omega^2/\kappa^2) F_1 = 0 \qquad (22)$$

whose solution, compatible with the implied condition that $\lim_{\xi \to \infty} \eta(\xi,t) = 0$ (hence $\lim_{\xi \to \infty} F_1(\xi) = 0$) is

$$F_1(\xi) = A e^{-\sqrt{\frac{\omega}{2\kappa}}\xi} \sin \sqrt{\frac{\omega}{2\kappa}}\xi + B e^{-\sqrt{\frac{\omega}{2\kappa}}\xi} \cos \sqrt{\frac{\omega}{2\kappa}}\xi , \qquad (23)$$

and from (23) and the first of (20) we get for F_2

$$F_2(\xi) = -A e^{-\sqrt{\frac{\omega}{2\kappa}}\xi} \cos \sqrt{\frac{\omega}{2\kappa}}\xi + B e^{-\sqrt{\frac{\omega}{2\kappa}}\xi} \sin \sqrt{\frac{\omega}{2\kappa}}\xi . \qquad (24)$$

Finally, from (24), (21) and (19) we obtain after some simplification

$$\eta(\xi,t) = \frac{a}{\sin \alpha} e^{-\sqrt{\frac{\omega}{2\kappa}}\xi} \sin \left(\omega t - \sqrt{\frac{\omega}{2\kappa}}\xi\right) \qquad (25)$$

Conclusions.

From (25) it is seen that the amplitude decays exponentially with ξ and with $\sqrt{\omega}$. Hence for fixed position, ground-water levels will be more affected by low frequency (long period) waves than by high frequency waves. For example, for a $\kappa = 3.28 \, ft^2/sec$, corresponding to a coarse sand and a permeable stratum depth, H_0, of 200 feet, the amplitude of oscillations of the ground-water at a $\xi = 3{,}000$ feet will be about 14% of the tide amplitude (period equal to one-half day). For wind induced waves, on the other hand, with a period of 5 minutes, the amplitude is negligible at $\xi = 3{,}000$ feet (all other parameters remaining the same), and the position at which the amplitude is 14% of the tide amplitude is $\xi = 80$ feet. Hence it is seen that the penetration distance of the wave into the beach is strongly dependent upon wave frequency, so that wind induced waves need not be included in consideration of ground-water level fluctuations for distances of 1,000 feet or more from the shore. Further, and as a corollary to the above, in the vicinity where wind induced waves may be an important consideration, only the first few terms in the Fourier series resolution need be included in computations, since the later terms in the series correspond to higher frequency waves, which we have seen are negligible. Thus, regardless of the exact wave form, Fig. 3, the response h(x,t) is given with sufficient accuracy if the boundary condition on the sloped beach is taken to be $H_{max} \sin \omega_0 t$ $(\zeta(0,t) = \frac{H_{max}}{\sin \alpha} \sin \omega_0 t)$.

Finally, it is noted that the solution (25) depends on a single parameter $\varkappa = \frac{KH_0}{n}$. The determination of $\varkappa$ may be accomplished simply by comparing the time of occurance of high water in the ocean with time of occurance of high water at a test hole at $\xi = \xi_0$. Since high water occurs when the sine term in (25) is unity, $\varkappa$ may readily be determined by setting the argument equal to $\frac{\pi}{2}$. The value of $\varkappa$ thus obtained may then easily be checked by comparing the magnitude of the high water at the two locations.

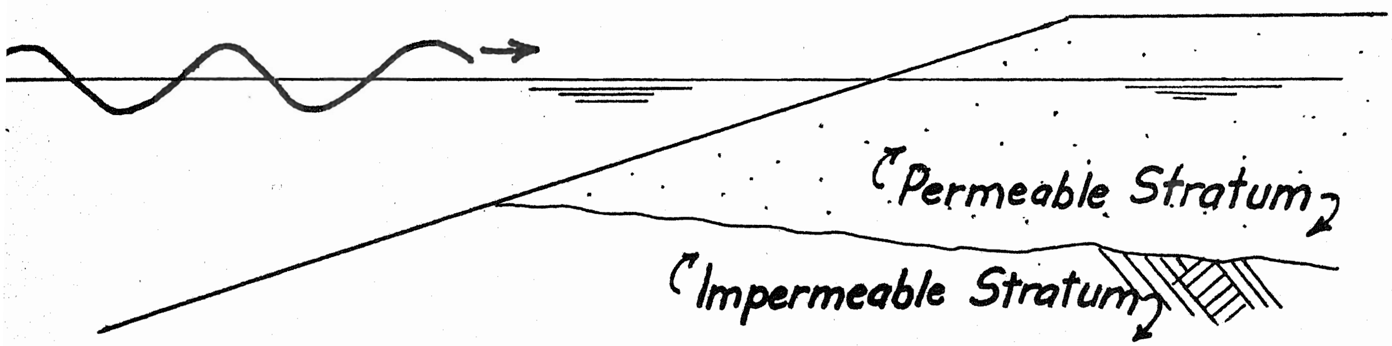
Permeable Stratum
Impermeable Stratum

FIG. 1

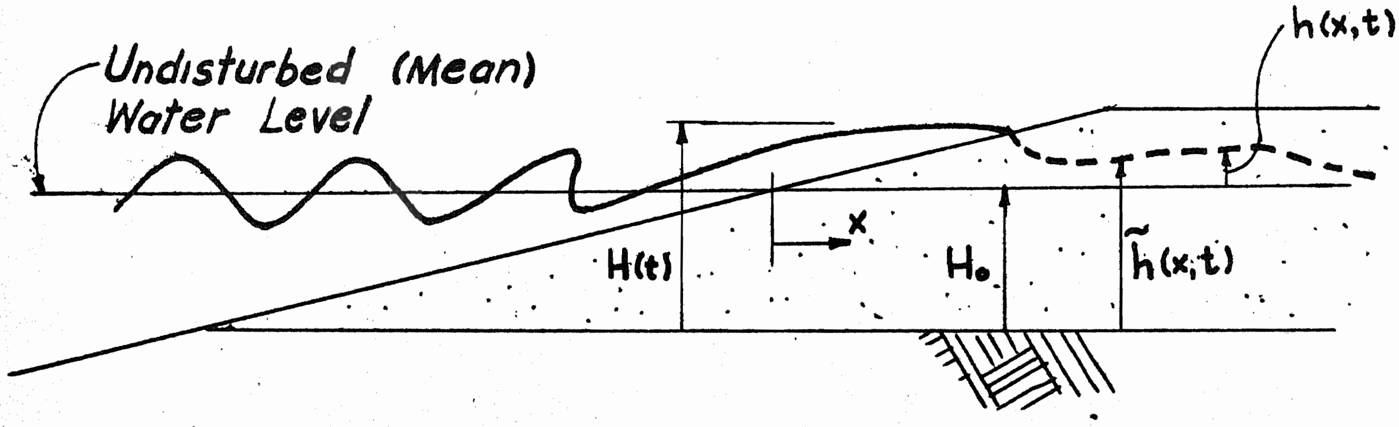
Undisturbed (Mean)
Water Level
h(x,t)
H(t)
x
H₀
h̃(x,t)

FIG. 2

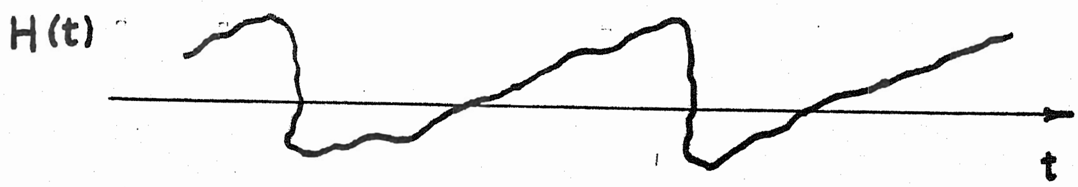

FIG. 3

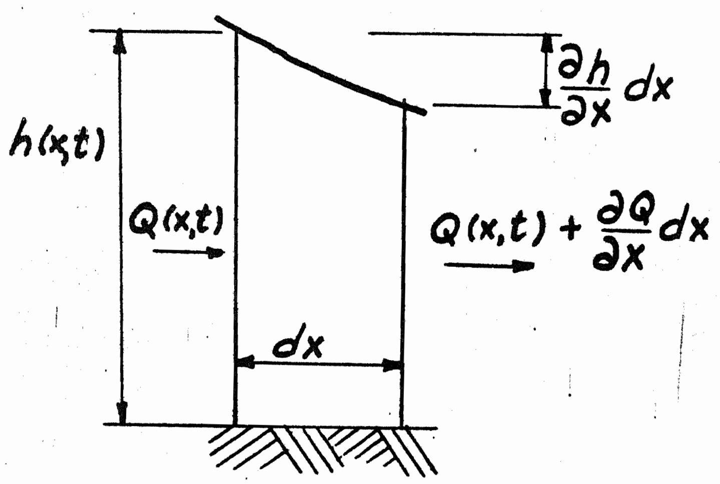

FIG. 4

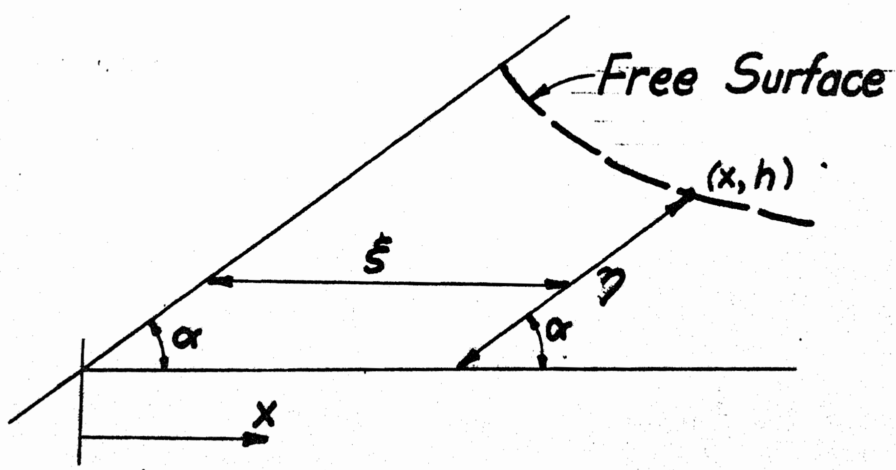

FIG. 5