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THE INFLUENCE OF POISSON'S RATIO ON THE VIBRATIONAL  
SPECTRUM

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## Abstract

This note concerns the normal modes of vibration of an elastic body subject to standard boundary conditions. The Poisson coefficient enters into the problem in a rather complex way, both in the differential equations and in the boundary conditions. To simplify this situation the Rayleigh-Ritz variational principle is introduced in order to define the normal mode frequencies as stationary values. This leads to a perturbation formula for the frequencies as a function of the Poisson coefficient. In some cases it is found that this formula can be evaluated exactly. In any case the formula furnishes upper and lower bounds for the variation.

\* Prepared while the author was visiting professor at the State University of New York at Stony Brook.

## The Influence of Poisson's Ratio on the Vibrational Spectrum

### 1. Introduction

If an elastic rod is stretched by  $r$  percent of its length then experiment shows that its diameter decreases by  $\sigma r$  percent. The constant  $\sigma$  is termed Poisson's ratio and it serves to determine the increase in volume. Clearly if  $\sigma$  were .5 there would be no change in volume. However measurement shows that  $\sigma$  is in the range  $0 < \sigma < .5$ . For metals  $\sigma$  is about .3 while for rubber  $\sigma$  is almost .5.

A problem of frequent occurrence in technology concerns the change in the vibrational spectrum of a mechanical system resulting from a change in the Poisson coefficient. For example such a problem arises when the behavior of a system is to be inferred from a scale model. However, it may not be feasible to construct a model with material having the same value of Poisson's ratio.

In this note the vibrational spectrum of an elastic system is analyzed by means of the calculus of variations. A perturbation formula is developed which relates the variation in frequency of a normal mode of vibration resulting from a variation in Poisson's ratio. To use this formula in a precise way it is necessary to know a factor  $q$  depending on the mode shape. However certain inequalities are easily deduced from the perturbation formula.

Before taking up the general case of a three dimensional body it seems best to treat the two dimensional limiting case of a thin flat plate. The problem of the transverse vibration of a plate is somewhat simpler to present because it is governed by a single differential equation rather than a system of three differential equations. Moreover, the plate problem is sufficiently important to warrant special attention.

## 2. Dimensional Analysis of the Vibrating Plate

The vibrating part of many mechanical systems may be accurately described as a clamped plate. The dynamical properties of such systems are developed in the elements of the theory of elasticity [1, p. 250]. Considerable simplification results because: (1) the plate is regarded as flat and (2) because the thickness is a fraction of the surface dimensions so the plate is thin. Under these simplifying hypotheses the theory shows that a normal mode of vibration of a thin flat plate is determined by the following biharmonic wave equation,

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{\rho h f^2 w}{D} .$$

Here:  $x$  and  $y$  are orthogonal coordinates in the plane of the plate,

$w = w(x,y)$  is the deflection of the plate normal to its plane,

$\rho$  is the density of the plate,

$D$  is the flexural rigidity,

$h$  is the thickness of the plate,

$f$  is the angular frequency.

In the derivation of the above equation it is shown that the flexural rigidity is given by the formula

$$D = \frac{E h^3}{12(1-\sigma^2)} .$$

Here:  $E$  is Young's modulus of elasticity,

$\sigma$  is Poisson's ratio.

Of equal importance are the boundary conditions. The three main types of boundary conditions are: clamped edge, hinged edge, and free edge. The derivation gives the following mathematical relations which must hold on the boundary [1, p. 251].

$$\begin{array}{l}
 \text{Clamped edge} \left\{ \begin{array}{l} w = 0 \\ \frac{\partial w}{\partial n} = 0 \end{array} \right. , \\
 \\
 \text{Hinged edge} \left\{ \begin{array}{l} w = 0 \\ \frac{\partial^2 w}{\partial n^2} + \sigma \frac{\partial^2 w}{\partial t^2} = 0, \end{array} \right. \\
 \\
 \text{Free edge} \left\{ \begin{array}{l} \frac{\partial^2 w}{\partial n^2} + \sigma \frac{\partial^2 w}{\partial t^2} = 0 \\ \frac{\partial^3 w}{\partial n^3} + (2-\sigma) \frac{\partial^3 w}{\partial n \partial t^2} = 0. \end{array} \right.
 \end{array}$$

Here  $n$  denotes a normal direction and  $t$  denotes a tangential direction to the boundary. Both these directions are in the plane of the plate.

A problem of concern is how the normal frequencies depend on the following properties: (1) the scale, (2) the thickness  $h$ , (3) the modulus of elasticity  $E$  (4) Poisson's ratio  $\sigma$ , and (5) the density  $\rho$ .

To treat this problem it is assumed that the plates have similar plan form but differ in scale. The scale is determined by a linear dimension  $L$  which gives the "diameter" in a certain direction.

Let the basic differential equation be multiplied by  $L^4$  and let  $X = x/L$ ;  $Y = y/L$ , and  $\lambda = \rho h^2 D^{2-1} L^4$ . This gives

$$\frac{\partial^4 w}{\partial X^4} + 2 \frac{\partial^4 w}{\partial X^2 \partial Y^2} + \frac{\partial^4 w}{\partial Y^4} = \lambda w ,$$

a dimensionless equation. The corresponding boundary conditions for a free edge are,

$$\frac{\partial^2 w}{\partial N^2} + \sigma \frac{\partial^2 w}{\partial T^2} = 0 , \quad \frac{\partial^3 w}{\partial N^3} + (2-\sigma) \frac{\partial^3 w}{\partial N \partial T^2} = 0$$

where  $N = n/L$  and  $T = t/L$ .

Suppose that the boundary value problem is solved for these dimensionless equations. The normal modes of vibration may be denoted as  $w_1, w_2, w_3, \dots$  with the corresponding eigenvalues:

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \dots .$$

Then the following formula gives the  $m$ -th angular frequency  $f_m$  for a plate of linear dimension  $L$ , thickness  $h$ , density  $\rho$  and elastic modulus  $E$

$$f_m^2 = \frac{\lambda_m E h^2}{12\rho(1-\sigma^2)L^4}$$

The dimensionless differential equation does not contain the variables  $h, L, \rho, E$  or  $\sigma$ . The dimensionless boundary condition for a clamped edge does not contain these variables either. Thus if the plate is clamped all around the eigenvalue  $\lambda_m$  is not a function of the variables  $h, L, \rho, E$  or  $\sigma$ . Then we can form the frequency ratio for two plates whose

plan forms are geometrically similar. Then  $\lambda_m$  cancels and we obtain the scaling formula

$$\frac{f_m}{f_m^*} = \left(\frac{\rho^*}{\rho}\right)^{\frac{1}{2}} \left(\frac{E}{E^*}\right)^{\frac{1}{2}} \left(\frac{1-\sigma^{*2}}{1-\sigma^2}\right)^{\frac{1}{2}} \left(\frac{h}{h^*}\right) \left(\frac{L^*}{L}\right)^2 .$$

It is worth noting, in connection with this formula, that the elastic constants of metals have considerable variation. For example steel has  $E = 20 \times 10^{11}$  dynes per  $\text{cm}^2$  and  $\sigma = 0.3$  while gold has  $E = 8 \times 10^{11}$  and  $\sigma = 0.4$ .

If there are hinged edges or free edges then the boundary conditions contain Poisson's ratio. Thus in this case the eigenvalue  $\lambda_m$  is expected to be a function of  $\sigma$ . Moreover it can not be supposed that eigenvalue ratios such as  $\lambda_1/\lambda_2$  are independent of  $\sigma$ .

If free edges are present then the above scaling formula holds rigorously if  $\sigma^* = \sigma$ . Presumably the scaling formula is reasonably accurate if  $\sigma^*$  is close to  $\sigma$ . To attack this question the next part of this paper will show how to estimate the variation in frequency due to variation of Poisson's ratio.



### 3. Variational Treatment of the Plate

We have seen that an eigenvalue  $\lambda$  is determined by solving the bi-harmonic wave equation for the deflection  $w$ ,

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \lambda w$$

subject to one of the boundary conditions: clamped edge, hinged edge, or free edge. It is desired to find the variation in  $\lambda$  resulting from a variation in  $\sigma$ . It seems most difficult to get a hold on the problem when it is formulated in this direct way. However the calculus of variations affords an equivalent but more tractable formulation. This formulation, termed the Rayleigh-Ritz method, will now be stated.

Let two expressions  $V$  and  $T$  be defined by the following integrals over the area of the plate.

$$V(w) = \iint (w_{xx}^2 + w_{yy}^2 + 2w_{xy}^2 + 2\sigma w_{xx} w_{yy} - 2\sigma w_{xy}^2) dx dy$$

$$T(w) = \iint w^2 dx dy$$

Here  $V$  is proportional to the elastic energy of a state with deflection  $w$ . Likewise  $T$  is proportional to the kinetic energy if  $w$  were a state of velocity rather than displacement. Then the eigenvalues  $\lambda_n$  are the stationary values of

$$\lambda = \frac{V(w)}{T(w)} \quad (\text{The Rayleigh quotient}).$$

Here  $w$  and its variation are required to satisfy the clamped boundary condition and  $w = 0$  for the hinged boundary. In approximate calculations it is not necessary to satisfy the free boundary condition. This is so

because the free boundary condition turns out to be a "natural boundary condition" as defined in the calculus of variations.

The variational definition of eigenvalues is very useful for numerical computation. In particular the smallest eigenvalue is simply the minimum of the quotient  $V/T$  for the smooth function  $w$  satisfying the clamped boundary condition. The variational method will now be applied to determine the rate of change of frequency with respect to Poisson's ratio.

Theorem 1. Let  $f$  be the frequency of a normal mode of vibration of a thin flat plate which is clamped on part of its boundary, hinged on another part of its boundary, and free on the remaining part of its boundary. Then the rate of change  $f$  with respect to Poisson's ratio  $\sigma$  is given by the formula

$$\frac{1}{f} \frac{df}{d\sigma} = \frac{\sigma}{1-\sigma^2} + \frac{p}{1+2cp}$$

where  $p$  is a dimensionless shape factor defined as

$$p = \frac{\iint (w_{xx} w_{yy} - w_{xy}^2) dx dy}{\iint (w_{xx}^2 + w_{yy}^2 + 2w_{xy}^2) dx dy}$$

Here  $w(xy)$  is the deflection of the normal mode of vibration of concern.

Proof. A variation  $\delta\sigma$  in  $\sigma$  gives rise to a variation  $\delta\lambda$  in  $\lambda$  and a variation  $\delta w$  in  $w$  so

$$\frac{d\lambda}{d\sigma} = \frac{2 \iint (w_{xx} w_{yy} - w_{xy}^2) dx dy}{T(w)} + \lim_{\delta w \rightarrow 0} \left[ \frac{V(w+\delta w)}{T(w+\delta w)} - \frac{V(w)}{T(w)} \right].$$

But the variation of  $V/T$  vanishes when  $w$  is a natural mode so the last term is zero and

$$\frac{1}{\lambda} \frac{d\lambda}{d\sigma} = \frac{2 \iint (w_{xx} w_{yy} - w_{xy}^2) dx dy}{V(w)} = \frac{2p}{1+2cp}.$$

From the formula for  $f$  we see that

$$\frac{2}{f} \frac{df}{d\sigma} = \frac{2\sigma}{1-\sigma^2} + \frac{1}{\lambda} \frac{d\lambda}{d\sigma}.$$

This is seen to complete the proof of the theorem.

To use the formula of this theorem it is necessary to know the second derivatives of  $w$  in order to compute  $p$ . There are three conceivable ways to obtain this information: (1) An analytical solution  $w$  might be available for a certain value of  $\sigma$ . (2) The deflection  $w$  could be obtained by experiment. (3) The deflection  $w$  could be approximated by the Rayleigh-Ritz method.

In the Rayleigh-Ritz method we assume a mode shape depending on a set of parameters. Then the Rayleigh quotient is minimized with respect to these parameters. This gives an upper bound to the lowest eigenvalue  $\lambda$ . The minimizing deflection is taken to be an approximation and  $p$  can be computed. A similar procedure is available for higher eigenvalues.

Corollary 1. The variational formula

$$\frac{1}{f} \frac{df}{d\sigma} = \frac{\sigma}{1-\sigma^2}$$

holds for a plate clamped on all edges.

Proof: If a function satisfies the boundary condition  $w = 0$  it follows that  $\partial w / \partial n = 0$ . Hence the conditions  $w = 0$  and  $\partial w / \partial n = 0$  together imply  $\partial w / \partial x = 0$  and  $\partial w / \partial y = 0$ . Then integration by parts with respect to  $x$  gives

$$\iint w_{xx} w_{yy} dx dy = - \iint w_x w_{xyy} dx dy$$

because the boundary term has  $w_x$  as a factor and so vanishes. Likewise integration by parts with respect to  $y$  gives

$$\iint w_{xy}w_{xy}dxdy = - \iint w_xw_{xyy}dxdy.$$

Subtracting these two equations gives

$$\iint (w_{xx}w_{yy} - w_{xy}^2) dxdy = 0.$$

Hence  $p = 0$  and the proof is complete. This confirms the result obtained in Section 2 by a different argument.

Corollary 2. The variational inequality

$$\frac{-1}{2(1+\sigma)} \leq \frac{1}{f} \frac{df}{d\sigma} \leq \frac{1}{2(1-\sigma)}$$

holds under the hypothesis of Theorem 1.

Proof: At any point Q of the plate let

$$N = w_{xx}w_{yy} - w_{xy}^2$$

$$D = w_{xx}^2 + w_{yy}^2 + 2w_{xy}^2.$$

But clearly

$$D + 2N = (w_{xx} + w_{yy})^2$$

$$D - 2N = (w_{xx} - w_{yy})^2 + 4w_{xy}^2.$$

Then the following inequality holds between the integrand N in the numerator of p and the integrand D in the denominator of p

$$-D \leq 2N \leq D.$$

Consequently  $-1 \leq 2p \leq 1$  and substitution of this inequality in the relation of Theorem 1 completes the proof of Corollary 2.

Corollary 3. For a rectangular beam  $df/d\sigma = 0$ .

Proof: A long rectangular plate may be regarded as a beam. We take the x axis along the middle of the beam. By symmetry we see that  $w_{xy} = 0$  along the x axis. To the degree of approximation employed in beam theory it may be assumed that  $w_{xy} \equiv 0$ . The boundary condition along the edges of the beam is  $w_{yy} = -\sigma w_{xx}$ . In the beam theory approximation it follows that  $w_{yy} \equiv -\sigma w_{xx}$ . Making these approximations we see that

$$p = \frac{-\sigma \iint w_{xx}^2 dx dy}{(1+\sigma^2) \iint w_{xx}^2 dx dy} = \frac{-\sigma}{1+\sigma^2}.$$

Substitution in Theorem 1 completes the proof.

#### 4. Vibration of a Three Dimensional Body

This section concerns the normal modes of vibration of a three-dimensional elastic body. The treatment in general except for the restriction that the material is isotropic and homogeneous. Again the problem of concern is the variation in natural frequencies due to a variation in Poisson's ratio.

We begin by a formulation of the equation of elasticity; for the purpose of definiteness we follow the notation of Courant-Hilbert [1, p. 268]. Suppose that the body in question occupies a region  $G$  in  $(x_1, x_2, x_3)$  space with piecewise smooth boundary surface  $\Gamma$ . Let  $(u_1, u_2, u_3)$  denote a small deformation of each point  $(x_1, x_2, x_3)$  from the rest position. Then the strains are defined as the tensor

$$\epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

The dilatation is defined as

$$\epsilon = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}.$$

If  $S_{ij}$  is the stress tensor, then Hooke's law, stating that stress is proportional to strain, is

$$S_{ij} = a \epsilon_{ij} + b \epsilon \delta_{ij}$$

Here  $\delta_{ij}$  is the Kronecker symbol and  $a$  and  $b$  are constants. By applying these relations to the stretching of a rod it is easy to show that

$$a = \frac{E}{1+\sigma} \quad b = \left( \frac{E}{1+\sigma} \right) \left( \frac{\sigma}{1-2\sigma} \right)$$

where  $E$  is Young's modulus and  $\sigma$  is Poisson's ratio.

The condition for equilibrium in the interior of G is

$$\Sigma \frac{\partial S_{ij}}{\partial x_j} + P_i = 0$$

where  $P_i$  denotes the body force density components. The equilibrium condition on the boundary  $\Gamma$  of G is

$$\Sigma S_{ij} n_j - p_i = 0 .$$

Here  $n_j$  denotes the exterior normal components and  $p_i$  denotes the surface force density components (tractions).

It now follows from Newton's equation that a normal mode of free vibration satisfies the equations

$$\Sigma \frac{\partial S_{ij}}{\partial x_j} = -\rho f^2 u_i$$

where  $\rho$  is the mass density and  $f$  is the angular frequency of vibration. Of course, the type of boundary condition must be specified. There are two types of boundary conditions commonly considered; and defined as follows:

(a) Fixed boundary,  $u_1 = u_2 = u_3 = 0$  ,

(b) Free boundary,  $p_1 = p_2 = p_3 = 0$  .

There are also two kinds of boundary conditions introduced by Somigliana :

(c) Normal fixed,  $u_n = 0, p_t = 0$  ,

(d) Tangential fixed,  $u_t = 0, p_n = 0$  .

Here  $u_n$  denotes the normal component of the displacement and  $p_t$  denotes the tangential component of the surface traction. Further information on

Somigliana boundary conditions is given in reference [2].

By a mixed boundary condition we shall mean that over a part  $\Gamma_a$  of the surface the boundary condition (a) holds, over a part  $\Gamma_b$  of the surface the boundary condition (b) holds, etc. Note that in a mixed boundary condition  $\sum p_i u_i = 0$  on the boundary.

The potential energy of a state of equilibrium is

$$V(u) = \frac{1}{2} \int_G \sum \epsilon_{ij} S_{ij} dx - \int_G \sum P_i u_i dx - \int_{\Gamma} \sum p_i u_i ds$$

Suppose that a mixed boundary condition is in force so that the integrand of the last integral vanishes. Moreover suppose that there are no body forces so the potential energy becomes

$$V(u) = \frac{a}{2} \int_G \sum \epsilon_{ij}^2 dx + \frac{b}{2} \int_G \epsilon^2 dx .$$

Also of concern is the kinetic energy quadratic form

$$T(u) = \frac{\rho}{2} \int_G \sum u_i^2 dx .$$

Then the square of the normal frequencies are the stationary values of the Rayleigh quotient

$$f^2 = \frac{V(u)}{T(u)}$$

The reason being, of course, that the above differential equations defining a normal mode of vibration are simply the Euler equations of the Rayleigh variational principle.

Theorem 2. Let  $f$  be the frequency of a normal mode of vibration corresponding to an arbitrary mixed boundary condition. Then the rate of change of  $f$  with respect to Poisson's ratio is given by the formula

$$\frac{2}{f} \frac{df}{d\sigma} = - \frac{1}{(1+\sigma)} + \frac{1}{\sigma(1-2\sigma)} - \frac{1}{\sigma(1-2\sigma+q\sigma)}$$

where  $q$  is a dimensionless shape factor defined as



$$q = \int_G \epsilon^2 dx / \int_G \Sigma \epsilon_{ij}^2 dx$$

Here  $\epsilon_{ij}$  is the strain tensor of the normal mode of concern.

Proof: Since  $f$  is a stationary value it follows that the variation in  $u_i$  due to a variation in  $\sigma$  can be ignored. The reasoning is analogous to that used in Theorem 1 so

$$\frac{2}{f} \frac{df}{d\sigma} = \frac{1}{V} \frac{dV}{d\sigma}$$

$$2V = b(a/b + q) \int \Sigma \epsilon_{ij}^2 dx$$

$$\log V = -\log(1 + \sigma) - \log(\sigma^{-1} - 2) + \log(\sigma^{-1} - 2 + q) + \text{const.}$$

$$\frac{1}{V} \frac{dV}{d\sigma} = -\frac{1}{(1+\sigma)} + \frac{1}{\sigma^2(\sigma^{-1}-2)} - \frac{1}{\sigma^2(\sigma^{-1}-2+q)}$$

This is seen to complete the proof.

Corollary 4. Let a normal mode of vibration be a pure shear. Then

$q = 0$  and

$$\frac{2}{f} \frac{df}{d\sigma} = -\frac{1}{(1+\sigma)}$$

Moreover under scaling

$$f = L \left[ \frac{E}{2\rho(1+\sigma)} \right]^{\frac{1}{2}} = L c_T$$

where  $L$  is a "wavelength" and  $c_T$  is the transverse velocity of sound.

Proof: The tensor  $\epsilon_{ij}$  is a pure shear if the trace vanishes; in other words  $\epsilon = 0$ . A pure shear may also be described as an incompressible flow because  $\text{div } u = 0$ . The torsional vibration of cylinders are examples of pure shear modes.

Since  $\epsilon = 0$  it follows that  $S_{ij} = a \epsilon_{ij}$

Substitution in the equations of motion gives

$$\frac{a}{2} \Sigma \frac{\partial^2 u_i}{\partial x_j^2} + \frac{a}{2} \Sigma \frac{\partial^2 u_i}{\partial x_i \partial x_j} = -\rho f^2 u_i$$

The second summation vanishes and so

$$c_T^2 \Delta u_i = -f^2 u_i .$$

Thus a pure shear mode satisfies the ordinary wave equation. Reducing this equation to dimensionless form completes the proof.

Lemma 1. Let  $\epsilon_{ij}$  be the strain tensor corresponding to a displacement  $u$  satisfying the condition that the normal component of  $u$  vanishes on the

boundary. Then

$$\int_G \Sigma \epsilon_{ij}^2 dx = \int_G (\text{div } u)^2 dx + \frac{1}{2} \int_G (\text{curl } u)^2 dx .$$

Proof: By definition

$$\Sigma \epsilon_{ij}^2 - \epsilon^2 = 2(\epsilon_{12}^2 - \epsilon_{11}\epsilon_{22} + \epsilon_{23}^2 - \epsilon_{22}\epsilon_{33} + \epsilon_{31}^2 - \epsilon_{33}\epsilon_{11})$$

The first two terms on the right can be written as

$$2\epsilon_{12}^2 - 2\epsilon_{11}\epsilon_{22} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right)^2 - 2 \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} =$$

$$\frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} \right)^2 + 2 \frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} - 2 \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} .$$

Adding such expression gives the identity:

$$\Sigma \epsilon_{ij}^2 = (\text{div } u)^2 + \frac{1}{2} (\text{curl } u)^2 + \Sigma \left( \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_j} \right) .$$

The last summation can be written as

$$\Sigma \frac{\partial}{\partial x_i} \left( \frac{\partial u_i}{\partial x_j} u_j - u_i \frac{\partial u_j}{\partial x_j} \right) = \text{div } r$$

where the vector  $r$  is defined as

$$r_i = \Sigma \frac{\partial u_i}{\partial x_j} u_j - u_i \epsilon_{jj} .$$

To show that the integral of  $\text{div } r$  over the region  $G$  vanishes it is sufficient to show that the normal component of  $r$  vanishes at an arbitrary point of the boundary. At such a point choose the coordinate system so that the  $x_1$  axis is along the normal deviation. Then

$$r_1 = \sum \frac{\partial u_1}{\partial x_j} u_j - u_1 \epsilon$$

If boundary condition (a) holds this obviously vanishes. If (c) holds then  $u_1 = 0$  and if the boundary is smooth at this point then  $\partial u_1 / \partial x_2$  and  $\partial u_1 / \partial x_3$  also vanishes so  $r_1 = 0$ . This is seen to complete the proof.

Corollary 5. Let a normal mode of vibration be such that the displacement  $u$  is irrotational in the region and such that the normal component of  $u$  vanishes on the boundary. Then  $q = 1$  and

$$\frac{2}{f} \frac{df}{d\sigma} = - \frac{1}{(1+\sigma)} + \frac{1}{\sigma(1-2\sigma)} - \frac{1}{\sigma(1-\sigma)}$$

Moreover under scaling

$$f = L \left[ \frac{E(1-\sigma)}{\rho(1+\sigma)(1-2\sigma)} \right]^{\frac{1}{2}} = L c_L$$

where  $L$  is a "wavelength" and  $c_L$  is the longitudinal velocity of sound.

Proof: Since  $u$  is irrotational  $\text{curl } u = 0$ . It then follows from Lemma 1 that  $q = 1$ .

An irrotational vector can be written as  $u = \text{grad } \phi$  where  $\phi$  is a scalar

Thus

$$\epsilon_{ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j} \quad \text{and} \quad \epsilon = \Delta \phi.$$

The equation of motion becomes

$$a \sum \frac{\partial^3 \varphi}{\partial x_j^2 \partial x_i} + b \frac{\partial}{\partial x_i} \Delta \varphi = -f^2 \rho \frac{\partial \varphi}{\partial x_i} .$$

This may be written as

$$c_L^2 \Delta u_i = -f^2 u_i .$$

Thus an irrotational mode satisfies the ordinary wave equation. Reducing this equation to dimensionless form completes the proof.

Corollary 6. Suppose a normal mode satisfies a boundary condition in which the normal component of the displacement vanishes on the boundary then

$0 < q < 1$  and

$$-\frac{1}{(1+\sigma)} \leq \frac{2}{f} \frac{df}{d\sigma} \leq -\frac{1}{(1+\sigma)} + \frac{1}{\sigma(1-2\sigma)} - \frac{1}{\sigma(1-\sigma)}$$

Proof: By definition it is clear that  $q \geq 0$ . On the other hand Lemma 1 gives  $q \leq 1$ . Then  $0 \leq q \leq 1$ . Moreover experiment shows that  $0 < \sigma < \frac{1}{2}$  so the right side of the formula of Theorem 2 is a monotone function of  $q$ . Putting  $q = 0$  gives the first inequality and putting  $q = 1$  gives the second inequality.

Corollary 7. Let a normal mode satisfy an arbitrary mixed boundary condition.

Then

$$-\frac{1}{(\sigma+1)} \leq \frac{2}{f} \frac{df}{d\sigma} \leq \frac{2}{(1-2\sigma)}$$

Proof: By the Cauchy inequality

$$\epsilon^2 = \left( \sum_{ij} \epsilon_{ij} \right)^2 \leq 3 \sum_{ii} \epsilon_{ii}^2 \leq 3 \sum_{ij} \epsilon_{ij}^2 .$$

Thus  $0 \leq q \leq 3$ . Substituting  $q = 0$  and  $q = 3$  into the formula of Theorem 1 yields the proof.

It is worth noting that Corollary 7 would apply to a plate. However, it is seen that Corollary 2 gives a better upper bound.

## 5. Discussion

This note was suggested by problems encountered by engineers in scaling models. The writer is indebted to H. C. Nathanson, W. E. Newell, G. Mott, and A. C. Hagg of the Westinghouse Research Laboratories for discussions concerning such questions. Such models may be smaller or longer than the actual machine. Thus the model of a steam turbine rotor is smaller. On the other hand, the model for a resonant gate transistor is larger.

The treatment given here shows that there is no difficulty in scaling Young's modulus. It is only Poisson's coefficient which leads to difficult questions of scaling. However, in some cases, such as the beam and the fully clamped plate, the problem is easily resolved.

The treatment given here indicates that it would be worth while to consider other special cases. For example Raleigh studied inextensional vibrations of shells and the frequencies for cylinders and spheres are given in Love's "Elasticity", pp. 513-514 of the 4th edition. In a recent paper Ross and Matthews [3] have treated domes and have obtained frequencies for different kinds of modes. For modes of bending type they obtained formulas (2), (20), and (41). For high frequency modes of membrane type they obtained formula (43).

In the analysis given here it has been assumed that the material has isotropic and homogeneous elastic properties. However, it is apparent from the derivation that the theorems still hold even if the mass density is not uniform in the body.

In this paper the approach to the problem has been by the way of avoiding boundary integrals. Presumably other results could be obtained by phrasing the problem in terms of such integrals. This latter approach proved quite successful in a somewhat similar problem [4], [5].

## References

1. R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. I (1953) Interscience, New York.
2. R. J. Duffin, "Analytic Continuation in Elasticity", *Jour. of Rational Mech. and Anal.* 5 (1956) pp. 939-950.
3. E. W. Ross Jr. and W. T. Matthews "Frequencies and Mode Shapes for Axisymmetric Vibration of Shells" *Jour. of Appl. Mech.* 89 (1967) pp. 73-80.
4. R. J. Duffin and A. Schild "The Effort of Small Constraints on Natural Vibrations", *Proc. Fifth Symposium in Applied Mathematics*, Amer. Math. Soc. (1954) pp. 155-163.
5. R. J. Duffin and A. Schild "On the Change of Natural Frequencies Induced by Small Constraints", *Jour. of Rational Mech. and Anal.* pp. 731-758 (1957).



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LIGHT DISPLACEMENT MULTIHULL SAILBOATS

by

W. S. Bradfield

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