STATE UNIVERSITY OF NEW YORK AT STONY BROOK

COLLEGE OF ENGINEERING

AFCRL-65-799 Report # 37

A PHASE-INTEGRAL APPROXIMATION FOR THE CURRENT DISTRIBUTION ALONG A LOG-PERIODIC ANTENNA

Richard B. Kieburtz

College of Engineering
State University of New York at Stony Brook
Stony Brook, L.I., New York

Technical Report No. 37

Scientific Report No. 1 on
Contract No. AF19(628)-4144
Project No. 4600
Task No. 460004

16 March 1965

Prepared
for

AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
OFFICE OF AEROSPACE RESEARCH
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Reprinted from IEEE TRANSACTIONS
ON *ANTENNAS AND PROPAGATION*
Volume AP-13, Number 5, September, 1965
pp. 813-814

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Introduction

In the analysis of certain types of log-periodic (LP) antennas, considerable information has been obtained by studying the dispersion curve, or k-β diagram, of a uniformly periodic prototype structure.[1] The geometry of an LP structure, such as that indicated in Fig. 1, can be generated from a uniformly periodic geometry by a spatial

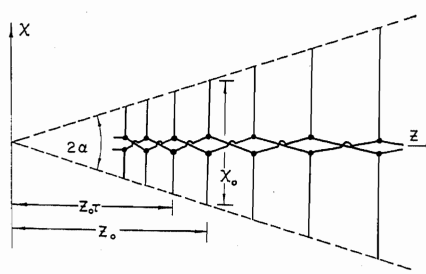

Fig. 1. Log-periodic dipole array.

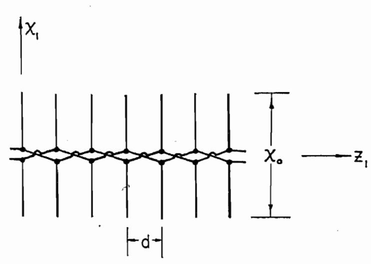

Fig. 2. Periodic prototype dipole array.

transformation of the form

$$ds = \tau^{z_1/d} ds$$

where d is the period along the z_1 axis of the prototype.

When the scaling parameter τ is close to unity, the LP geometry over a distance of several cell lengths is very nearly similar to that of the uniformly periodic structure. Under such circumstances, it is reasonable to expect that the fields in adjacent cells of an LP antenna structure will have similar functional dependence on the spatial coordinates, and that they will be related by a complex phase constant which is nearly equal to the complex phase shift per cell for the prototype structure having the same average electrical dimensions. A method of analysis which may then be applicable is the use of a phase-integral (WKB) approximation for the complex phase which gives the dominant behavior of the fields along the structure. This approximation gives the phase as

$$\phi(z) = \int_0^z \beta_0(z')dz'$$

where the propagation constant $\beta_0(z')$ is that obtained for the uniformly periodic prototype whose electrical dimensions agree with those of the LP structure at the point $z=z'$. It is the purpose of this Communication to establish conditions for the validity of the phase-integral approximation applied in this way to LP antenna problems.

The Phase-Integral Approximation

In order to define a propagation constant, it is necessary to refer to some measurable field component which is a continuous function of the distance z along the structure. Such a quantity might be the current along a two-wire transmission line which feeds an array of dipole elements. We shall confine our discussion to LP structures for which such a measurable quantity can be found, and denote it by $I(z)$. Let us assume that $I(z)$ admits a representation as a product of a quasi-periodic function with a phase-integral term,

Manuscript received April 2, 1965. The work reported here was supported by the Air Force Cambridge Research Labs., Office of Aerospace Research, under Contract AF 19(628)-4144.

[1] R. Mittra and K. E. Jones, "Theoretical Brillouin (k-β) diagram for monopole and dipole arrays and their application to log-periodic antennas," *IEEE Trans. on Antennas and Propagation*, vol. AP-12, pp. 533–540, September 1964.

$$I(z) = \psi(z) \exp\left[-j\int_0^z \beta(z')dz'\right]. \quad (1)$$

The quasi-periodic function $\psi(z)$ is assumed to be similar in form from one cell to the next, if its argument is scaled appropriately to account for the scaling of cell dimensions. Thus, $\psi(z)$ is required to satisfy, approximately, the relation

$$\psi(z)\big|_{z_0\tau_{m+1}<z<z_0\tau_m} \simeq \psi_0(z\tau^m) \quad (2)$$

over a number of cells about the original cell $-M<m<M$. Here, $\psi_0(z)$ is the function $\psi(z)$ evaluated at the original cell, for which $z_0\tau<z<z_0$. The similarity relation (2) will be approximately satisfied then

$$\beta \ll \frac{2\pi}{d}. \quad (3)$$

The argument for this restriction is as follows. If we consider the current variation along the periodic prototype structure, it can be represented by a Fourier series,

$$I_p(z_1) = \sum_{n=-\infty}^{\infty} a_n \exp\left[-j(\beta + 2n\pi/d)z_1\right]. \quad (4)$$

The relative amplitude coefficients a_n/a_0 of those space harmonics for which $2n\pi/d \gg \beta$ will be largely independent of frequency, as little error is incurred by making the quasi-static approximation $\beta=0$ in the three-dimensional wave functions corresponding to these space harmonics. If the cell length d is sufficiently small so that all but the $n=0$ space harmonics have relative amplitudes insensitive to β, then the functional form of the current variation within a unit cell can be approximated by a quasi-static function multiplied by a phase function $e^{-j\beta_0 z_1}$. The quasi-static function depends only on the boundaries and not on frequency, and will therefore be similar for all cells for which the boundary conditions are similar. Thus, a necessary condition that the current variations over adjacent cells of an LP structure be similar in form, apart from differing phase factors, is furnished by (3). This condition is not sufficient by itself, however, as the boundary conditions for each cell depend not only on its length, but also on the nature of the antenna element loading the transmission line at the cell, and thus, may not be similar.

Since the variation of boundary conditions will also lead to a variation in β_0, we can further examine the validity of the phase-integral formula in terms of the variability of β. Define the Fourier transform of the current $I(z)$ as given by (1)

$$\begin{aligned} A(\lambda) &= \int_0^\infty I(z)\exp\left[-j\lambda z\right]dz \\ &= \int_0^\infty \psi(z) \\ &\quad \cdot \exp\left[-j\left\{\lambda z + \int_0^z \beta(z')dz'\right\}\right] dz. \quad (5)\end{aligned}$$

In view of the approximate scaling relation (2) which is presumed to hold over a number of cells about the original cell, the transform $A(\lambda)$ can be approximated by

$$\begin{aligned} A(\lambda) \simeq \int_{z_0\tau}^{z_0} \psi_0(z) \sum_{m=-\infty}^{\infty} \tau^m \exp\Big[-j\Big\{&\lambda\tau^m z \\ &+ \int_0^{z\tau^m} \beta(z')dz'\Big\}\Big] dz. \quad (6)\end{aligned}$$

To evaluate (6), we may use the principle of stationary phase. The stationary phase point is defined by the condition

$$\frac{d}{dz}\left\{\lambda\tau^m z + \int_0^{z\tau^m} \beta(z')dz'\right\} = 0 \quad (7)$$

which leads to the stationary phase point z_λ given by

$$\lambda + \beta(z_\lambda) = 0. \quad (8)$$

Expanding the phase term in the integrand of (6) in powers of $(z-z_\lambda)$, and neglecting powers higher than the second gives

$$\begin{aligned} A(\lambda) = \exp&\left[-j\int_0^{z_\lambda}\beta(z')dz'\right] \\ &\cdot \int_{z_0\tau}^{z_0} \psi_0(z) \sum_{m=-\infty}^{\infty} \tau^m \\ &\cdot \exp\left[-j\frac{\beta'}{2}(\tau^m z - z_\lambda)^2\right] dz \quad (9)\end{aligned}$$

where

$$\beta' = \frac{d\beta}{dz}\Big|_{z=z_\lambda}.$$

If β' is small enough, the exponential of the residual phase function for $m=0$ remains essentially constant over the interval $z_0\tau<z<z_0$, and the integral takes the average of ψ_0 over the original cell, multiplied by the cell length. If β' is not small, then the value of the integral will depend on the functional form of ψ_0 as well as on its average value. Thus the phase-integral formula is useful when

$$\frac{\beta'}{2} \ll \frac{2\pi}{d^2}. \quad (10)$$

In this case the phase-integral approximation is obtained by using the phase constant of the $n=0$ space harmonic for the periodic prototype structure. The coefficient a_0 of this space harmonic also represents the average value of the wave function (apart from the exponential of the phase), just as does the integral of (9).

Conclusion

We have established, as conditions for the applicability of the phase-integral approximation for the dependence of the current along an LP antenna, that the phase constant must satisfy the restrictions (3) and (10). Physically, these restrictions correspond to requiring that the length of cells be kept electrically short, so that only one space harmonic is primarily responsible for the radiation,[2] and to restricting the fractional change per cell in the phase constant to a small amount. Fortunately these restrictions are satisfied by the majority of practical LP antennas, which might be analyzed by exploiting their similarities to their uniformly periodic prototypes.

Richard B. Kieburtz
Dept. of Electrical Sciences
State University of New York
Stony Brook, N. Y.

[2] This condition was first suggested to this author in a discussion by R. Mittra.

APPENDIX

PARAMETERS OF THE L.P. GEOMETRY

The basic description of the L.P. geometry can be recast in terms of a set of parameters which all pass to finite, non-zero limits as the L.P. geometry is replaced by its uniformly periodic prototype. Some conventional parameters for planar L.P. antennas include the scaling factor τ which gives the transformation of linear dimensions from one cell to the next, and the apex angle 2α which describes the envelope of the structure. When this set of parameters is used, the location of the center of any cell is described by its distance z_o from the apex. However, as the L.P. geometry is mapped into that of a periodic structure, the apex angle 2α tends to zero while the distance z_o from the apex to a cell of fixed linear dimension becomes infinite.

It is therefore convenient to utilize a new set of parameters. Choosing the location of a particular cell of the L.P. structure as the origin of the z-axis, we represent the electrical breadth of the antenna element occupying that cell by a parameter $\kappa = k_o x_o$. This parameter is then used as the unit of distance, normalized to free-space wavelength. The breadth of the antenna element occupying a cell which is m cells displaced from the origin is then given by $\kappa\tau^m$. The distance parallel to the z-axis from the original cell can be given in terms of the breadth κ by defining a length-to-breadth factor δ. By comparing the old with the new parameters, it is seen that a connection is furnished by the relation

$$\delta = \frac{(\tau^{-1/2} - \tau^{1/2})}{\tan \alpha}$$

Now as the L.P. geometry is mapped continuously into the periodic geometry, the similarity relation between cells becomes a congruence relation and $\tau \to 1$, the electrical breadth of the structure becomes uniform so that $\kappa\tau^m = k_o x_o$ for all m, and the factor δ becomes the ratio of cell length to cell width, $\delta = d/x_o$. We then have the uniformly periodic prototype whose dimensions are equivalent to those of the L.P. structure at the point where its breadth is given by κ. Note that the parameters of the original cell are preserved under the transformation, with the sole exception of τ.

Having established the close correspondence of the L.P. and uniformly periodic geometries, we should note that the phase-integral approximation for the field variation along the L.P. structure cannot be made to yield an exact result, even if the propagation constant for the prototype is known exactly at all frequencies. The environment of a cell of an L.P. structure never consists of identical cells, as it does for a periodic structure. Thus insofar as the complex phase shift between cells of the L.P. structure depends explicitly on mutual coupling between cells, as reflected in the mathematical description by dependence on τ, it cannot be predicted exactly on the basis of information obtained from the prototype structure.

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Equation (2) should read:

$$\psi(z)\Big|_{z_o\tau^{m+1}<z<z_o\tau^m} \simeq \psi_o(z\tau^m)$$

Equation (6): The upper limit on the second integral should be $z\tau^m$.

Unclassified
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DOCUMENT CONTROL DATA - R&D

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1. ORIGINATING ACTIVITY *(Corporate author)* College of Engineering State University of New York at Stony Brook Stony Brook, Long Island, New York	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

A Phase-Integral Approximation for the Current Distribution Along a Log-Periodic Antenna

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Scientific Report Interim

5. AUTHOR(S) *(Last name, first name, initial)*

Kieburtz, Richard B. Dr.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
Sept. 1965		1

8a. CONTRACT OR GRANT NO. AF19(628)-4144 b. PROJECT AND TASK NO. 4600-04 c. DOD ELEMENT 62405304 d. DOD SUBELEMENT 674600	9a. ORIGINATOR'S REPORT NUMBER(S) Tech. Rept. No. 37 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)* AFCRL-65-799

10. AVAILABILITY/LIMITATION NOTICES Qualified requestors may obtain copies of this report from DDC. Other persons or organizations should apply to the Clearinghouse for Federal Scientific and Technical Information*

11. SUPPLEMENTARY NOTES IEEE Trans. on Ant. and Prop. v. AP-13, pp. 813-814, Sept. 65	12. SPONSORING MILITARY ACTIVITY Hq. AFCRL, OAR (CRD) United States Air Force L.G.Hanscom Field, Bedford, Mass.

13. ABSTRACT

The variation of current with distance from the apex along the feed line of a log-periodic antenna is considered. The current distribution is found to be given approximately by a phase-integral formula in which the differential phase increment is related to the propagation constant on the prototype periodic structure. Conditions for the validity of the phase-integral approximation are investigated. These conditions are that the spacing between adjacent radiating elements is much less than the wavelength, and that the logarithmic increment in the phase constant from one cell of the structure to the next is much smaller than unity.

DD FORM 1 JAN 64 1473

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
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