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SOME THEOREMS ON UNIMODULAR BOUNDED FUNCTIONS AND THEIR APPLICATION TO IMPEDANCE SYNTHESIS

by

IRVING GERST

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Abstract

This paper presents the initial results of an investigation into the question of a canonical product for a real, rational, unimodular bounded function. Here, a study is made of the possibility of factoring such a function, f(s), into a product of functions of the same type, in which each factor corresponds to an impedance having just two kinds of elements. Such a factorization implies an immediate and complete realization of the impedance corresponding to f(s) by means of a transformerless network. As a preliminary, several theorems are proved which serve to characterize those unimodular bounded functions which correspond to RC or RL impedances. These results are then used to determine classes of unimodular bounded functions in which the desired factorization can be carried out. Examples which illustrate the procedures are given.

Footnotes

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(1) In defining a general u.b.f., these conditions are required only for $|s| < 1$, but for rational functions they follow for $|s| = 1$ as well.

(2) Throughout the paper, we will assume all functions are real and rational unless stated otherwise.

(3) As usual, sgn x, the signum of x, is defined as $x/|x|$ for real x, $x \neq 0$.

(4) If an $\alpha_i = 1$ or -1, then $1/f^*(1)$ or $1/f^*(-1)$ respectively is to be interpreted as ∞. Because of the alternation property, these cases cannot occur simultaneously.

SOME THEOREMS ON UNIMODULAR BOUNDED FUNCTIONS AND THEIR APPLICATION TO IMPEDANCE SYNTHESIS

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1. Introduction

Let f(s) be a real, rational, and unimodular bounded function (u.b.f.), i.e. f(s) analytic and $|f| \leq 1$ on the unit disc[1] $|s| \leq 1$. As is known, the factorization of f(s) into u.b.f.'s is related to the synthesis of the positive real, rational function (p.r.f.) Z(p), which corresponds to f(s) via the bi-unique transformation

$$F(p) = \frac{Z(p)-1}{Z(p)+1} \quad , \tag{1}$$

$$f(s) = F(p) \quad , \quad s = \frac{p-1}{p+1} \; . \tag{2}$$

More precisely, if $f(s) = f_1(s)\, f_2(s)$ where all functions[2] are u.b., and $Z_1(p)$, $Z_2(p)$ are the p.r.f.'s corresponding to $f_1(s)$ and $f_2(s)$ respectively, then the balanced bridge whose opposite pairs of arms are Z_1, $1/Z_1$ and $1/Z_2$, Z_2 respectively, has an impedance equal to Z(p) (Cf.[2],p.162). This process lends itself to iteration if the factorization of f(s) can be continued.

In certain cases a complete realization of Z(p) can be obtained in this way from a factorization of f(s) alone. In this paper we consider what is probably the simplest situation where this is so. Namely, we investigate conditions under which f(s) can be written as a product of u.b.f.'s in which each factor corresponds to a p.r.f. arising from a network having just two kinds of elements. Since such p.r.f.'s are immediately realizable, the possibility exists for obtaining a simpler transformerless synthesis of Z(p) in this case than that provided by existing standard procedures.

[1] In defining a general u.b.f., these conditions are required only for $|s| < 1$, but for rational functions they follow for $|s| = 1$ as well.

[2] Throughout the paper, we will assume all functions are real and rational unless stated otherwise.

In order to study the specified type of factorization for f(s), it is first necessary to ascertain the form of those u.b.f.'s which correspond to LC, RC, and RL impedances respectively. We shall designate such u.b.f.'s as LC, RC and RL u.b.f.'s respectively. Now, the form of an LC u.b.f. is known (Cf.[2],p.161). It is given by

$$g(s) = \pm \prod_{i=1}^{t} \frac{s-s_i}{1-\bar{s}_i s} \quad , \quad |s_i|<1, \ (i=1,2,\ldots,t), \tag{3}$$

where non-real s_i occur in conjugate complex pairs. Further, the role of LC u.b.f.'s in factorization is clear [1]. They act as unit functions in the sense that a factorization f(s) = g(s)h(s) with f(s) a u.b.f. and g(s) an LC u.b.f. implies that h(s) is also a u.b.f. We may therefore restrict our discussion to RC and RL u.b.f.'s (but note the remark in Section 5).

No results analogous to those for LC u.b.f.'s seem to be available for RC or RL u.b.f.'s. We therefore proceed first to characterize these functions. This development which takes up most of the paper, is given in Sections 2 and 3 and the results are embodied in Theorems 1 and 2. The latter are of interest in themselves. Then, in Section 4, we discuss factorization into RC and RL u.b.f.'s. Some extensions and several examples complete the paper.

As we have already indicated by our structuring of eqs. (1) and (2), the transformation from Z(p) to f(s) may be considered to be accomplished in two steps. The first of these, eq. (1), yields a function F(p) which is real, rational and unimodular bounded in the right half-plane $Re(p) \geq 0$. We shall refer to F(p) also as a u.b.f. In any discussion of u.b.f.'s it will be clear from the independent variable employed (p or s), whether the right half-plane or the unit disc is intended.

Functions F(p) which are u.b. stand in the same relation to the corresponding Z(p) with regard to factorization as does the u.b.f. f(s). It is a matter of choice which class is used. We have therefore derived, incidentally, Theorems 3 and 4 which relate to RC and RL u.b.f.'s F(p), and from which Theorems 1 and 2 follow directly using eq. (2). However, in Section 4, our results are stated for the u.b.f. f(s) only, with an indication of how the corresponding results for u.b.f.'s F(p) may be formulated.

The methods used in this paper are similar to those employed in a previous paper [3] which was concerned with another problem. Reference will be made to this paper on occasion for additional details. We also cite a partial list of papers [4]-[9], other than those already mentioned, which essentially make use of u.b.f.'s. Of these, the paper by Reza [6] comes closest to the present one in that an alternative argument for proving part of condition (b) of our Theorem 4 could be made using his results.

2. The Zeros and Poles of RC and RL u.b.f.'s

In this section, we establish certain conditions which must be satisfied by the zeros and poles of RC and RL u.b.f.'s, F(p) or f(s). In particular, we show that in all cases the zeros and poles are real and that they alternate.

Consider the non-constant RC impedance

$$Z(p) = k\,\frac{P(p)}{Q(p)}\,,\quad P(p) = \prod_{i=1}^{n}(p+\delta_i)\,,\quad Q(p) = \prod_{j=1}^{n}(p+\gamma_j)\,, \tag{4}$$

where $k > 0$ and

$$0 \leq \gamma_1 < \delta_1 < \gamma_2 \ldots < \gamma_n < \delta_n\,. \tag{5}$$

Here δ_n may be lacking, i.e. the last zero may be at ∞. For simplicity, we assume throughout the argument that δ_n is finite, indicating later any modification which is to be made if this is not the case.

Applying eq. (1) to Z(p) in eq. (4), we get the u.b.f. F(p) given by

$$F(p) = \frac{kP(p) - Q(p)}{kP(p) + Q(p)} .$$

Let

$$N(p) = kP(p) - Q(p) , \quad D(p) = kP(p) + Q(p) , \qquad (6)$$

and denote the zeros of N(p) and D(p) by $-\eta_i$ and $-\lambda_j$ respectively. For $k > 0$, N and D can have no common zero. For if $p = \rho$ were one, then a consideration of $N(\rho) + D(\rho)$ and $N(\rho) - D(\rho)$ would imply successively that $\rho = -\delta_i$ for some i, and that $\rho = -\gamma_j$ for some j, contrary to (5). Thus the $-\eta_i$ and the $-\lambda_j$ are respectively the zeros and poles of F(p).

We next locate the $-\eta_i$ and the $-\lambda_j$. Let γ_r be one of the γ's. Then from eqs. (6) and (4), we have

$$N(-\gamma_r) = k\prod_{i=1}^{n}(-\gamma_r+\delta_i) ;$$

and thus (5) implies that[3]

$$\operatorname{sgn} N(-\gamma_r) = (-1)^{r+1}, \quad (r=1, 2, \ldots, n) .$$

Similarly,

$$\operatorname{sgn} N(-\delta_r) = (-1)^{r+1} , \quad (r = 1, 2, \ldots, n) .$$

It follows that N(p) changes sign in each interval $(-\gamma_{r+1}, -\delta_r)$, $r=1,2,\ldots,n-1$. Hence, each such interval contains at least one real zero of N(p). Thus we have located n-1 real zeros, $-\eta_i$, of N(p). If k=1, N(p) is of degree n-1 and we have all of the $-\eta_i$ (one η_i has become infinite). If $k \neq 1$, then the one remaining $-\eta_i$ must be real, and it follows from the preceding information on the signs taken by N(p) that it must lie outside of the interval $(-\delta_n, -\gamma_1)$. Its location may be made more precise by considering N(p) at the real values $\pm\infty$.

[3] As usual, sgn x, the signum of x, is defined as $x/|x|$ for real x, $x \neq 0$.

If $k < 1$, we have sgn $N(\infty) = -1$. Hence, there is a sign change in $(-\gamma_1, \infty)$ and the remaining $-\eta_i$ lies there. Similarly, when $k > 1$, sgn $N(-\infty) = (-1)^n$, which implies that the remaining $-\eta_i$ is in $(-\infty, -\delta_n)$.

The polynomial $D(p)$ is treated in the same way. We find that sgn $D(-\gamma_r) = (-1)^{r+1}$ and sgn $D(-\delta_r) = (-1)^r$, for $r = 1, 2, \ldots, n,$. Thus each interval $(-\delta_r, -\gamma_r)$, $r = 1, 2, \ldots, n$, contains exactly one real zero, $-\lambda_j$, of $D(p)$.

Summarizing, one of the following two order relations holds for the δ_i, γ_j and the η's and λ's if the latter are each indexed in increasing numerical order.

$$0 \leq \gamma_1 < \lambda_1 < \delta_1 < \eta_1 < \gamma_2 < \ldots < \gamma_n < \lambda_n < \delta_n < \eta_n; \tag{7}$$

$$\eta_1 < \gamma_1 < \lambda_1 < \delta_1 < \eta_2 < \gamma_2 < \ldots < \gamma_n < \lambda_n < \delta_n . \tag{7$'$}$$

In (7), η_n may be missing, i.e., may be at ∞, in which case $F(p)$ has only n-1 finite zeros.

The stated order relations can easily be shown to hold when δ_n is missing in (4), if we interpret δ_n as ∞. In this case only (7)$'$ is possible.

We have now shown that an RC u.b.f. $F(p)$ must be of the form

$$F(p) = \varkappa \frac{\prod_{i=1}^{m} (p+\eta_i)}{\prod_{j=1}^{n} (p+\lambda_j)} , \tag{8}$$

where $\varkappa$ is real; $m = n-1$, or n; the η_i and λ_j are distinct real numbers with the latter all positive; the η_i and λ_j separate each other according to one of the following two schemes:

$$0 < \lambda_1 < \eta_1 < \lambda_2 < \eta_2 < \ldots < \lambda_n < \eta_n ; \tag{9}$$

$$\eta_1 < \lambda_1 < \eta_2 < \lambda_2 < \ldots < \eta_n < \lambda_n . \tag{9$'$}$$

In (9), η_n may be missing. Note that all the η_i and λ_j are positive except possibly η_1 in (9)$'$

Now replace p in (8) using the transformation of eq. (2). We get

$$f(s) = c \frac{\prod_{i=1}^{M} (s+\alpha_i)}{\prod_{j=1}^{N} (s+\beta_j)} , \tag{10}$$

where f(s) is u.b., c is a real constant expressible in terms of $\varkappa$, η_i and λ_j, and α_i and β_j are given by

$$\alpha_i = \frac{1+\eta_i}{1-\eta_i} , \ \beta_j = \frac{1+\lambda_j}{1-\lambda_j} , \tag{11}$$

when $\eta_i \neq 1$ and $\lambda_j \neq 1$. If $\eta_i = 1$ or $\lambda_j = 1$, α_i or β_j may be interpreted as ∞ and there is no corresponding factor $(s+\alpha_i)$ or $(s+\beta_j)$ in (10). Also when $m = n-1$ in (8), i.e., when $\eta_n = \infty$, a factor $(s+1)$ appears in the numerator of (10). This factor can be considered as arising from (11) by taking $\eta_i = \eta_n = \infty$. We see then that there are three possibilities for the integers M, N in (10), viz: $M = N = n$; $N = n$, $M = n-1$; or $N = n-1$, $M = n$, where n is a positive integer.

It follows from (11), (9) and (9)′ that the α_i and β_j are all real and distinct but they may take both positive and negative values. Furthermore, from the positivity of the λ_j, we have $|\beta_j| > 1$, as was to be expected for a u.b.f. f(s). Consideration of the linear transformation $(1+p)/(1-p)$ which is involved in (11) shows that the alternation property of the η_i and λ_j is preserved in the mapped sets α_i and β_j. However, the mapping may effect a cyclical permutation in the numerical order in which the α_i and β_j appear as compared to the corresponding η_i and λ_j. For example, if

$$\ldots \ \eta_{k-1} < \lambda_k < \eta_k < \lambda_{k+1} < \eta_{k+1} \ \ldots$$

are five consecutive terms of the η, λ sequence and if $\lambda_k < 1 < \eta_k$, then (11) implies that the α,β sequence will start with the negative terms $\alpha_k < \beta_{k+1} < \alpha_{k+1}$ and end with the positive terms $\alpha_{k-1} < \beta_k$.

Therefore, let us re-index each of the sets (α_i), (β_j) in increasing numerical order. The combined set of the α's and β's when arranged as an increasing sequence will then exhibit one of the two following order relations:

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_n < \beta_n , \quad (12)$$

$$\beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \ldots < \beta_n < \alpha_n . \quad (12)'$$

In both (12) and (12)′ the last term may be missing, i.e., we may have $\beta_n = \infty$ or $\alpha_n = \infty$.

This completes our development of the conditions which must be satisfied by the zeros and poles of an RC u.b.f. f(s).

The case of an RL u.b.f. is now easily treated. For, any RL impedance may be written as 1/Z(p) with Z(p) as in (4). But by (1) and (2) 1/Z corresponds to -F(p) and to -f(s), where F(p) and f(s) are the RC u.b.f.'s. Thus the conditions on the zeros and poles remain as before also for RL u.b.f.'s.

3. The Representation Theorems

Having obtained necessary conditions for the α_i and β_j of the RC (or RL) u.b.f. f(s) in (10), we next consider the real multiplicative constant c which appears in that formula. Since $|f(s)| \leq 1$ for $|s| \leq 1$, we have, in particular, that $|f(1)| \leq 1$ and $|f(-1)| \leq 1$. Let $f^*(s) = f(s)/c$. Then it is necessary that the following inequality hold[4]:

$$|c| \leq \mathrm{Min}\left[\left|\frac{1}{f^*(1)}\right| , \left|\frac{1}{f^*(-1)}\right| \right] .$$

Since RC and RL u.b.f.'s just differ by a factor equal to -1, this condition holds for both classes of u.b.f.'s.

We will show that the necessary conditions obtained thus far are also sufficient, i.e., that the following theorem holds.

[4] If an $\alpha_i = 1$ or -1, then $1/f^*(1)$ or $1/f^*(-1)$ respectively, is to be interpreted as ∞. Because of the alternation property, these cases cannot occur simultaneously.

Theorem 1. Necessary and sufficient conditions that a non-constant rational function in lowest terms given by

$$f(s) = c \frac{\prod_{i=1}^{M} (s+\alpha_i)}{\prod_{j=1}^{N} (s+\beta_j)}$$

be either an RC or RL u.b.f. are

(i) the α_i and β_j are real and distinct;

(ii) for each $j = 1, 2, \ldots, N$, $|\beta_j| > 1$;

(iii) $M = N = n$, or $M = n$, $N = n-1$, or $M = n-1$, $N = n$, n a positive integer;

(iv) the α_i and β_j, each indexed according to increasing numerical value, exhibit one of the order relations (12) or (12)$'$;

(v) c is real and subject to the inequality

$$|c| \leq \text{Min} \left[\frac{1}{|f^{*}(1)|} , \frac{1}{|f^{*}(-1)|} \right],$$

where $f^{*}(s) = f(s)/c$.

We can also say something more precise about the range of c, and we can distinguish between an RC and RL u.b.f. . For this purpose, it is convenient to classify the order relations (12) and (12)$'$ into three types. We do this in the following way, which is dictated by considerations arising later in the proof.

Let S be the ordered sequence of the α_i and β_j to which we have added as a last term $\alpha_n = \infty$ or $\beta_n = \infty$ in case f(s) has a zero or a pole at infinity respectively. Then S may be considered to be the juxtaposition of two sub-sequences S_1 and S_2, where S_1 contains all those terms of S which lie in the semi-closed interval $(-\infty, -1]$, and S_2 contains all terms of S which lie in the semi-closed interval $(-1, \infty]$. Symbolically, we can write $S = (S_1, S_2)$. Of course either S_1 or S_2 could be vacuous. Form the new sequence $S' = (S_2 . S_1)$. It is clear that S_1 represents a cyclical permutation of S. We can then state the following:

Definition: S is said to be of Type I if S' begins with a β, of Type II if S' begins with an $\alpha \geq 1$, and of Type III if S' begins with an α such that $-1 < \alpha < 1$. For example, consider the following three S sequences:

S: $\alpha_1 = -4$, $\beta_1 = -3$, $\alpha_2 = -1$, $\beta_2 = \infty$;

S: $\alpha_1 = -3$, $\beta_1 = -2$, $\alpha_2 = 3$, $\beta_2 = \infty$;

S: $\beta_1 = -4$, $\alpha_1 = -3$, $\beta_2 = -2$, $\alpha_2 = 0$, $\beta_2 = 3$.

Then the corresponding sequences S' would be respectively:

S': $\beta_2 = \infty$, $\alpha_1 = -4$, $\beta_1 = -3$, $\alpha_2 = -1$;

S': $\alpha_2 = 3$, $\beta_2 = \infty$, $\alpha_1 = -3$, $\beta_1 = -2$;

S': $\alpha_2 = 0$, $\beta_2 = 3$, $\beta_1 = -4$, $\alpha_1 = -3$, $\beta_2 = -2$.

The original sequences S would therefore be of Types I, II and III respectively.

We can now formulate the result to which we have already alluded, using the classification of the sequence S and the function $f^*(s)$ previously defined

Theorem 2. Let f(s) be as in Theorem 1, with c real; and let (i)-(iv) of that theorem hold for the α_i and β_j. Then f(s) is an RC u.b.f. if and only if c satisfies the following conditions:

(a) if S is of Type I then $0 < |c| \leq |1/f^*(-1)|$ and

$\operatorname{sgn} c = \operatorname{sgn} f^*(1)$, $\alpha_i \neq -1$, $(i=1,2,\ldots,M)$,

$\operatorname{sgn} c = \left.\dfrac{f^*(s)}{s-1}\right|_{s=1}$, one $\alpha_i = -1$;

(b) if S is of Type II then $0 < |c| \leq |1/f^*(1)|$ and

$\operatorname{sgn} c = -\operatorname{sgn} f^*(1)$;

(c) if S is of Type III then

$$0 < |c| \leq \text{Min} \left[\left|\frac{1}{f^*(1)}\right| , \left|\frac{1}{f^*(-1)}\right|\right] ,$$

and

$\operatorname{sgn} c = -\operatorname{sgn} f^*(1)$.

Corollary. Necessary and sufficient conditions that f(s) be an RL u.b.f. are given by those of Theorem 2 if c is replaced by -c throughout.

Theorems 1 and 2 will follow directly when we have established their counterparts which characterize RC and RL u.b.f.'s F(p). These are next described.

Theorem 3. Necessary and sufficient conditions that a non-constant, rational function in lowest terms given by

$$F(p) = \nu \frac{\prod_{i=1}^{m} (p+\eta_i)}{\prod_{j=1}^{n} (p+\lambda)}$$

be either an RC or RL u.b.f. are

(i) the η_i and λ_j are real and distinct;

(ii) for each $j = 1, 2, \ldots, n$, $\lambda_j > 0$;

(iii) $m = n$, or $m = n-1$, n a positive integer;

(iv) the η_i and λ_j, each indexed according to increasing numerical value, exhibit one of the order relations (9) or (9)';

(v) $\varkappa$ is real and subject to the inequality

$$0 < \varkappa \le \mathrm{Min}\left[\left|\frac{1}{F^*(0)}\right| \;,\; \left|\frac{1}{F^*(\infty)}\right|\right] ,$$

where $F^*(p) = F(p)/\varkappa$.

To state the analogue of Theorem 2, we again first classify the order relations (9), (9)′ into types. Let S^* be the ordered sequence of η_i and λ_j.

Definition: The sequence S^* is said to be of Type I, II or III respectively, if S^* begins with a λ, an $\eta \ge 0$, or an $\eta < 0$ respectively.

Thus, if (9) holds, S^* is of Type I, while (9)′ yields an S^* of either Type II or III.

Theorem 4. Let F(p) be as in Theorem 3, with $\varkappa$ real; and let (i)-(iv) of that theorem hold for the η_i and λ_j. Then F(p) is an RC u.b.f. if and only if $\varkappa$ satisfies one of the following conditions:

(a) if S^* is of Type I, then $0 < \varkappa \le \prod_{j=1}^{n} \lambda_j / \prod_{i=1}^{m} \eta_i$;

(b) if S^* is of Type II, then $-1 \le \varkappa < 0$;

(c) if S^* is of Type III, then

$$- \mathrm{Min}\left[-\prod_{j=1}^{n} \lambda_j / \prod_{i=1}^{m} \eta_i \;,\; 1\right] \le \varkappa < 0 .$$

Corollary. Necessary and sufficient conditions that F(p) be an RL u.b.f. are given by those of Theorem 4 if $\varkappa$ is replaced by $-\varkappa$ throughout.

So as not to interrupt the development of our results on factorization at this point, we give the proofs of Theorems 1 - 4 in an Appendix. However, the following remark is pertinent.

Remark: In Theorems 1 and 2 we have not postulated that f(s) is a u.b.f. It is a consequence of our proofs that this is so if and only if c is in the specified ranges. Because of this fact we may draw another conclusion from our results.

For given arbitrary zeros and poles with the latter outside the unit circle, a rational function f(s), as in (10), is a u.b.f. if and only if its multiplicative constant c is subject to the inequality

$$0 \leq |c| \leq 1/M , \qquad (13)$$

where $M = \operatorname{Max}_{|s|=1} |f^*(s)|$, $f^* = f/c$. This follows directly from the Maximum Modulus Theorem applied to f(s) in the unit disc.

In particular, for the f(s) of Theorem 2, (13) and the proof of Theorem 2 imply that

$$\operatorname{Max}_{|s|=1} |f^*(s)| = \begin{cases} |f^*(-1)| , & S \text{ of Type I,} \\ |f^*(1)| , & S \text{ of Type II,} \\ \operatorname{Max}\,[|f^*(1)|, |f^*(-1)|], & S \text{ of Type III .} \end{cases}$$

It is easy to prove directly the first two of these formulas. The third seems a bit more recondite. Similar results hold for F(p) of Theorem 4.

4. Factorization Theorems

We next consider the application of the results of the preceding sections to the factorization of certain classes of u.b.f.'s. We discuss only the case of u.b.f.'s f(s). At the end of this section, we indicate briefly how our results can be carried over to u.b.f.'s F(p). It is to be understood throughout that all of our products are finite products.

According to Theorem 1, a u.b.f. which is a product of RC and RL u.b.f.'s can have only real zeros and poles. We establish a partial converse of this statement in

Theorem 5. Let the real, non-constant, rational function $f(s)$, in the canonical product form (10), have real zeros and poles, with the latter outside the unit circle. Then there exists a constant $c_o > o$ such that for c real and $o < |c| \leq c_o$, $f(s)$ is either an RC or RL u.b.f. or is a product of such functions.

Proof: If the zeros and poles of $f(s)$ alternate, then c_o may be taken as $\text{Min}\,[1/|f^*(1)|\ ,\ 1/|f^*(-1)|]$, $f^*(s) = f(s)/c$. By Theorem 1, $f(s)$ will be either an RC or RL u.b.f. if $0 < |c| < c_o$.

If the alteration property does not hold, then consider the ordered, increasing sequence, S, of the α_i and β_j in (10) each suitably re-indexed. If there are equal α's or equal β's, they are to be assigned separate index numbers in the sequence. Also, if ∞ is a zero or a pole of $f(s)$ of order m, it is to be entered m times at the end of S as an α or β respectively.

The sequence S may be partitioned into disjoint sub-sequences $S^{(i)}$, $(i=1,2,\ldots,r)$ such that each $S^{(i)}$ has its α's and β's alternating and S is the union of the $S^{(i)}$. For example, one such decomposition is to take each α_i separately and each β_j separately as sub-sequences. Corresponding to each such sub-sequence $S^{(i)}$, form the rational function $f_i(s) = c_i f_i^*(s)$ where $f_i^*(s)$ has its zeros and poles at the $-\alpha_i$ and $-\beta_j$ respectively of $S^{(i)}$, and has multiplicative constant 1. Let

$$c_{io} = \text{Min}\left[\left|\frac{1}{f_i^*(1)}\right| , \left|\frac{1}{f_i^*(-1)}\right|\right], \ (i=1,\ 2,\ \ldots,\ r)\ .$$

If now, we take $c_o = \pi_{i=1}^r c_{io}$, then for any real c, such that $o < |c| \leq c_o$ we can find real c_i, $(i=1,\ 2,\ \ldots,\ r)$ so that $c = \pi_{i=1}^r c_i$, and for each i, $o < |c_i| \leq c_{io}$. By Theorem 1, each $f_i(s)$ with this choice of c_i, is either an RC or an RL u.b.f. Evidently $f(s) = \pi_{i=1}^r f_i(s)$. This completes the proof of Theorem 5.

Theorem 5 establishes that the factorization of a real, rational function, $f(s)$, having only real zeros and poles is always possible in the stated form for sufficiently small $|c|$. The question arises as to how large we can take $|c|$, i.e., what is the value of the positive constant, c_o^*, such that factorization in terms of RC and RL u.b.f.'s is possible for $o < |c| < c_o^*$ and impossible for $|c| > c_o^*$? The existence of such a constant c_o^* can be established using the ideas of Theorem 5, and the fact that, in any event, $|c| \leq 1/M$ where $M = \operatorname{Max}_{|s|=1} |f^*(s)|$, $f^* = f/c$, for $f(s)$ to be u.b. In order to consider this question, we must allow a more general type of decomposition of the sequence S than that used in Theorem 5. Namely, in terms of the corresponding function, $f(s)$, we must make provision for a situation where $f(s) = f_1(s)\, f_2(s)$ and $f_1(s)$ has a zero and $f_2(s)$ a pole at the same point $s = -s_o$. Then s_o does not appear in S. Thus, we are motivated to make the following definition.

Let $S^{(1)}$ and $S^{(2)}$ be two numerically ordered sequences of α's and β's. By the <u>product</u>, $S^{(1)} \times S^{(2)}$ we mean a third numerically ordered sequence S whose terms consist of all the terms of both $S^{(1)}$ and $S^{(2)}$, suitably re-indexed, except that if an α (β) of $S^{(1)}$ is equal to a β (α) of $S^{(2)}$, both terms are deleted from S, (cf. [3] p. 66). We write $S = S^{(1)} \times S^{(2)}$ and speak of the factorization of S into the factors $S^{(1)}$ and $S^{(2)}$. The product thus defined is seen to be both commutative and associative.

The determination of c_o^* is still open in the case of a general sequence S. In practice, one can try different decompositions of S in order to improve the constant c_o of Theorem 5. However, for certain special classes of sequences, S, the value of c_o^* can be given explicitly as the following theorem shows.

<u>Theorem 6</u>. Let $f(s)$ be given as in Theorem 5. If the sequence S correspondong to $f(s)$ can be factored into a product of sequence $S^{(i)}$, $(i=1,2,\ldots, r)$ such that (a) each $S^{(i)}$ is of Type I, or (b) each $S^{(i)}$ is of Type II, then $f(s)$.

is factorable into RC and RL u.b.f.'s for all values of c for which f(s) is u.b., i.e., for $o < |c| \leq 1/M$, $M = \max_{|s|=1} |f^*(s)|$, $f^* = f/c$. In addition, $M = |f^*(-1)|$ or $M = |f^*(1)|$ according as (a) or (b) holds respectively.

<u>Proof:</u> Suppose (a) holds. We employ the construction of Theorem 5 but invoke (a) of Theorem 2 and its corollary. We have $f(s) = \prod_{i=1}^{r} f_i(s)$ for $o < |c| \leq \prod_{i=1}^{r} |1/f_i^*(-1)|$. But by the remark at the end of Section 3

$$\max_{|s|=1} |f_i^*(s)| = |f_i^*(-1)| \text{ , } (i= 1, 2, \ldots, r).$$

Hence

$$M = \max_{|s|=1} |f^*(s)| = \max \prod_{i=1}^{r} |f_i^*(s)| = \prod_{i=1}^{r} |f_i^*(-1)| = f^*(-1) \text{ ,}$$

and the theorem follows.

The case when (b) holds is treated in the same way except that (b) of Theorem 2 and its corollary are used. This completes the proof of Theorem 6.

In applying Theorem 6, it would be helpful to know when the sequence S can be factored into sequences of the specified type. Such conditions can be given. They are best expressed in terms of the sequence $S' = (S_2, S_1)$ which has been defined earlier in connection with Theorem 2. The same definition carries over to the more general sequences S considered here. We also make use of the "excess function," E(x), which has already occurred in [3] in connection with another problem.

<u>Definition:</u> In any sequence of α's and β's let E(x), where x is any term in the sequence, denote the difference between the number of β's in the sequence up to and including x and the number of α's in the sequence up to and including x.

Then we can state

Theorem 7: S can be factored into a finite product of sequences of Type I if and only if $E(x) \geq o$ for every term x of the sequence S′.

Theorem 8: S can be factored into a finite product of sequences of Type II if and only if no α_i is in the open interval (-1, 1) and $E(x) \leq o$ for every term of the sequence S′.

Theorem 9: Let S satisfy the conditions for factorization given in Theorem 7 or Theorem 8. Then S may be factored into a product of $m = \mathrm{Max}_{x \in S'} |E(x)|$ but no smaller number of sequences of the specified type.

The proofs of these theorems may be carried out by induction and because of their straightforward nature, will be omitted.

It is clear from the relation between Theorems 1 and 2, and Theorems 3 and 4, as brought out in the Appendix, that results corresponding to Theorems 5-9 can immediately be written down for a function F(p) given by (8) where the Π_i and λ_j are real with the latter positive. One simply replaces S and S′ by S^*, c by $\varkappa$. $f^*(1)$ or $f^*(-1)$ by $F^*(\infty)$ or $F^*(o)$ respectively.

5. Discussion

In this section, we list a number of miscellaneous remarks about various aspects of the preceding investigation.

(a) It is clear that condition (a) of Theorem 6 can be weakened to allow S also to have sub-sequences $S^{(i)}$ of Type III as long as the associated function $f_i^*(s)$ is such that

$$\mathrm{Min}\left[\left|\frac{1}{f_i^*(1)}\right|, \left|\frac{1}{f_i^*(-1)}\right|\right] = \left|\frac{1}{f_i^*(-1)}\right| .$$

A similar remark applies to condition (b) with the above minimum now being $|1/f_i^*(1)|$.

(b) Although f(s) in Theorems 5 and 6 has real zeros and poles, it is easily seen that the corresponding impedance Z(p) can have zeros and poles which are complex as well.

(c) In Theorems 1 and 2 we can get a variety of forms for u.b.f.'s by forming the composite function $f[g(s)]$ where $g(s)$ is u.b. For example, replacing s by s^2 gives a particularly simple form which still has a network interpretation since s^2 is an LC u.b.f.

(d) It can be shown that none of the classes of u.b.f.'s considered in this paper, including the composite functions mentioned in the previous remark correspond to so-called singular impedances $Z(p)$, i.e., impedances such that $Z(i\omega_0) = ai$, a, ω_0 real and $\neq 0$.

(e) If $f(s)$ is a u.b.f. having real zeros and poles, where some of the zeros lie in the open interval $(-1,1)$, then $f(s)$ can be written as a product $f(s) = g(s)h(s)$ where $g(s)$ is an LC u.b.f. and $h(s)$ is u.b. having real zeros and poles, with none of the zeros in $(-1,1)$. This procedure obviates the consideration of Type III sequences but, in general, it may complicate the realization of $f(s)$. An example will suffice to illustrate the technique.

Consider $f(s) = c(s+1/3)(s+5)/(s+2)(s+4)$. We can write $f(s)=g(s)h(s)$ where $g(s) = 3(s+1/3)/(s+3)$, $h(s)= c(s+3)(s+5)/3(s+2)(s+4)$. Then $g(s)$ is an LC u.b.f. by (3), and $h(s)$ is an RC u.b.f. by Theorem 2(a) when $0 < c \leq 9/8$.

By Theorem 5, we have the simpler alternate factorization into $f(s) = f_1(s)\ f_2(s)$ where $f_1(s) = c_1(s+1/3)/(s+2)$ and $f_2(s) = c_2(s+5)/(s+4)$, $c = c_1 c_2$, $0 < |c_1| \leq 3/2$, $0 < |c_2| \leq 3/4$.

6. Examples

1. Consider the impedance

$$Z(p) = \frac{31.5p^3 + 42.1p^2 + 16.7p + 1.7}{34.5p^3 + 54.9p^2 + 29.3p + 5.3}.$$

Transforming $Z(p)$ into $f(s)$ by (1) and (2) and factoring $f(s)$, we find

$$f(s) = \frac{(s-2)(s+4)(s+5)}{10(s+2)(s+3)(s+4.5)} .$$

By Theorem 5, since the zeros and poles of $f(s)$ are all real, factorization of $f(s)$ into RC and RL u.b.f.'s is possible for some range of the multiplicative constant. We will see whether the given multiplicative constant, 1/10, is in this range.

The sequence S for $f(s)$ is $S: \alpha_1 = -2,\ \beta_1 = 2,\ \beta_2 = 3,\ \alpha_2 = 4,\ \beta_3 = 4.5,\ \alpha_3 = 5$. Hence, the sequence S' is

$$S': \beta_1 = 2,\ \beta_2 = 3,\ \alpha_2 = 4,\ \beta_3 = 4.5,\ \alpha_3 = 5,\ \alpha_1 = -2 .$$

We next calculate the function $E(x)$ for S'. We have $E(\beta_1) = 1$, $E(\beta_2) = 2$, $E(\alpha_2) = 1$, $E(\beta_3) = 2$, $E(\alpha_3) = 1$, $E(\alpha_1) = 0$. Since $E(x) \geq 0$ for every term of S', Theorem 7 applies, and Theorem 9 then indicates that S can be factored into a product of two sequences of Type I. Further, by Theorem 6, since $|f^*(-1)| = 36/7$, we get the range $0 < |c| \leq 7/36$ for the multiplicative constant. Thus the given $f(s)$ is factorable.

There are many ways of factoring S into the product of two Type I sequences, $S = S^{(1)} \times S^{(2)}$. We choose $S^{(1)}: \beta_1^{(1)} = 3,\ \alpha_1^{(1)} = 5$;

$$S^{(2)}: \alpha_1^{(2)} = -2,\ \beta_1^{(2)} = 2,\ \alpha_2^{(2)} = 4,\ \beta_2^{(2)} = 4.5;$$

and form $f(s) = f_1(s)f_2(s)$, where

$$f_1(s) = c_1 \frac{(s+5)}{(s+3)},\quad f_2(s) = c_2 \frac{(s-2)(s+4)}{(s+2)(s+4.5)} .$$

Here $c_1 c_2$ must be 1/10, and by Theorem 2

$$0 < |c_1| \leq |1/f_1^*(-1)| = 1/2 ,$$

$$0 < |c_2| \leq |1/f_2^*(-1)| = 7/18 .$$

We choose $c_1 = 1/2$, $c_2 = 1/5$. The impedances corresponding to f_1 and f_2 are then

$$Z_1(p) = \frac{7p+4}{p} , \quad Z_2(p) = \frac{15.5p^2 + 12.4p + 1.7}{17.5p^2 + 19.6p + 5.3} .$$

By Theorem 2(a), since sgn $f_1^*(1) = +1$ and sgn $f_2^*(1) = -1$, we know that Z_1 will be an RC impedance and Z_2 an RL impedance. $Z(p)$ is then realized by a balanced bridge shose opposite pairs of arms are Z_1, $1/Z_1$ and $1/Z_2$, Z_2 respectively.

2. Consider the following function, $f(s)$, having real zeros and poles:

$$f(s) = c \, \frac{(s-1)(s+1)}{(s-2)(s-3)} .$$

Let us determine a range for c, as guaranteed by Theorem 5, for which $f(s)$ is factorable into RC and RL u.b.f.'s. We have

$$S: \beta_1 = -3, \beta_2 = -2, \alpha_1 = -1, \alpha_2 = 1,$$

so that

$$S': \alpha_2 = 1, \beta_1 = -3, \beta_2 = -2, \alpha_1 = -1 .$$

Then

$$E(\alpha_2) = -1, E(\beta_1) = 0, E(\beta_2) = 1, E(\alpha_1) = 0 .$$

Since $E(x)$ is neither non-negative or non-positive, a factorization of S into factors which are either exclusively of Type I or Type II is not possible. We must therefore employ the general procedure of Theorem 5 and try various factorizations of the sequence S.

We shall choose two factorizations out of those which are possible.

(a) Let $f(s) = f_1(s)f_2(s)$, where

$$f_1(s) = c_1\frac{s-1}{s-2}\ ,\ f_2(s) = c_2\ \frac{s+1}{s-3}\ .$$

The sequence $S^{(1)}$ for f_1 is of Type I. Hence, by Theorem 2(a),

$$0 < |c_1| \leq |1/f_1^*(-1)| = 1.5\ .$$

As for the sequence $S^{(2)}$, it is of Type II. Hence, by Theorem 2(b),

$$0 < |c_2| \leq |1/f_2^*(1)| = 1\ .$$

Since $c = c_1 c_2$, $f(s)$ will be factorable for $0 < |c| \leq 1.5$.

(b) Let $f(s) = f_1(s)f_2(s)$ where

$$f_1(s) = c_1\ \frac{s-1}{s-3}\quad ,\quad f_2 = c_2\ \frac{s+1}{s-2}$$

Again we have $S^{(1)}$ of Type I and $S^{(2)}$ of Type II, so that

$$0 < |c_1| \leq |1/f^*(-1)| = 2\ ,$$

$$0 < |c_2| \leq |1/f^*(1)\ | = .5\ ,$$

and, therefore, $f(s)$ is factorable in this way for $0 < |c| \leq 1$.

It is conjectured that the factorization in example 2(a) provides the maximum range of c, i.e., $c_o^* = 1.5$. On the other hand, it is easily found that $\underset{|s|=1}{\text{Max}} |f^*(s)| = .4$. Thus $f(s)$ is a u.b.f. for $0 \leq |c| \leq 2.5$, so that factorization into RC and RL u.b.f.'s is not possible for the full range of c.

Appendix

Proof of Theorems 3 and 4.

We remark that Theorem 3 is implied by Theorem 4 and its corollary. We have only to see whether condition (v) of Theorem 3 is implied by conditions (a) and (b) of Theorem 4 and its corollary, since condition (c) except for notation is already equivalent to (v).

Now if S^* is of Type I then

$$0 < 1/F^*(0) = \prod_{j=1}^{n}\lambda_j / \prod_{i=1}^{m}\eta_i < 1 = 1/F^*(\infty)$$

when m = n; and when m = n - 1

$$0 < 1/F^*(0) < \infty = 1/F^*(\infty).$$

If S^* is of Type II,

$$1/F^*(0) = \prod_{j=1}^{n}\lambda_j / \prod_{i=1}^{m}\eta_i > 1 = 1/F^*(\infty) .$$

Thus, in every case, condition (v) of Theorem 3 follows. We therefore address ourselves to the proof of Theorem 4 and its corollary.

We already know from Section 2 that conditions (i) - (iv) of Theorem 4 and its corollary are necessary and that $\varkappa$ is a real constant. There remains the proof of sufficiency for (i) - (iv) and the exact specification of the multiplicative constant $\varkappa$ for RC and RL u.b.f.'s.

Applying (1) to F(p) gives

$$Z(p) = \frac{\prod_{j=1}^{n}(p+\lambda_j) + \varkappa \prod_{i=1}^{m}(p+\eta_i)}{\prod_{j=1}^{n}(p+\lambda_j) - \varkappa \prod_{i=1}^{m}(p+\eta_i)} . \qquad (14)$$

Since $Z(p)$ in (14) is replaced by $1/Z(p)$ when $\varkappa$ is replaced by $-\varkappa$, any condition on $\varkappa$ characterizing $F(p)$ as an RC u.b.f. will give a corresponding condition on $-\varkappa$ for $F(p)$ to be an RL u.b.f. and vice versa. Thus, it will suffice to establish our conditions for $\varkappa$ positive, in order to establish both Theorem 4 and its corollary.

As before, it follows that the numerator and denominator of the fraction in (14) have no common factors when $\varkappa \neq 0$. If $Z(p)$ is represented as in (4) with P and Q relatively prime, then we can take the $-\delta_i$ and the $-\gamma_j$ as the zeros of

$$\prod_{j=1}^{n} (p+\lambda_j) + \varkappa \prod_{i=1}^{m} (p+\eta_i) = 0 \qquad (15)$$

and

$$\prod_{j=1}^{n} (p+\lambda_j) - \varkappa \prod_{i=1}^{m} (p+\eta_i) = 0 \qquad (16)$$

respectively.

We now consider $\varkappa$ to be a real variable. Then eq. (16) defines p as an algebraic function of $\varkappa$ having n branches. Denote these by $-\gamma_j(\varkappa)$, $(j = 1, 2, \ldots, n)$. We shall investigate the behavior of these branches (i.e. the root locus of (16)) as $\varkappa$ varies along the real line from 0 to $+\infty$.

First, note that argument of Section 2 may be repeated here for eq. (16) to show that for real $\varkappa$ all of its roots are real. Now in [3], p.56, it has been proved that <u>when the $-\gamma_j(\varkappa)$ are real, they are strictly monotonic functions of $\varkappa$</u>. To determine the direction in which the $-\gamma_j(\varkappa)$ vary we proceed as follows. If $\varkappa = 0$, we see from (16), that by properly indexing the $-\gamma_j(\varkappa)$, we have $-\gamma_j(0) = -\lambda_j$, $(j = 1, 2, \ldots, n)$. Let q be one of 1, 2, ..., n. In the neighborhood of $\varkappa = 0$ we can represent $-\gamma_q(\varkappa)$

by a Maclaurin series:

$$- \gamma_q(\varkappa) = -\lambda_q + a_{1q}\varkappa + \cdots \quad , \tag{17}$$

where

$$a_{1q} = \frac{dp}{d\varkappa} \Big|_{\varkappa = 0}^{p = -\lambda_q} \quad .$$

From (16), we find by differentiating implicitly that

$$a_{1q} = \frac{\prod_{i=1}^{m} (-\lambda_q + \eta_i)}{\prod_{\substack{j=1 \\ j \neq q}}^{n} (-\lambda_q + \lambda_j)} \quad . \tag{18}$$

From this point on, the argument depends upon the type of sequence S^* under consideration. First, let S^* be of Type I. Then, from (18) and (9)

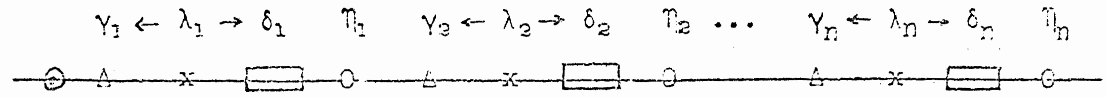
$$\operatorname{sgn} a_{1q} = \frac{(-1)^{q-1}}{(-1)^{q-1}} = +1 \quad ,$$

for each q = 1, 2, ..., n. Hence (17) shows that for small $\varkappa$, as $\varkappa$ increases positively from 0, $-\gamma_q(\varkappa)$ increases from the value $-\lambda_q$. By the italicized statement above, $-\gamma_q(\varkappa)$ will be increasing with increasing $\varkappa$ for all real $\varkappa$.

When $\varkappa \to +\infty$, the roots of equation (16) are the $-\eta_i$ (including one root at ∞ if $m = n - 1$). Hence as $\varkappa$ ranges positively from 0 to $+\infty$, $\gamma_j(\varkappa)$ decreases from λ_j to some η_i. In view of the order relation (9), the only way this can occur and still maintain the monotonic character of the variation is that $\gamma_j(\varkappa)$, (j = 1, 2, ..., n), varies from λ_j to η_{j-1}, while γ_1 decreases from λ_1 to $-\infty$ and then from $+\infty$ to η_n.

Replacing $\varkappa$ by $-\varkappa$, the same argument shows that roots of eq. (15), call them $-\delta_j(\varkappa)$, which are equal to $-\gamma_j(-\varkappa)$, increase from λ_j to η_j,

($j = 1, 2, \ldots, n$). Figure 1 represents the manner in which both δ's and γ's vary.



Root Locus for Type I Sequence
FIGURE 1

For small positive $\varkappa$, since the δ_j and γ_j are both close to the λ_j, it is clear that the order relation (5) holds. As $\varkappa$ increases positively (5) will continue to hold as long as γ_1 remains non-negative. Now γ_1 becomes zero when $\varkappa = \varkappa_o$ in (16) where

$$\varkappa_o = \prod_{j=1}^{n} \lambda_j \Big/ \prod_{i=1}^{m} \eta_i \quad .$$

Thus when $0 < \varkappa \leq \varkappa_o$, the order relation (5) holds.

We must still show that k in (4) is positive. When m = n, (4) and (14) yield $k = (1+\varkappa)/(1-\varkappa)$. Since $0 < \varkappa_o < 1$ for a sequence of Type I, $k > 0$. When $m = n - 1$, $k = 1$.

Thus Z(p) given by (14) is an RC impedance. This completes the proof of the sufficiency of condition (a) in Theorem 4 and its corollary.

We next show that for $\varkappa > \varkappa_o$ or for $\varkappa < 0$, Z(p) in (14) is not an RC impedance. This is clear for $\varkappa > \varkappa_o$ as the preceding argument shows that (5) will be violated. In fact, for $\varkappa > \varkappa_o$, Z(p) is not even positive real.

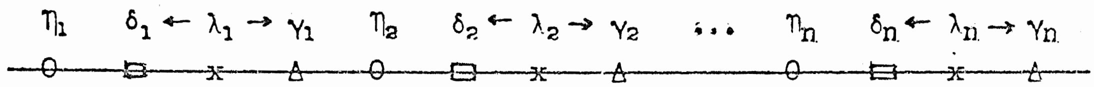
When m = n, this follows for $1 > \varkappa > \varkappa_o$ since γ_1 is then negative; and for $\varkappa > 1$ since $k = (1+\varkappa)/(1-\varkappa)$ is negative. When $\varkappa = 1$,
$k = 2/\sum_{j=1}^{n}(\lambda_j - \eta_j) < 0$.

For $m = n - 1$, γ_1 is negative for all $\varkappa > \varkappa_o$.

Next, suppose Z(p) in (14) were an RC impedance for any $\varkappa = \varkappa_1 < 0$. Then the F(p) with $\varkappa = -\varkappa_1 > 0$ would correspond to 1/Z(p), i.e. would be a RL impedance. This is impossible in view of what has just been about (14) for $\varkappa > 0$. Thus condition (a) of Theorem 4 and its corollary has been proved necessary.

The other two cases, namely S^* of Type II and III are treated in exactly the same way. We omit the details but indicate the salient features of the argument.

For S^* of both Types II and III, the monotonic variation of the δ's and γ's as κ increases from 0 through positive values is shown schematically in Figure 2.



Root Locus for Type II and III Sequence

FIGURE 2

For small positive κ, δ_i and γ_i will be close to λ_i and hence the δ's and γ's will be in the order relation for an RL-impedance which is

$$0 \leq \delta_1 < \gamma_1 < \delta_2 \ldots < \delta_n < \gamma_n \ , \tag{19}$$

where γ_n may be ∞. Since both the δ's and γ's approach the η's when $\kappa \to \infty$, we see from Figure 2 that (19) will hold, up to and including that value of κ for which first $\delta_1 = 0$ or $\gamma_n = \infty$. Only the latter possibility can occur when S^* is of Type II, since $\delta_1 > \eta_1 \geq 0$ for all $\kappa > 0$. As $\gamma_n = \infty$ when $\kappa = 1$, the admissible range for κ is then $0 < \kappa \leq 1$.

For S^* of Type III, $0 < \kappa \leq \text{Min}\left[-\prod_{j=1}^{n} \lambda_j / \prod_{i=1}^{m} \eta_i , 1 \right]$,

where the first quantity in the bracket represents the value of κ which yields $\delta_1 = 0$. That $k > 0$ follows as before. In this way we arrive at conditions (b) and (c) in the corollary of Theorem 4.

This completes the proof of Theorem 4 and its corollary.

Proof of Theorem 1 and 2.

We apply the transformation (2) to f(s). This gives an F(p) as in (8) where

$$\varkappa = c \frac{\prod_{i=1}^{M} (1+\alpha_i)}{\prod_{j=1}^{N} (1+\beta_j)}, \quad \text{if no } \alpha_i = -1 ;$$

$$\varkappa = -2c \frac{\prod_{i=1}^{M}{}' (1+\alpha_i)}{\prod_{j=1}^{N} (1+\beta_j)}, \quad \text{if an } \alpha_i = -1.$$

(Here π' means that the factor corresponding to $\alpha_i = -1$ is omitted.) The sequence S^* formed by suitably indexing the η's and λ's is seen to be the map, term by term, of the sequence $S' = (S_2, S_1)$ previously defined, under the transformation $\eta_i = (\alpha_i - 1)/(\alpha_i + 1)$, $\lambda_j = (\beta_j - 1)/(\beta_j + 1)$. Furthermore, the three types of sequences S go into the similarly numbered types of sequences S^*. (This correspondence, in fact, was the basis of our classification of S). It may thus be verified that conditions (i) - (iv) of Theorem 1 imply that the corresponding conditions (i) - (iv) of Theorem 3 hold for the function F(p) just obtained. In view of the bilinearity of the transformation (2), the converse is also true. We may therefore apply Theorems 3, 4, and the corollary to Theorem 4, to obtain the required necessary and sufficient conditions which c must satisfy.

We restrict our discussion to the application of Theorem 4 to F(p), and there to the case when S is of Type I. The application of Theorem 3 and the discussion of the remaining cases in Theorem 4 are treated in exactly the same way.

By Theorem 4(a), for an RC u.b.f., when no $\alpha_i = -1$, we must have

$$0 < \varkappa = c\,\frac{\prod_{i=1}^{M}(1+\alpha_i)}{\prod_{j=1}^{N}(1+\beta_j)} \leq \frac{\prod_{j=1}^{n}\lambda_j}{\prod_{i=1}^{m}\eta_i} = \prod_{j=1}^{N}\frac{(\beta_j-1)}{(\beta_j+1)}\cdot\prod_{i=1}^{M}\frac{(\alpha_i+1)}{(\alpha_i-1)}\,.$$

Thus, $0 < c \leq 1/f^*(-1)$ if sgn $f^*(1) = +1$, and $1/f^*(-1) \leq c < 0$ if sgn $f^*(1) = -1$. This statement is equivalent to condition (a) given in Theorem 2, when no $\alpha_i = -1$.

If an $\alpha_i = -1$, then again by Theorem 4(a), we get

$$0 < \varkappa = -2c\,\frac{\prod_{i=1}^{M}{}'(1+\alpha_i)}{\prod_{j=1}^{N}(1+\beta_i)} \leq \frac{\prod_{j=1}^{n}\lambda_j}{\prod_{i=1}^{m}\eta_i} = \prod_{j=1}^{N}\frac{(\beta_j-1)}{(\beta_j+1)}\cdot\prod_{i=1}^{M}{}'\frac{(\alpha_i+1)}{(\alpha_i-1)}\,.$$

Hence

$$0 < c \leq \frac{\prod_{j=1}^{N}(\beta_j-1)}{-2\prod_{i=1}^{M}{}'(\alpha_i-1)} = \frac{1}{f^*(-1)}\,,$$

if sgn $\left[-2\prod_{i=1}^{M}{}'(1+\alpha_i)/\prod_{j=1}^{N}(1+\beta_j)\right] = +1$; all signs in the inequality are reversed if the quantity in the bracket is negative. This statement is equivalent to (a) of Theorem 4 when an $\alpha_i = -1$.

This completes our discussion of Theorems 1 and 2.

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