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STABILITY OF SYSTEMS WITH NONLINEAR DAMPING

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Abstract

Of concern are the set of differential equations which are the equations of motion of a passive mechanical system. Of course any actual system of this type has damping. If the differential equations have constant coefficients then it is an elementary problem to show that any effect of the initial conditions on the motion dies out at an exponential rate. The main problem treated in this paper relates to the situation in which the damping is not constant but is variable between limits. This problem is analyzed by a special method making use of the Laplace transform. It results, as is to be expected, that any effect of the initial condition on the motion dies out exponentially.

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STABILITY OF SYSTEMS WITH NONLINEAR DAMPING

R. J. Duffin

1. Introduction

This paper is concerned with the forced vibration of electrical or mechanical system with damping. More precisely it is a study of the effect of replacing constant linear damping with variable nonlinear damping. The stability property assumed for the constant linear system is that any effect of the initial conditions on the velocity decays exponentially to zero. From a physical standpoint, it is reasonable to expect that the same stability property holds when the constant damping is replaced by damping varying arbitrarily between constant limits. The purpose of this paper is to give a mathematical proof of such stability.

This paper is one of a series of papers on the subject of nonlinear networks. In particular, the papers [1], [2], [3] and [4] listed below are also concerned with the stability problem studied here. The present paper makes no appeal to this previous work because the hypotheses are somewhat different. The hypotheses in this paper are in some ways less general than assumed in [4] and in other ways more general. The essential difference is that in [4] it was assumed that resistance was present in every circuit of the network while here some circuits may be without resistance. This means that the resistance matrix may be singular.

In [4] the stability proof depended on the construction of a function of Liapunov type. By contrast, the stability proof developed here does not make use of the methods of Liapunov.

The type of nonlinearity permitted in the damping is the same as that introduced in [1] and termed a "quasilinear replacement." The same concept was used in the other papers of this series. A quasilinear replacement arises in an electrical network when resistors obeying Ohm's law are replaced by nonlinear conductors whose differential resistance lies between positive limits. Minty [5] has termed such nonlinear conductors "monotone resistors" and has given far-reaching applications of these concepts [6]. Browder [7], Zarantello [8] and others have developed similar theories.

2. Equations of motion with constant coefficients

Of concern are electrical or mechanical systems with damping. First attention is confined to systems with constant damping; later the constant damping will be replaced by variable damping. The systems with constant damping are assumed to obey the following vector differential equation

$$L\frac{d^2q}{dt^2} + T\frac{dq}{dt} + R\frac{dq}{dt} + Sq = e \tag{1}$$

Here L, T, R and S are n by n matrices whose matrix elements are real constants. The vectors q and e have n components. In the usual mechanical interpretation (1) is the Lagrange equation of motion for the system. Thus q is the displacement vector, $q' = dq/dt$ is the velocity vector, and $q'' = d^2q/dt^2$ is the acceleration vector. Then Lq'' is the inertia force, Tq' is the gyroscopic force, Rq' is the damping force, Sq is the spring force, and e is the applied force. Energy considerations demand that L, R, and S be symmetric semi-definite matrices. On the other hand, T is skew symmetric. The electrical network interpretation of the differential equation (1) is equally familiar. Then q' is the current vector and e is the applied electromotive force. The matrix T appears when the network contains gyrators, a concept introduced by Tellegen.

Attention is restricted in this paper to those systems having no undamped free motions. For an equation with constant coefficients the condition that there be no undamped motion is completely specified by the following theorem.

Theorem 1. A necessary and sufficient condition that all solutions of the homogeneous equation

$$Lq'' + Tq' + Rq' + Sq = 0 \tag{2}$$

be such that the velocity vector q' approaches zero as $t \to +\infty$ is that

$$z^{-N} \det (z^2 L + zT + zR + S) \neq 0 \tag{3}$$

for Re $(z) \geq 0$. Here N is the nullity of S.

Proof. In this theorem the matrices L, T, R, and S can be completely arbitrary. If

$$\det (z^2 L + zT + zR + S) = 0 \tag{4}$$

there is a non-zero vector q_0 such that

$$(z^2 L + zT + zR + S)\, q_0 = 0. \tag{5}$$

Thus $q = q_0 e^{zt}$ solves (2). If $z \neq 0$ and Re $(z) \geq 0$ then $q' = zq_0 e^{zt}$ does not vanish at $+\infty$. If $z = 0$ and if (4) has multiplicity greater than N then it is shown in reference [9] that (2) has a solution of the form

$$q = q_1 + tq_0 , \quad q_0 \neq 0 \tag{6}$$

Then $q' = q_0$ does not vanish at $+\infty$. The necessity of condition (3) is thereby proved.

To prove the sufficiency note that if $z = 0$ then there is a set of N independent constant vectors satisfying (5). These vectors are solutions of (2) but have zero velocity. According to [9] these are the only solutions corresponding to $z = 0$. Also according to [9] the solution of the equation can be written as a finite sum of terms of the form $q_m t^m e^{zt}$. If Re $(z) < 0$ then this term and its derivative vanish as $t \to +\infty$.

This proves Theorem 1. Moreover the proof has the following corollary. Suppose that $q_1(t)$ and $q_2(t)$ are two solutions of the non-homogeneous equation (1).

Then their difference satisfies the homogeneous equation (2) and so

$$\|q_1'(t) - q_2'(t)\| e^{ct} \to 0 \quad \text{as } t \to +\infty. \tag{7}$$

Here c is a positive constant which depends only on the coefficients of the differential equation. In other words the following statement holds.

<u>Stability Property 1.</u> <u>Any effect of initial conditions on the velocity decays exponentially to zero</u>. The main goal of this paper is to extend this property to systems with nonlinear damping.

3. <u>Equation of motion with variable damping</u>

The linear equation (1) is now replaced by the nonlinear equation

$$Lq'' + Tq' + V(q') + Sq = e. \tag{8}$$

The only change is that the constant linear damping term Rq' is replaced by a variable nonlinear term $V(q')$. We term this a <u>quasilinear replacement</u> if $V(y)$ is a continuous vector function which satisfies the following condition,

$$V(y_1) - V(y_2) = U \cdot (y_1 - y_2) \tag{9}$$

where U is a symmetric matrix satisfying the quadratic form relation

$$D^{-1} \ (Ry,y) \le (Uy,y) \le D \ (Ry,y) \tag{10}$$

for a positive constant D independent of the vectors y_1, y_2, and y and the time t. The physical interpretation of a quasilinear replacement in an electrical network is the replacement of constant linear resistors by variable nonlinear resistors whose differential resistance lies between positive limits. A proof of this is given in [1].

It is permissable to have V depend on the time explicitly as well as implicitly through q'. In particular equation (8) includes the linear equation

$$Lq'' + Tq' + Uq' + Sq = e \tag{8a}$$

where U is a matrix which is a continuous function of the time and which satisfies inequality (10).

4. An energy inequality

Suppose that there are two solutions, q_1 and q_2 of the nonlinear differential equation (8) corresponding to the same applied force e but differing in initial conditions at time $t = 0$. Then let $w = q_1 - q_2$ and it is seen that w satisfies the homogeneous equation

$$Lw'' + Tw' + Uw' + Sw = 0. \tag{11}$$

In this equation all coefficients are constant except for U which is a function of the time because of its dependence on q_2' and q_1'. It shall be assumed that q_2' and q_1' are continuous functions of the time. Thus w' and U are continuous functions of the time. In what follows the only other property assumed about U is that it satisfies the inequality (10). It is desired to show that Stability Property 1 holds. This will be deduced by means of a series of lemmas.

Lemma 1. *If* w *is a solution of equation* (11) *then*

$$\int_0^a e^{2ct}(Uw',w')dt \le c\int_0^a e^{2ct}[(Lw',w') + (Sw,w)]dt + B. \tag{12}$$

Here a *and* c *are arbitrary constants and the constant* B *depends only on* w.

Proof. Multiplying the differential equation (11) by w' leads to the relation

$$(Uw',w') = -(1/2)\,[(Lw',w') + (Sw,w)]'.$$

According to the hypotheses the function on the left is continuous so we may multiply by e^{2ct} and integrate. Integration by parts then gives

$$\int_0^a e^{2ct}(Uw',w')\,dt = c\int_0^a e^{2ct}\,[(Lw',w') + (Sw,w)]dt$$
$$-(e^{2ct}/2)\,[(Lw',w') + (Sw,w)]\Big|_0^a.$$

But $e^{2ca}\,[(Lw',w') + (Sw,w)]_{t=a} \ge 0$ because L and S are positive semi-definite. This proves the lemma with $2B = [(Lw',w') + (Sw,w)]_{t=0}$.

The integral on the left of inequality (12) may be regarded as a weighted sum of "dissipated energy." The integral on the right may be regarded as a weighted sum of "kinetic energy" plus "potential energy." This is a principle lemma in the proof; the following lemma is a corollary.

Lemma 2. The functions (Uw',w'), (Rw',w'), and $\|Rw'\|^2$ are of class $L_1(0,\infty)$.

Proof: Let c = 0 in Lemma 1 then

$$\int_0^a (Uw',w')\, dt \leq B.$$

Allowing a to approach infinity proves the first statement of the lemma. Then hypothesis (10) gives $(Rw',w') \leq D\,(Uw',w')$. This proves the second statement of this lemma. To prove the third statement we note that if P is any positive semi-definite matrix

$$A_1^{-1}\|Px\|^2 \leq (Px,x) \leq A_1\|Px\|^2. \tag{13}$$

Where A_1 is a positive constant dependent on P but not on x. Thus taking $P = R$ and $x = q'$ completes the proof.

5. Reduction of variable damping to constant damping

The homogeneous equation with variable damping can be reduced to an inhomogeneous equation with constant damping by virtue of Lemma 3 to follow. This reduction permits application of well known methods of analysis developed for differential equations with constant coefficients.

Lemma 3. The differential equation (11) may be written as

$$Lw'' + Tw' + Rw' + Sw = Rf \tag{14}$$

where f is a continuous vector in the range of R and

$$\|f\|^2 \leq A_2(Uw',w') \tag{15}$$

for a constant A_2 independent of w.

Proof: By a standard argument it follows from hypothesis (10) that U and R have the same range. In particular Rw' and Uw' are both in the range of R. Thus we can solve the following equation for f

$$Rf = Rw' - Uw'$$

by the formula

$$f = R^{+} (Rw' - Uw').$$

Here R^{+} denotes the inverse of R in the range subspace. Then f is in the range of R^{+} and this is the same as the range of R because R is semi-definite. Clearly f is continuous and if A_3 is the norm of R^{+} then

$$\|f\| \leq A_3\|Rw' - Uw'\| \leq A_3\|Rw'\| + A_3\|Uw'\|.$$

This inequality together with inequalities (10) and (13) gives (15) and the proof is complete.

It is desired to solve the equation (14) for w in terms of f. Since the equation (14) has constant coefficients the Laplace transform furnishes a good way to do this.

6. Analysis via the Laplace transform

The following lemma is needed to determine the algebraic structure of the Laplace transform.

Lemma 4. *Let* $G = (zL + T + R + z^{-1}S)^{-1}R$; *then* G, zLG, *and* $z^{-1}SG$ *are uniformly bounded matrices for* $\operatorname{Re}(z) \geq 0$.

Proof. First suppose z is real and in the range $1 \leq z < \infty$. Then according to Theorem 1 relation (3) holds and G exists. Let h be an arbitrary vector and let $x = Gh$ so

$$(zL + T + R + z^{-1}S)x = Rh.$$

Since T is skew symmetric $(Tx,x) = 0$ and

$$z(Lx,x) + (Rx,x) + z^{-1}(Sx,x) = (Rh,x).$$

On the right side we use the inequality

$$(Rh,x) = (Rx,h) \leq \|h\| \, \|Rx\|.$$

On the left side we apply inequality (13) to the three semi-definite matrices. This yields

$$z \, \|Lx\|^2 + \|Rx\|^2 + z^{-1}\|Sx\|^2 \leq A_1 \|h\| \, \|Rx\|.$$

From this inequality it is apparent that $\|Rx\|$ is uniformly bounded as $z \to +\infty$. It then follows that $\|Lx\| = O z^{-\frac{1}{2}}$ and $\|Sx\| = O z^{\frac{1}{2}}$.

From the definition of x it follows that

$$\|Tx\| \leq z \, \|Lx\| + \|Rx\| + z^{-1}\|Sx\| + \|Rh\|.$$

Then employing the bounds obtained for the terms on the right we find $\|Tx\| = O z^{\frac{1}{2}}$. Thus $\|(L + T + R + S)x\| = O z^{\frac{1}{2}}$, and since $L + T + R + S$ is a nonsingular constant matrix we see that $\|x\| = O z^{\frac{1}{2}}$. Since h is an arbitrary vector it follows that a matrix element of G, say G_{ij}, must satisfy $|G_{ij}| = O z^{\frac{1}{2}}$. But G_{ij} is a rational function of z so if G_{ij} were unbounded it would increase at least as rapidly as $|z|$. Thus actually G_{ij} is uniformly bounded as z approaches $+\infty$.

To show that G_{ij} is uniformly bounded for z in the range $0 < z \leq 1$ let $z' = 1/z$ and the above argument can be repeated with L and S interchanging roles. Thus we have shown that G_{ij} does not have a pole at $z = \infty$ and also at $z = 0$. But by condition (3) of Theorem 1 it follows that G_{ij} has no poles for Re $z \geq 0$ and $|z| > 0$. Therefore, it may be concluded that G_{ij} is uniformly bounded for $\mathrm{Re}(z) \geq 0$. This proves the statement of the lemma concerning G.

By what has just been proved zLG is bounded at the origin. Write

$$zLG = -TG - RG - z^{-1}SG + (zL + T + R + z^{-1}S)G$$

$$= -TG - RG - z^{-1}SG + R.$$

The four matrices on the right are uniformly bounded at infinity so this proves the statement of the lemma for zLG. A symmetrical argument demonstrates the statement of the lemma concerning $z^{-1}SG$ and the proof is complete.

Lemma 5. Equation (14) has a solution w_1 such that w_1 and w_1' are continuous.

Proof: In this lemma it is assumed that f is given a priori as a function of the time. It follows from Lemma 2 and Lemma 3 that $f \in L_2(0,\infty)$. Thus the integral

$$\varphi(z) = \int_0^\infty e^{-zt} f(t)\, dt \qquad (16)$$

exists for $\mathrm{Re}(z) > 0$. Of course, φ is the Laplace transform of f and is symbolized as $\varphi = \mathcal{L}(f)$. Since the matrix elements of G are uniformly bounded rational functions it follows that there is a continuous function u, such that $\mathcal{L}(u) = G\varphi$. Let $w_1 = \int_0^t u\, dt$ so $w_1' = u$ and $\mathcal{L}(w_1) = z^{-1}G\varphi$.

$$\mathcal{L}(Lw_1' + Tw_1 + Rw_1 + S\int_0^t w_1\, dt) =$$
$$(L + z^{-1}T + z^{-1}R + z^{-2}S)\, G\varphi = z^{-1}R\varphi.$$

Then by the uniqueness theorem of the Laplace transform for continuous functions we have

$$Lw_1' + Tw_1 + Rw_1 + S\int_0^t w_1\, dt = \int_0^t Rf\, dt.$$

Differentiating this completes the proof.

Lemma 6. The solution w_1 of Equation 14 satisfies:

$$Lw_1' = \int_0^t K_1(s)\, f(t-s)\, ds,$$

$$Sw_1 = \int_0^t K_2(s)\, f(t-s)\, ds,$$

where $K_1(t)$ and $K_2(t)$ are continuous matrix functions such that $K_1(t) = 0\, e^{-kt}$ and $K_2 = 0\, e^{-kt}$ for any positive constant k such that $-k$ exceeds the real part of any pole of G.

Proof. Lemma 4 shows that both zLG and LG are uniformly bounded for $\mathrm{Re}(z) \geq 0$. This implies that actually $LG = 0(1 + |z|)^{-1}$ for $\mathrm{Re}(z) \geq 0$. Then by the elementary properties of the Laplace transform $LG = \mathcal{L}(K_1)$

where K_1 is a continuous matrix function such that $K_1 = O\, e^{-kt}$.

It also follows from Lemma 4 that SG and $z^{-1}SG$ are uniformly bounded for $\text{Re}(z) \geq 0$. Hence $z^{-1}SG = O(1 + |z|)^{-1}$ and so $z^{-1}SG = \mathcal{L}(K_2)$ where K_2 is a continuous matrix function such that $K_2 = O\, e^{-kt}$.

It is a consequence of the proof of Lemma 5 that $LG\varphi = \mathcal{L}(Lw_1')$ and $z^{-1}SG\varphi = \mathcal{L}(Sw_1)$. The inversion of these relations by the convolution theorem of the Laplace transform completes the proof.

7. Application of a convolution inequality

The following lemma is needed to carry on the analysis; it is a general property of convolutions.

Lemma 7. *Let $f(t) \in L_2(0,a)$ and let $K(t) \in L_1(0,a)$ where $a > 0$. Then the convolution*

$$g(t) = \int_0^t K(s)\, f(t-s)\, ds$$

satisfies the inequality

$$\int_0^a e^{2ct}\|g(t)\|^2 dt \leq \left[\int_0^a e^{ct}\|K(t)\|\, dt\right]^2 \int_0^a e^{2ct}\|f(t)\|^2 dt$$

where c is any real constant.

Proof. The existence of $g(t)$ for almost all t as a measurable function is well known. First consider the special case when K and f are real valued non-negative functions and $c = 0$. Define $f(t) = 0$ for $t \leq 0$. Then the following three relations actually are equalities:

$$g(t) \leq \int_0^a K(s)\, f(t-s)\, ds,$$

$$[g(t)]^2 \leq \int_0^a \int_0^a K(s)\, K(r)\, f(t-s)\, f(t-r)\, ds\, dr,$$

$$\int_0^a [g(t)]^2 dt \leq \int_0^a \int_0^a K(s)\, K(r) \int_0^a f(t-s)\, f(t-r)\, dt\, ds\, dr.$$

But by the Schwarz inequality

$$\left[\int_0^a f(t-s)\, f(t-r)\, dt\right]^2 \leq \int_0^a [f(t-s)]^2 dt \int_0^a [f(t-r)]^2\, dt.$$

Since $f(t) = 0$ if $t < 0$ then

$$\int_0^a [f(t-s)]^2\,dt = \int_0^{a-s} [f(x)]^2\,dx \le \int_0^a [f(x)]^2\,dx.$$

Combining the last three inequalities gives

$$\int_0^a [g(t)]^2\,dt \le \left[\int_0^a K(t)\;dt\right]^2 \int_0^a [f(t)]^2\,dt.$$

This is seen to prove the lemma in the special case.

In the vector case it is assumed, of course, that the norm of a matrix is defined so that

$$\|K\,f\| \le \|K\|\;\|f\|.$$

Then by the definition of g

$$\|g(t)\| \le \int_0^a \|K(s)\|\;\|f(t-s)\|\;ds.$$

Since norms are non-negative scalar functions the argument given above holds and proves the lemma in the case $c = 0$. But for arbitrary c

$$[e^{ct}g(t)] = \int_0^t [e^{cs}\,K(s)]\;[e^{c(t-s)}f(t-s)]\;ds$$

and this is seen to complete the proof.

<u>Lemma</u> 8. <u>Let</u> w_1 <u>be the solution of equation</u> (14). <u>Let</u> a <u>and</u> c <u>be constants such that</u> $a > 0$ <u>and</u> $c \le k$ <u>where</u> $-k$ <u>exceeds the real part of any pole of</u> $G(z)$. <u>Then</u>

$$\int_0^a e^{2ct}[(w_1', Lw_1') + (w_1, Sw_1)]dt \le A_4 \int_0^a e^{2ct}\;\|f\|^2\,dt$$

<u>where the constant</u> A_4 <u>is independent of</u> a <u>and</u> c.

<u>Proof</u>. Direct application of Lemmas 6 and 7 gives

$$\int_0^a e^{2ct}\;\|Lw_1'\|^2\,dt \le \left[\int_0^a e^{ct}\;\|K_1\|dt\right]^2 \int_0^a e^{2ct}\;\|f\|^2\,dt.$$

But by definition of c

$$\int_0^a e^{ct}\,\|K_1\|\,dt \le \int_0^a e^{kt}\,\|K_1\|\,dt \le \int_0^\infty e^{kt}\,\|K_1\|\,dt.$$

This last integral is independent of a and c. It converges because of the definition of k. Moreover since L is a positive semi-definite matrix the inequality (13) applies and gives

$$\int_0^a e^{2ct}(w_1', Lw_1')\,dt \le A_5 \int_0^a e^{2ct}\,\|f\|^2\,dt$$

where A_5 does not depend on a or c. A strictly analogous argument shows that

$$\int_0^a e^{2ct}(w_1, Sw_1)\,dt \le A_6 \int_0^a e^{2ct}\,\|f\|^2\,dt.$$

Adding these inequalities proves the lemma with $A_4 = A_5 + A_6$.

8. <u>The main stability theorem</u>

To prove the stability property of w it is first necessary to show that the last lemma can be modified so as to hold for w.

<u>Lemma</u> 9. <u>Let</u> w <u>be a solution of equation</u> (11). <u>Let</u> a <u>and</u> c <u>be positive constants and let</u> $c \le k$. <u>Then</u>

$$\left[\int_0^a e^{2ct}[(w', Lw') + (w, Sw)]\,dt\right]^{\frac{1}{2}} \le B_1\left[\int_0^a e^{2ct}(w', Uw')\,dt\right]^{\frac{1}{2}} + B_2$$

<u>where</u> B_1 <u>and</u> B_2 <u>are constants independent of</u> a <u>and</u> c <u>and</u> B_1 <u>is independent of</u> w <u>as well</u>.

<u>Proof</u>: Let $x = w - w_1$. Then x is a solution of the homogeneous equation (2). Let us use the notation

$$\|w\|_{ac} = \left[\int_0^a e^{2ct}[(w', Lw') + (w, Sw)]\,dt\right]^{\frac{1}{2}}.$$

Clearly the triangle inequality applies to such a norm. Thus

$$\|w\|_{ac} = \|w_1 + x\|_{ac} \le \|w_1\|_{ac} + \|x\|_{ac}.$$

Moreover

$$\|x\|_{ac} \leq \|x\|_{ak} \leq \|x\|_{\infty k} = B_2 .$$

Here B_2 is finite because all solutions of the homogeneous equation (12) are such that $e^{kt}x'$ and $e^{kt}x''$ are in $L_2(0,\infty)$. However $Sx = -Lx'' - Tx' - Rx'$ so $e^{kt}\, Sx$ is also in $L_2(0,\infty)$.

Lemma 3 gives $\|f\|^2 \leq A_2(w',Uw')$ so according to Lemma 8

$$\|w_1\|_{ac} \leq (A_4A_2)^{\frac{1}{2}} \left[\int_0^a e^{2ct}(w',Uw')dt\right]^{\frac{1}{2}}.$$

This completes the proof with $B_1 = (A_4A_2)^{\frac{1}{2}}$.

The following is the main stability theorem.

Theorem 2. Suppose that all solutions of the differential equation with real constant coefficients

$$L \frac{d^2x}{dt^2} + T\frac{dx}{dt} + R\frac{dx}{dt} + Sx = 0$$

are such that the velocity vector $dx/dt \to 0$ as $t \to +\infty$. Here L, R, and S are symmetric semi-definite matrices and T is a skew-symmetric matrix. Let $V(dx/dt)$ be a quasilinear replacement of $R\,dx/dt$ and suppose that for $t \geq 0$ the vectors $q(t)$ and $e(t)$ satisfy the nonlinear differential equation

$$L\frac{d^2q}{dt^2} + T\frac{dq}{dt} + V(\frac{dq}{dt}) + Sq = e.$$

Then the effect of initial conditions on the velocity decays exponentially to zero in the following sense. If q_1 and q_2 are two solutions with continuous first derivatives then

$$\int_0^\infty e^{2ct} \left\|\frac{dq_1}{dt} - \frac{dq_2}{dt}\right\|^2 dt < \infty \tag{18}$$

where c is a positive constant independent of q.

Proof. We let $w = q_1 - q_2$ and substitute the inequality of Lemma 9 into the inequality of Lemma 1 and obtain

$$\int_0^a e^{2ct}(w',Uw')dt \le c\left[B_1\left[\int_0^a e^{2ct}(w',Uw')dt\right]^{\frac{1}{2}} + B_2\right]^2 + B.$$

Choose c such that $cB_1^{\frac{1}{2}} < 1$. Then it is apparent that the integral on the left must converge as $a \to +\infty$. It then follows from Lemma 3 that $f\, e^{ct} \in L_2(0,\infty)$. Since the matrix G is uniformly bounded for Re $(z) \ge -k$ it follows that $G = G_0 + G_1$ where G_0 is a constant and $G_1 = O(1+|z|)^{-1}$. Since $\mathcal{L}(w_1') = G_0\varphi + G_1\varphi$ we see that

$$w_1' = G_0 f(t) + \int_0^t K_3(s)\, f(t-s)\, ds. \tag{19}$$

Here, just as in Lemma 6, the matrix function satisfies $K_3(t) = 0\, e^{-kt}$. Multiply relation (19) by e^{ct} then the first term on the right is in $L_2(0,\infty)$. The convolution inequality of Lemma 7 can be applied to the second term of (19) and consequently it is in $L_2(0,\infty)$. Thus $e^{ct}w_1' \in L_2(0,\infty)$. But $w' = w_1' + x'$ and we already know that $e^{ct}x' \in L_2(0,\infty)$. Hence $e^{ct}w' \in L_2(0,\infty)$ and the proof of Theorem 2 is complete.

Theorem 2 shows that the system has a stability property such that the effect of initial conditions on the velocity decays exponentially to zero. More precisely relation (18) formulates the exponential decay as convergence in mean rather than pointwise convergence as originally stated in relation (7).

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