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A TABLE OF DISTRIBUTIONAL MELLIN TRANSFORMS: I

by

T. A. Loughlin

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Task No. 562806

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ABSTRACT

A list of distributional Mellin transforms is derived. Singular distributions that are considered include delta functionals and pseudofunctions obtained from Hadamard's Finite Part. The ordinary distributions include algebraic and elementary transcendental functions. Proofs or derivations are provided for all results.

TABLE OF CONTENTS

INTRODUCTION

The occurrence of distributions or generalized functions is becoming increasingly more prevalent in modern analysis today. The generalized concepts of distribution theory are being put to use in all fields of theoretical work. This has created a need of extending the various mathematical tables to include functions of this type.

The task of expanding tables to make them commensurate with the developing use of distribution theory is only in its infancy. In the literature, only extensive tables of distributional Laplace transforms may be found [1, 12]. The Mellin transform. $\mathfrak{M}\left\{f(x)\right\} = \int_0^\infty f(x)\, x^{s-1}\, dx = F(s)$, and most others have been treated thus far essentially only in a classical manner. In [5, 6], several distributions in the latter operations are considered by means of forming the Cauchy Principle Value of a divergent integral. This, though, is the extent of the availability of distributions in the existing tables of operational transforms.

The object of this investigation is to begin a table of Mellin Transforms that will be comparable to those existing for the Laplace transformation ([1, 12]) in the distributional sense. Many of the singular distributions that will be considered are formed by extracting Hadamard's Finite Part ([1], section 1 - 4) from a divergent integral. The delta functional and series of delta functionals are also included. Algebraic and elementary transcendental functions comprise the ordinary distributions that are considered. Following the list of transforms, proofs or derivations are provided for all results.

The symbols used for the higher functions and the majority of notations adopted are those used by Erdélyi et al in [3-6]. A reference accompanies each function in the glossary indicating where further representations and formulas using it may be found.

A number found in brackets, [ ], will denote a reference included in the bibliography. Braces, { }, will enclose an equation number. "P_f" indicates a pseudofunction ([1], section 1-4). "F_p" represents Hadamard's finite part of an integral.

GLOSSARY OF SYMBOLS

a, b -- Real positive numbers

x, t, σ, -- Real numbers

j, k, n, -- integers

α, β, η, z, s, -- complex numbers

$$(\alpha)_k \begin{cases} = 1, \quad k = 0, \alpha \neq 0 \\ = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+k-1), \quad k = 1,2,3,\ldots \end{cases}$$

Factorial Function {[7], section 18}

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt \qquad \text{Re } z > 0$$

Gamma Function {[3], section 1.1-1.4}

$$\Gamma(a, z) = \int_z^\infty e^{-t}\, t^{a-1}\, dt$$

Incomplete Gamma Function {[4], section 9.1 and 9.2}

$$B(\alpha, \beta) = \int_0^1 (1-x)^{\alpha-1}\, x^{\beta-1}\, dx \qquad \text{Re } \alpha > 0, \text{ Re } \beta > 0$$

$$= \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Beta Function {[3], Section 1.5}

$$\gamma = \lim_{N\to\infty}\left(\sum_{j=1}^{N}\frac{1}{j} - \log n\right) = 0.5772156649\ldots$$

Euler's Constant

$$C = e^{\gamma}$$

$$\psi(k+1) = 1 + 1/2 + 1/3 + \ldots 1/k - \gamma$$

Logarithmic derivative of Gamma Function {[3], section 1.7}

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

$$\psi'(z) = \frac{d\,\psi(z)}{dz} = \zeta(2, z)$$

Polygamma Function {[3], section 1.16}

$\zeta(\alpha, \beta) = \sum_{N=0}^{\infty} (\beta + n)^{-\alpha}$ $\quad \beta \neq 0, -1, -2$ $\quad \mathrm{Re}\,\alpha > 1$ — Generalized Zeta Function {[3], section 1.10}

$\zeta(\alpha) = \zeta(\alpha, 1) \quad \mathrm{Re}\,\alpha > 1$ — Riemann's Zeta Function {[3], section 1.12}

$${}_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n (\beta_2)_n \cdots (\beta_q)_n} \frac{z^n}{n!}$$

Generalized Hypergeometric Function {[7], chapter 5.}

$p \leq q + 1$; p & q integers; $\beta_i \neq 0, -1, -2, \ldots$

$$F(\alpha, \beta; \eta; z) = {}_2F_1(\alpha, \beta; \eta; z) \qquad \eta \neq 0, -1, -2, \ldots$$

Hypergeometric Function {[7], chapter 4}

$$\Phi(\alpha, \beta; z) = {}_1F_1(\alpha; \beta; z)$$

$$\Psi(\alpha, \beta; z) = \frac{\Gamma(1-\beta)}{\Gamma(\alpha-\beta+1)} \Phi(\alpha, \beta; z) + \frac{\Gamma(\beta-1)}{\Gamma(\alpha)} z^{1-\beta} \Phi(\alpha-\beta+1, 2-\beta; z)$$

Confluent Hypergeometric Functions {[3], chapter 6.}

β - non-integral

See reference for representation when β is integral in Ψ function.

$$\Phi(z, \alpha, \beta) = \sum_{n=0}^{\infty} (\beta + n)^{-\alpha} z^n$$

{[3], section 1.11}

$\beta \neq 0, -1, -2, \ldots$ and either $|z| < 1$,

or $|z| = 1$ and $\mathrm{Re}\,\alpha > 0$

$G(z) = \psi(1/2 + z/2) - \psi(z/2)$ {[3], section 1.8}

$W_{k,\mu}(z) = e^{-z/2} x^{c/2}\, \Psi(a, c; z)$ Whittaker Function

$a = 1/2 - k + \mu; \; c = 2\mu + 1$ {[3], section 6.9}

$$1_+(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$ Heaviside's Step Function

$\delta(x)$ ———————————— Delta Functional

$$\langle f(x), g(x) \rangle = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx$$

LIST OF MELLIN TRANSFORMS

$$\mathcal{M} f(x) = \int_0^\infty f(x) \; x^{s-1} \, dx = F(s) \qquad \text{for } \sigma_1 < \text{Re } s < \sigma_2$$

1. General Operational Formulas*

	$f(x)$	$F(s) = \int_0^\infty f(x) \, x^{s-1} \, dx$	
1-1	$f(ax)$	$a^{-s} F(s)$	$\sigma_1 < \text{Re } s < \sigma_2$
1-2	$f(1/x)$	$F(-s)$	$-\sigma_2 < \text{Re } s < -\sigma_1$
1-3	$f(x^a) \quad a > 0$	$1/a \; F(s/a)$	$a\sigma_1 < \text{Re } s < a\sigma_2$
1-4	$f(x^{-a}) \quad a > 0$	$1/a \; F(-s/a)$	$-a\sigma_2 < \text{Re } s < a\sigma_1$
1-5	$(\log x)^n f(x)$	$\dfrac{d^n F(s)}{ds^n}$	$\sigma_1 < \text{Re } s < \sigma_2$
1-6	$x^\alpha f(x)$	$F(s + \alpha)$	$(\sigma_1 - \text{Re } \alpha) < \text{Re } s < (\sigma_2 - \text{Re } \alpha)$
1-7	$f^{(n)}(x)$	$(-1)^n (s-1)(s-2)\ldots(s-n) F(s-n)$	$(\sigma_1 + n) < \text{Re } s < (\sigma_2 + n)$
1-8	$\left(\dfrac{x\,d}{dx}\right)^n f(x)$	$(-1)^n \, s^n \, F(s)$	$\sigma_1 < \text{Re } s < \sigma_2$
1-9	$\left(\dfrac{d}{dx} x\right)^n f(x)$	$(-1)^n (s-1)^n F(s)$	$\sigma_1 < \text{Re } s < \sigma_2$

e.g. if n = 2,

$$\frac{d}{dx}\left[\frac{x\,d}{dx}\left(x \, f(x)\right)\right]$$

* This listing is obtained from [2], pp. 58 - 59

1-10	$x^{\alpha} f^{(n)}(x)$	$(-1)^{n}(s+\alpha-1)(s+\alpha-2)\ldots(s+\alpha-n)F(s+\alpha-n)$	$(\sigma_1+n-\mathrm{Re}\,\alpha) < \mathrm{Re}\ s < (\sigma_2+n-\mathrm{Re}\,\alpha)$
1-11	$\dfrac{d^n}{dx^n}\left[x^{\alpha} f(x)\right]$	$(-1)^{n}(s-1)(s-2)\ldots(s-n)F(s+\alpha-n)$	$(\sigma_1+n-\mathrm{Re}\,\alpha) < \mathrm{Re}\ s < (\sigma_2+n-\mathrm{Re}\,\alpha)$

2. Algebraic Functions

	$f(x)$	$F(s)=\int_0^{\infty} f(x)\, x^{s-1}\, dx$	
2-1	$1_+(x-a)\, x^{\alpha}$	$\dfrac{-a^{\alpha+s}}{\alpha+s}$ $\quad s\neq-\alpha$	$-\infty < \mathrm{Re}\ s < -\mathrm{Re}\,\alpha$
2-2	$\mathrm{Pf}\ 1_+(x-a)(x-a)^{\eta}$, $\eta \neq -1,-2,\ldots$	$\dfrac{a^{s+\eta}\,\Gamma(\eta+1)\,\Gamma(-s-\eta)}{\Gamma(1-s)}$	$s\neq 1,\ s+\eta\neq 0,1,2,\ldots$ $-\infty < \mathrm{Re}\ s < -\mathrm{Re}\,\eta$
2-3	$\mathrm{Pf}\ 1_+(x-a)\, 1_+(b-x)(x-a)^{\eta}$, $a<b \quad \eta\neq-1,-2,\cdots$	$\dfrac{a^{s+\eta}\,\Gamma(-s-\eta)\,\Gamma(\eta+1)}{\Gamma(1-s)} + \dfrac{b^{s+\eta}}{s+\eta} F(-\eta,-s-\eta;\ 1-s-\eta; a/b)$	$s\neq 1$ $s+\eta\neq 0,1,2,\ldots$
2-4	$\mathrm{Pf}\ \dfrac{1_+(x-a)}{(x-a)}$	$a^{s-1}\left[\log a/C - \psi(1-s)\right]$	$-\infty < \mathrm{Re}\ s < 1$
2-5	$\mathrm{Pf}\ \dfrac{1_+(x-a)\,1_+(b-x)}{(x-a)}$, $a<b$	$a^{s-1}\left[\log \dfrac{a}{C} - \psi(1-s) - \left(\dfrac{b}{a}\right)^{s-1} \Phi\left(\dfrac{a}{b},1,\ 1-s\right)\right]$	$s\neq 1,2,3,\ldots$

2-6	$Pf\,\dfrac{1_+(x-a)}{(x-a)^k}$ $\quad k=1,2,3,\ldots$	$\dfrac{(-1)^{k+1}\,(1-s)_{k-1}\,a^{s-k}}{(k-1)!}\left[\log a - \psi(k-s) + \psi(k)\right]$ $\quad -\infty < \mathrm{Re}\, s < k$
2-7	$Pf\,\dfrac{1_+(x-a)1_+(b-x)}{(x-a)^k}$ $a<b,\ k=1,2,3,\ldots$	$\dfrac{(-1)^{k+1}\,(1-s)_{k-1}\,a^{s-k}}{(k-1)!}\left[\log a + \psi(k) - \psi(k-s) - \left(\frac{b}{a}\right)^{s-k}\Phi\left(\frac{a}{b},1,k-s\right)\right]$ $-\dfrac{b^{s-1}}{(b-a)^{k-1}}\displaystyle\sum_{\nu=0}^{k-3}\dfrac{(1-s)_\nu\,(k-2-\nu)!\,\left(\frac{a}{b}-1\right)^\nu}{(k-1)!}$ $\dfrac{(-1)^{k+1}\,(1-s)_{k-2}\,b^{s-k+1}}{(b-a)(k-1)!}$ $s\neq k,k+1,k+2,\ldots$
2-8	$Pf\,\dfrac{1_+(x-a)(a+b)}{(x-a)(x+b)}$ $a>b$	$a^{s-1}\left[\log\frac{a}{C} - \psi(1-s) - \Phi\left(-\frac{b}{a},1,1-s\right)\right]$ $-\infty < \mathrm{Re}\, s < 2 \quad s\neq 1$
2-9	$Pf\,\dfrac{1_+(x-a)2a}{(x-a)(x+a)}$	$a^{s-1}\left[\log\frac{a}{C} - \psi(1-s) - \frac{1}{2}G(1-s)\right]$ $-\infty < \mathrm{Re}\, s < 2 \quad s\neq 1$
2-10	$Pf\,\dfrac{1_+(x-a)(a+b)}{(x-a)(x+b)}$ $a<b$	$a^{s-1}\left[\log\frac{a}{c} - \psi(1-s)\right.$ $\left.-\left(\frac{a}{b}\right)^{1-s}\pi\csc\pi s + \frac{a}{b}\Phi\left(-\frac{a}{b},1,s\right)\right]$ $-\infty < \mathrm{Re}\, s < 2 \quad s\neq 1,0,-1\ldots$

2-11	$P_f \dfrac{1_+(x-a)(a+b)^2}{(x-a)^2\,(x+b)}$ $a > b$	$a^{s-1}\left\{\left[(s-1)(1+\frac{b}{a})-1\right]\left[\log\frac{a}{C}-\Psi(1-s)\right]\right.$ $\left.+\,s(1+\frac{b}{a})+\Phi(-\frac{b}{a},1,1-s)\right\}$ $-\infty < \mathrm{Re}\ s < 3 \qquad s \neq 1,2$
2-12	$P_f \dfrac{1_+(x-a)4a^2}{(x-a)^2(x+a)}$	$a^{s-1}\left\{\left[(s-1)(1+\frac{b}{a})-1\right]\left[\log\frac{a}{C}-\Psi(1-s)\right]\right.$ $\left.+\,s(1+\frac{b}{a})+\frac{1}{2}G(1-s)\right\} \qquad -\infty < \mathrm{Re}\ s < 3$ $s \neq 1,2$
2-13	$P_f \dfrac{1_+(x-a)(a+b)^2}{(x-a)^2(x+b)}$ $a < b$	$a^{s-1}\left\{\left[(s-1)(1+\frac{b}{a})-1\right]\left[\log\frac{a}{C}-\Psi(1-s)\right]\right.$ $+\,s(1+\frac{b}{a})+(\frac{a}{b})^{1-s}\pi\csc\pi s$ $\left.-\,\frac{a}{b}\Phi(-\frac{a}{b},1,s)\right\}$ $s \neq 1,2 \qquad -\infty < \mathrm{Re}\ s < 3 \qquad s \neq 0,-1,-2..$
2-14	$P_f \dfrac{1}{x-a}$	$-\,a^{s-1}\,\pi\,\mathrm{ctn}\,\pi s \qquad 0 < \mathrm{Re}\ s < 1$
2-15	$P_f \dfrac{1_+(x-a)(a-b)}{(x-a)(x-b)}$ $a > b$	$a^{s-1}\left[\log\frac{a}{C}-\Psi(1-s)-\Phi(\frac{b}{a},1,1-s)\right]$ $-\infty < \mathrm{Re}\ s < 2 \qquad s \neq 1$
2-16	$P_f \dfrac{1_+(x-a)(a-b)}{(x-a)(x-b)}$ $a < b$	$a^{s-1}\left[\log\frac{a}{C}-\Psi(1-s)\right.$ $\left.+\,(\frac{a}{b})^{1-s}\pi\,\mathrm{ctn}\,\pi s-\frac{a}{b}\Phi(\frac{a}{b},1,s)\right]$ $-\infty < \mathrm{Re}\ s < 2$ $s \neq 1, 0, -1, -2, \ldots$

2-17	$P_f \dfrac{1_+(x-a)(a-b)^2}{(x-a)^2(x-b)}$ $a > b$	$a^{s-1}\left\{\left[(s-1)(1-\frac{b}{a})-1\right]\left[\log\frac{a}{\mathfrak{C}} - \psi(1-s)\right] + s(1-\frac{b}{a}) + \Phi(\frac{b}{a}, 1, 1-s)\right\}$ $-\infty < \mathrm{Re}\ s < 3 \qquad s \neq 1, 2.$
2-18	$P_f \dfrac{1_+(x-a)(a-b)^2}{(x-a)^2(x-b)}$ $a < b$	$a^{s-1}\left\{\left[(s-1)(1-\frac{b}{a})-1\right]\left[\log\frac{a}{\mathfrak{C}} - \psi(1-s)\right] + s(1-\frac{b}{a}) - (\frac{a}{b})^{1-s}\pi \operatorname{ctn}\pi s + \frac{a}{b}\Phi(\frac{a}{b}, 1, s)\right\}$ $-\infty < \mathrm{Re}\ s < 3 \qquad s \neq 2, 1, 0, -1, -2, \ldots$
2-19	$P_f \dfrac{1_+(x-a)}{(x-b)}$ $a < b$	$b^{s-1}\left[(\frac{a}{b})^s \Phi(\frac{a}{b}, 1, s) - \pi \operatorname{ctn}\pi s\right]$ $-\infty < \mathrm{Re}\ s < 1 \qquad s \neq 0, -1, -2, \ldots$
2-20	$\dfrac{1_+(x-a)}{(x-b)}$ $a > b$	$a^{s-1}\ \Phi(\frac{b}{a}, 1, 1-s) \qquad -\infty < \mathrm{Re}\ s < 1$
2-21	$\dfrac{1}{x+a}$	$a^{s-1}\pi \csc \pi s \qquad 0 < \mathrm{Re}\ s < 1$
2-22	$\dfrac{1_+(x-a)}{(x+b)}$ $a > b$	$a^{s-1}\ \Phi(-\frac{b}{a}, 1, 1-s) \qquad -\infty < \mathrm{Re}\ s < 1$
2-23	$\dfrac{1_+(x-a)}{(x+a)}$	$\frac{1}{2}a^{s-1}\ G(1-s) \qquad -\infty < \mathrm{Re}\ s < 1$

2-24	$\dfrac{1_+(x-a)}{x+b}$ $\quad a<b$	$b^{s-1}\left[\pi \csc \pi s - \left(\frac{a}{b}\right)^s \Phi\left(-\frac{a}{b},1,s\right)\right]$ $-\infty < \mathrm{Re}\, s < 1 \quad s \neq 0,-1,-2,\ldots$
2-25	$P_f \dfrac{1_+(a-x)}{a-x}$	$a^{s-1}\left[\log \frac{a}{C} - \psi(s)\right] \qquad 0 < \mathrm{Re}\, s < \infty$
2-26	$P_f \dfrac{1_+(x-a)1_+(b-x)}{b-x}$ $a<b$	$b^{s-1}\left[\log \frac{b}{C} - \psi(s) - \left(\frac{a}{b}\right)^s \Phi\left(\frac{a}{b},1,s\right)\right]$ $s \neq 0,-1,-2,\ldots$
2-27	$P_f \dfrac{1_+(x-a)1_+(b-x)}{(b-x)^2}$ $a<b$	$(s-1)b^{s-2}\left[\psi(s-1) - \log \frac{b}{C} - 1 + \left(\frac{a}{b}\right)^{s-1} \left\{\Phi\left(\frac{a}{b},1,s-1\right) - \frac{1}{(s-1)(1-\frac{a}{b})}\right\}\right]$ $s \neq 1,0,-1,\ldots$
2-28	$\dfrac{1_+(b-x)}{x-a}$ $\quad a>b$	$\dfrac{-b^s}{a} \Phi\left(\frac{b}{a},1,s\right) \qquad 0 < \mathrm{Re}\, s < \infty$
2-29	$P_f \dfrac{1_+(b-x)}{(x-b)^2}$	$(s-1)b^{s-2}\left[\psi(s-1) - \log \frac{b}{C} - 1\right]$ $s \neq 1 \quad 0 < \mathrm{Re}\, s < \infty$
2-30	$\dfrac{1_+(b-x)}{(x-a)^2}$ $\quad a>b$	$b^{s-1}\left[\frac{1}{a-b} - \frac{(s-1)}{a} \Phi\left(\frac{b}{a},1,s-1\right)\right]$ $s \neq 1 \quad 0 < \mathrm{Re}\, s < \infty$

3. ELEMENTARY TRANSCENDENTAL FUNCTIONS

3-1	$1_+(x-a)\log(x-a)$	$\dfrac{a^s}{s}\left[\psi(-s) - \log \frac{a}{C}\right] \qquad -\infty < \mathrm{Re}\, s < 0$
3-2	$1_+(x-a)\log(x+b)$ $a>b$	$\dfrac{-a^s}{s}\left[\Phi\left(-\frac{b}{a},1,-s\right) + \log(a+b)\right] \qquad -\infty < \mathrm{Re}\, s < 0$

3-3	$1_+(x-a)\log(x+a)$	$-\frac{a^s}{s}\left[\frac{1}{2}G(-s)+\log(2a)\right]$ $\quad -\infty<\mathrm{Re}\ s<0$
3-4	$1_+(x-a)\log(x+b)$ $\quad a<b$	$\frac{a^s}{s}\left[\frac{a}{b}\Phi(-\frac{a}{b},1,s+1)+(\frac{b}{a})^s\pi\csc\pi s-\log(a+b)\right]$ $\quad -\infty<\mathrm{Re}\ s<0 \quad s\neq -1,-2,\ldots$
3-5	$1_+(x-a)\log\lvert x-b\rvert$ $\quad a>b$	$-\frac{a^s}{s}\left[\Phi(\frac{b}{a},1,-s)+\log(a-b)\right]$ $\quad -\infty<\mathrm{Re}\ s<0$
3-6	$1_+(x-a)\log\lvert x-b\rvert$ $\quad a<b$	$-\frac{a^s}{s}\left[(\frac{b}{a})^s(\pi\,\mathrm{ctn}-\pi s)+\log(b-a)+\frac{a}{b}\Phi(\frac{a}{b},1,s+1)\right]$ $\quad -\infty<\mathrm{Re}\ s<0 \quad s=-1,-2,-3,\ldots$
3-7	$1_+(x-a)\log\lvert x^2-b^2\rvert$ $\quad a>b$	$-\frac{a^s}{s}\left[\Phi(\frac{b^2}{a^2},1,\frac{s}{2})+\log(a^2-b^2)\right]$ $\quad -\infty<\mathrm{Re}\ s<0$
3-8	$1_+(x-a)\log\lvert x^2-a^2\rvert$	$\frac{a^s}{s}\left[\psi(-s)-\frac{1}{2}G(-s)-\log\frac{2a^2}{C}\right]$ $\quad -\infty<\mathrm{Re}\ s<0$
3-9	$1_+(x-a)\log\lvert x^2-b^2\rvert$ $\quad a<b$	$\frac{-a^s}{s}\left[\log(b^2-a^2)+(\frac{b}{a})^s\pi(\frac{1+\cos\pi s}{\sin\pi s})+(\frac{a}{b})^2\Phi(\frac{a^2}{b^2},1,1+\frac{s}{2})\right]$ $\quad -\infty<\mathrm{Re}\ s<0 \quad s\neq -1,-2,\ldots$
3-10	$1_+(x-a)\log^2(x-a)$	$\frac{a^s}{s}\left\{2(\psi(-s)+\gamma)(\log a-\frac{1}{s})-\log^2 a-\frac{\pi^2}{6}-\psi'(1-s)-(\gamma+\psi(1-s))^2\right\}$ $\quad -\infty<\mathrm{Re}\ s<0$
3-11	$P_f\,\frac{1_+(x-a)\log(x-a)}{(x-a)}$	$a^{s-1}\left\{(\psi(1-s)+\gamma)(\frac{1}{s-1}-\log a)+\log^2\sqrt{a}+\frac{\pi^2}{12}+\frac{\psi'(2-s)}{2}+\frac{1}{2}(\gamma+\psi(2-s))^2\right\}$ $\quad -\infty<\mathrm{Re}\ s<1$
3-12	$1_+(x-a)1_+(b-x)\log(x-a)$ $\quad a<b$	$\frac{a^s}{s}\left[\psi(-s)+\log\frac{C}{a}+(\frac{a}{b})^{-s}(\Phi(\frac{a}{b},1,-s,)+\log(b-a))\right]$ $\quad s\neq 0,1,2,\ldots$

3-13	$1_+(x-a)\, e^{-\alpha(x-a)}$ $\operatorname{Re}\alpha > 0$	$e^{\alpha a}\alpha^{-s}\Gamma(s) - \frac{a^s}{s}\Phi(1, s+1; \alpha a)$ $s \neq 0, -1, -2, \ldots$
3-14	$1_+(x-a)1_+(b-x)e^{-\alpha(x-a)}$ $a < b$	$\frac{e^{-\alpha(b-a)}b^s}{s}\Phi(1, s+1; \alpha b) - \frac{a^s}{s}\Phi(1, s+1; \alpha a)$ $s \neq 0, -1, -2, \ldots$
3-15	$P_f 1_+(x-a)(x-a)^{\eta}$ $e^{-\alpha(x-a)}$ $\operatorname{Re}\alpha > 0,\ \eta \neq -1, -2, \ldots$	$\Gamma(\eta+1)a^{s+\eta}\ \Psi(\eta+1, \eta+1+s; a\alpha)$ $-\infty < \operatorname{Re} s < \infty$
3-16	$1_+(x-a)\sin b(x-a)$	$\frac{b^{-s}}{2i}\left\{ e^{i[\frac{\pi s}{2} - ab]}\Gamma(s, -iab) - e^{-i[\frac{\pi s}{2} - ab]} \Gamma(s, iab)\right\}$ $-\infty < \operatorname{Re} s < 1$
3-17	$1_+(x-a)\cos b(x-a)$	$\frac{b^{-s}}{2}\left\{ e^{i[\frac{\pi s}{2} - ab]}\Gamma(s, -iab) + e^{-i[\frac{\pi s}{2} - ab]} \Gamma(s, iab)\right\}$ $-\infty < \operatorname{Re} s < 1$
3-18	$1_+(x-a)1_+(b-x)$ $\sinh \alpha(x-a)$ $a < b$	$\frac{\alpha a^{s+1}}{s(s+1)}{}_1F_2(1; \frac{s+2}{2}, \frac{s+3}{2}; \frac{\alpha^2 a^2}{4}) + \frac{b^s}{2s}\left\{ e^{-a\alpha} \Phi(s, s+1; \alpha b) - e^{a\alpha}\Phi(s, s+1; -\alpha b)\right\}$ $s \neq 0, -1, -2, \ldots$
3-19	$1_+(x-a)1_+(b-x)\cosh$ $\alpha(x-a)$ $a < b$	$\frac{-a^s}{s}{}_1F_2(1; \frac{s+1}{2}, \frac{s+2}{2}; \frac{\alpha^2 a^2}{4}) + \frac{b^s}{2s}\left\{ e^{-a\alpha} \Phi(s, s+1; \alpha b) + e^{a\alpha}\ \Phi(s, s+1; -\alpha b)\right\}$ $s \neq 0, -1, -2, \ldots$
3-20	$1_+(x-a)1_+(b-x)\log$ $(b-x)$ $a < b$	$\frac{-b^s}{s}\left[\psi(s+1) + \log\frac{a}{b} + (\frac{a}{b})^s(\log(b-a) + (\frac{a}{b}) \Phi(\frac{a}{b}, 1, s+1))\right]$ $s \neq 0, -1, -2, \ldots$

4. STEP AND DELTA FUNCTIONALS

4-1	$\delta^{(k)}(x-a)$ $k = 0,1,2,\ldots$	$(1-s)_k\, a^{s-1-k}$	$-\infty < \operatorname{Re} s < \infty$
4-2	$\sum_{\nu=1}^{\infty} \delta(x-a\nu)$	$a^{s-1}\, \zeta(1-s,1)$	$-\infty < \operatorname{Re} s < 0$
4-3	$\sum_{\nu=1}^{\infty} (-1)^{\nu}\, \delta(x-a\nu)$	$-a^{s-1}\, \Phi(-1,1-s,1)$	$-\infty < \operatorname{Re} s < 1$
4-4	$1_{+}(x-a)$	$-\frac{a^{s}}{s}$	$-\infty < \operatorname{Re} s < 0$
4-5	$\sum_{\nu=1}^{\infty} 1_{+}(x-a\nu)$	$-\frac{a^{s}}{s}\, \zeta(-s,1)$	$-\infty < \operatorname{Re} s < -1$
4-6	$\sum_{\nu=1}^{\infty} (-1)^{\nu}\, 1_{+}(x-a\nu)$	$\frac{a^{s}}{s}\, \Phi(-1,-s,1)$	$-\infty < \operatorname{Re} s < 0$

PROOFS AND DERIVATIONS OF RESULTS

5. Algebraic Functions

(5-1) Derivation of formula 2-1, $\mathcal{M}\left\{1_+(x-a)x^{\alpha}\right\} = -\frac{a^{\alpha+s}}{\alpha+s}$ $\quad s \neq \alpha \quad -\infty < \text{Re } s < -\text{Re } \alpha$, follows by direct integration.

(5-2) Derivation of formula 2-2,

$$\mathcal{M}\left\{P_f\ 1_+(x-a)(x-a)^{\eta}\right\} = \frac{a^{s+\eta}\ \Gamma(\eta+1)\Gamma(-s-\eta)}{\Gamma(1-s)} \qquad s \neq 1,\ s+\eta \neq 0,1,2,\ldots$$

$$-\infty < \text{Re } s < -\text{Re } \eta$$

$$\eta \neq -1,-2,\ldots$$

Consider first, $\text{Re } \eta > -1$, thus the P_f becomes classical,

$$\mathcal{M}\left\{P_f\ 1_+(x-a)(x-a)^{\eta}\right\} = \int_a^{\infty}(x-a)^{\eta}\, x^{s-1}\, dx = a^{s+\eta}\int_0^1 (1-y)^{\eta}\, y^{-s-\eta-1} dy$$

$$= a^{s+\eta}\ B(-s-\eta,\ \eta+1) \qquad \text{Re } \eta > -1,\ \text{Re } s < -\text{Re } \eta$$

$$= \frac{a^{s+\eta}\ \Gamma(\eta+1)\Gamma(-s-\eta)}{\Gamma(1-s)} \qquad \{5\text{-}2\text{-}1\}$$

Which follows from the change of variable, $y = \frac{a}{x}$, the definition of the Beta Function, and a standard result found for example in [3], page 9, eqn (5):

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

Now to extend $\{5\text{-}2\text{-}1\}$ to include all values of η except $\eta = -1, -2, \ldots$

<u>Prove:</u> $$\mathcal{M}\left\{ P_f\, 1_+(x-a)(x-a)^{\alpha-k} \right\} = \frac{a^{s+\alpha-k}\,\Gamma(\alpha-k+1)\,\Gamma(-s-\alpha+k)}{\Gamma(1-s)}$$

for $\quad \mathrm{Re}\, s < -\mathrm{Re}(\alpha-k)$

where $\quad -1 < \mathrm{Re}\,\alpha < 0 \qquad k = 0, 1, 2, \ldots$

<u>Proof:</u> From [1], section 2-5, eqn.(9), $\quad -1 < \mathrm{Re}\,\alpha < 0,$

$$\frac{d}{dx} P_f (x-a)^{\alpha-k} 1_+(x-a) = (\alpha-k)\; P_f\; (x-a)^{\alpha-k-1}\, 1_+(x-a) \qquad \{5\text{-}2\text{-}2\}$$

Thus: $$\frac{d^k}{dx^k} P_f\, (x-a)^{\alpha}\, 1_+(x-a) = \alpha(\alpha-1)\ldots(\alpha-k+1)\; P_f (x-a)^{\alpha-k} 1_+(x-a) \qquad \{5\text{-}2\text{-}3\}$$

Using $\{5\text{-}2\text{-}1\}$

$$\left\langle P_f\, 1_+(x-a)(x-a)^{\alpha}, x^{s-1} \right\rangle = \frac{a^{s+\alpha}\,\Gamma(\alpha+1)\,\Gamma(-s-\alpha)}{\Gamma(1-s)} = F_1(s) \qquad \{5\text{-}2\text{-}4\}$$
$$\mathrm{Re}\, s < -\mathrm{Re}\,\alpha$$

Also from formula 1-7

$$\left\langle \frac{d^k}{dx^k} P_f\, 1_+(x-a)(x-a)^{\alpha}, x^{s-1} \right\rangle = (-1)^k (s-1)(s-2)\ldots(s-k)\; F_1(s-k) \qquad \{5\text{-}2\text{-}5\}$$
$$\mathrm{Re}\, s < -\mathrm{Re}\,\alpha + k$$

Combining and rearranging $\{5\text{-}2\text{-}3\}$, $\{5\text{-}2\text{-}4\}$, and $\{5\text{-}2\text{-}5\}$,

$$\left\langle P_f\, (x-a)^{\alpha-k} 1_+(x-a), x^{s-1} \right\rangle = \frac{(1-s)(2-s)\ldots(k-s)\; a^{s-k+\alpha}\,\Gamma(\alpha+1)\,\Gamma(k-s-\alpha)}{(\alpha)\;(\alpha-1)\ldots(\alpha-k+1)\;\;\Gamma\,(1-s+k)}$$
$$\mathrm{Re}\, s < -\mathrm{Re}\,(\alpha-k)$$

or $\mathcal{M}\left\{P_f\ (x-a)^{\alpha-k}1_+(x-a)\right\} = \dfrac{a^{s+\alpha-k}\,\Gamma(\alpha-k+1)\,\Gamma(-s-\alpha-k)}{\Gamma(1-s)}$

$$\mathrm{Re}\ s < -\mathrm{Re}(\alpha-k) \qquad \{5\text{-}2\text{-}6\}$$

$$-1 < \mathrm{Re}\,\alpha < 0 \qquad k = 0,1,2,\ldots$$

Q.E.D.

By letting $\eta = \alpha - k$, formula 2-2 has been derived by comparing $\{5\text{-}2\text{-}1\}$ and $\{5\text{-}2\text{-}6\}$.

(5-3) Derivation of formula 2-3,

$$\mathcal{M}\left\{P_f\ 1_+(x-a)1\ (b-x)(x-a)^{\eta}\right\} = \frac{a^{s+\eta}\,\Gamma(-s-\eta)\,\Gamma(\eta+1)}{\Gamma(1-s)}$$

$$\frac{b^{s+\eta}}{s+\eta}F(-\eta,-s-\eta\ ;\ 1-s-\eta\ ;\ \frac{a}{b})$$

$$s \neq 1, \quad a < b \qquad s+\eta \neq 0,1,2,\ldots \qquad \eta \neq -1,-2,\ldots$$

First consider $\mathrm{Re}\ \eta > -1$, thus the pseudofunction reduces to classical form,

$$\mathcal{M}\left\{P_f\ 1\ (x-a)1\ (b-x)(x-a)\right\} = \lim_{\epsilon\to 0}\int_{a+\epsilon}^{b}(1-\frac{a}{x})^{\eta}x^{s+\eta-1}\,dx \qquad \mathrm{Re}\,\eta \geq 0$$

Now, $(1-\frac{a}{x})^{\eta} = \sum_{\nu=0}^{\infty}\frac{(-\eta)_\nu}{\nu!}(\frac{a}{x})^{\nu}$ is uniformly convergent over the interval $\epsilon + a \leq x \leq b$ with $\epsilon > 0$ and $\mathrm{Re}\ \eta > -1$. Using this in the the above and integrating term by term yields:

$$\mathcal{M}\left\{P_f\ 1_+(x-a)1\ (b-x)(x-a)^{\eta}\right\} = b^{s+\eta}\sum_{\nu=0}^{\infty}\frac{(-\eta)_\nu\,(\frac{a}{b})^{\nu}}{\nu!\,(s+\eta-\nu)} - \lim_{\epsilon\to 0}\ a^{s+\eta}\sum_{\nu=0}^{\infty}\frac{(-\eta)_\nu\,(\frac{a}{a+\epsilon})^{\nu}}{\nu!\,(s+\eta-\nu)}$$

$$s+\eta \neq 0,1,2,\ldots \qquad \mathrm{Re}\ \eta > -1$$

Since $\dfrac{1}{s+\eta-\nu}=\dfrac{(-s-\eta)_\nu}{(s+\eta)(1-s-\eta)_\nu}$, applying the definition $\{5\text{-}3\text{-}1\}$ of the Hypergeometric Function to the above gives:

$$\mathcal{M}\left\{P_f\ 1_+(x-a)1_+(b-x)(x-a)^\eta\right\} = \frac{b^{s+\eta}}{s+\eta}F(-\eta,-s-\eta;1-s-\eta;\frac{a}{b}) - \frac{a^{s+\eta}}{s+\eta}F(-\eta,-s-\eta;1-s-\eta;1) \qquad s+\eta \neq 0,1,2$$

$$\operatorname{Re}\eta > -1 \qquad \{5\text{-}3\text{-}2\}$$

"The interchange of the limit and summation is valid by Abel's Theorem."

Now, from [3], page 104, equation (46),

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad c \neq 0,-1,-2,\ldots \qquad \operatorname{Re} c > \operatorname{Re}(a+b) \qquad \{5\text{-}3\text{-}3\}$$

Putting $\{5\text{-}3\text{-}3\}$ in $\{5\text{-}3\text{-}2\}$ and simplifying,

$$\mathcal{M}\left\{P_f\ 1_+(x-a)1_+(b-x)(x-a)^\eta\right\} = \frac{b^{s+\eta}}{s+\eta}F(-\eta,-s-\eta;1-s-\eta;\frac{a}{b})$$

$$\frac{a^{s+\eta}\,\Gamma(-s-\eta)\Gamma(\eta+1)}{\Gamma(1-s)}$$

$$s+\eta \neq 0,1,2,\ldots \quad \operatorname{Re}\eta > -1, \quad s \neq 1 \qquad \{5\text{-}3\text{-}4\}$$

Now to extend $\{5\text{-}3\text{-}4\}$ to include all η except $\eta = -1,-2,\ldots$

<u>Prove:</u> $$\mathcal{M}\left\{P_f\ 1_+(x-a)1_+(b-x)(x-a)^{\alpha-1}\right\} = \frac{b^{s+\alpha-1}}{s+\alpha-1}F(1-\alpha,1-s-\alpha;2-s-\alpha;\frac{a}{b}) + \frac{a^{s+\alpha-1}\Gamma(1-s-\alpha)\Gamma(\alpha)}{\Gamma(1-s)}$$

$$-1 < \operatorname{Re}\alpha < 0 \qquad s+\alpha-1 \neq 0,1,2,\ldots \qquad s \neq 1$$

Proof:

From [1], section 2-5, equation (8),

$$\frac{d}{dx} P_f (x-a)^{\alpha} 1_+(x-a)1_+(b-x) = \alpha P_f (x-a)^{\alpha-1} 1_+(x-a)1_+(b-x) - (b-a)^{\alpha}\,\delta(x-b)$$

$$\operatorname{Re}\alpha < 0 \qquad \alpha \neq -1,-2,\ldots \qquad \{5\text{-}3\text{-}5\}$$

Using $\{5\text{-}3\text{-}4\}$,

$$\left\langle P_f\, 1_+(x-a)1_+(b-x)(x-a)^{\alpha} \right\rangle = \frac{b^{s+\alpha}}{s+\alpha} F(-\alpha,-s-\alpha;1-s-\alpha;\tfrac{a}{b})$$

$$\frac{a^{s+\alpha}\Gamma(-s-\alpha)\Gamma(\alpha+1)}{\Gamma(1-s)} = F_1(s)$$

$$s+\alpha \neq 0,1,2 \qquad -1<\operatorname{Re}\alpha<0 \qquad s\neq 1 \qquad \{5\text{-}3\text{-}6\}$$

From formula 1-7,

$$\left\langle \frac{d}{dx} P_f\,(x-a)^{\alpha} 1_+(x-a)1_+(b-x), x^{s-1} \right\rangle = (1-s)\,F_1(s-1) \qquad s+\alpha \neq 1,2,3,\ldots$$

$$\{5\text{-}3\text{-}7\}$$

Combining $\{5\text{-}3\text{-}5\}$, $\{5\text{-}3\text{-}6\}$, and $\{5\text{-}3\text{-}7\}$

$$\mathcal{M}\left\{P_f\, 1_+(x-a)1_+(b-x)(x-a)^{\alpha-1}\right\} = \frac{(b-a)^{\alpha} b^{s-1}}{\alpha} + \frac{(1-s)b^{s+\alpha-1}}{\alpha\,(s+\alpha-1)} F(-\alpha,1-s-\alpha;2-s-\alpha;\tfrac{a}{b})$$

$$+ \frac{(1-s)\,a^{s+\alpha-1}\,\Gamma(1-s-\alpha)\Gamma(\alpha+1)}{\alpha\;\Gamma(2-s)}$$

$$= \frac{a^{s+\alpha-1}\Gamma(1-s-\alpha)\Gamma(\alpha)}{\Gamma(1-s)} + \frac{b^{s+\alpha-1}(1-\frac{a}{b})^{\alpha}}{\alpha}$$

$$+ \frac{(1-s)b^{s+\alpha-1}}{\alpha\,(s+\alpha-1)} \sum_{\nu=0}^{\infty} \frac{(-\alpha)_\nu (1-s-\alpha)_\nu}{(2-s-\alpha)_\nu\, \nu!} \left(\frac{a}{b}\right)_\nu$$

$$s+\alpha \neq 1,2,3,\ldots \qquad s\neq 1 \qquad -1<\operatorname{Re}\alpha<0$$

$$\{5\text{-}3\text{-}8\}$$

Using $\{5\text{-}3\text{-}1\}$, with $\eta = \alpha - 1$,

$$\frac{b^{s+\alpha-1}(1-\frac{a}{b})^{\alpha}}{\alpha} + \frac{(1-s)b^{s+\alpha-1}}{\alpha(s+\alpha-1)} \sum_{\nu=0}^{\infty} \frac{(-\alpha)_\nu (1-s-\alpha)_\nu (\frac{a}{b})^\nu}{(2-s-\alpha)_\nu \, \nu!}$$

$$= \frac{b^{s+\alpha-1}}{\alpha} \left\{ \sum_{\nu=0}^{\infty} \frac{(-\alpha)_\nu (\frac{a}{b})^\nu}{\nu!} + \frac{(1-s)}{(s+\alpha-1)} \sum_{\nu=0}^{\infty} \frac{(-\alpha)_\nu (1-s-\alpha)(\frac{a}{b})^\nu}{(1-s-\alpha+\nu)\,\nu!} \right\}$$

$$= \frac{b^{s+\alpha-1}}{\alpha} \left[\sum_{\nu=0}^{\infty} \frac{(-\alpha)_\nu (\frac{a}{b})^\nu (\nu-\alpha)}{\nu!(1-s-\alpha+\nu)} \right] = \frac{b^{s+\alpha-1}}{s+\alpha-1} \sum_{\nu=0}^{\infty} \frac{(1-\alpha)_\nu (1-s-\alpha)^\nu (\frac{a}{b})^\nu}{\nu!(2-s-\alpha)_\nu}$$

Combining this with $\{5\text{-}3\text{-}8\}$,

$$\mathcal{M}\left\{ P_f\, 1_+(x-a) 1_+(b-x)(x+a)^{\alpha-1} \right\} = \frac{b^{s+\alpha-1}}{s+\alpha-1} F(1-\alpha, 1-s-\alpha; 2-s-\alpha; \frac{a}{b})$$

$$+ \frac{a^{s+\alpha-1}\Gamma(1-s-\alpha)\Gamma(\alpha)}{\Gamma(1-s)}$$

$s+\alpha-1 \neq 0,1,2,\ldots \quad s \neq 1 \qquad -1 < \alpha < 0$

Q.E.D. $\{5\text{-}3\text{-}9\}$

Now this proof can be repeated for $-2 < \mathrm{Re}\,\alpha < -1$, then $-3 < \mathrm{Re}\,\alpha < -2$, etc.

Thus by induction formula 2-3 is proven.

(5-4) Proof of formula 2-4

$$\mathcal{M}\left\{ P_f \frac{1_+(x-a)}{(x-a)} \right\} = a^{s-1}\left[\log \frac{a}{c} - \psi(1-s)\right] \qquad -\infty < \mathrm{Re}\, s < 1$$

Several preliminary results will be needed:

From [1], section 2-5, equation (20),

$$\left\langle P_f \frac{1_+(x-a)1_+(b-a)}{(x-a)^k}, x^{s-1} \right\rangle = \lim_{\epsilon \to 0^+} \left\{ \int_{a+\epsilon}^{b} \frac{x^{s-1}}{(x-a)^k} dx - \sum_{\nu=0}^{k-2} \frac{(-1)^\nu (1-s)_\nu a^{s-1-\nu}}{\nu! \, (k-1-\nu) \, \epsilon^{k-1-\nu}} \right.$$

$$\left. + (-1)^{k-1}(1-s)_{k-1} a^{s-k} \log \epsilon \right\}$$

{5-4-1}

In particular, $$\mathcal{M}\left\{ P_f \frac{1_+(x-a)}{(x-a)} \right\} = \lim_{\epsilon \to 0^+} \left\{ \int_{a+\epsilon}^{\infty} \frac{x^{s-1}}{(x-a)} dx + a^{s-1} \log \epsilon \right\}$$

{5-4-2}

From [9], page 8, eqn. (12-8A),

$$\int_{a+\epsilon}^{\infty} \frac{dx}{x(x-a)} = -\frac{1}{a} \log \epsilon + \frac{1}{a} \log(a+\epsilon) \qquad \{5\text{-}4\text{-}3\}$$

From [3], page 16, equation (13),

$$\psi(z) + \gamma = \int_0^1 (1-t^{z-1})(1-t)^{-1} dt \qquad \mathrm{Re}\, z > 0 \qquad \{5\text{-}4\text{-}4\}$$

Now, making the changes of variables $z = 1-s$ and $t = \frac{a}{x}$ {5-4-4} becomes

$$\psi(1-s) + \gamma = \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{\infty} \frac{a(1-a^{-s}x^{s})\, dx}{x(x-a)} = \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{\infty} \frac{a dx}{x(x-a)} - \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{\infty} \frac{a^{1-s} dx}{x^{1-s}(x-a)}$$

or rearranging,

$$\lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{\infty} \frac{dx}{x^{1-s}(x-a)} = \frac{1}{a^{1-s}} \lim_{\epsilon \to 0^+} \left\{ \int_{a+\epsilon}^{\infty} \frac{a\, dx}{x(x-a)} - \psi(1-s) - \gamma \right\} \qquad \mathrm{Re}\ s < 1$$

Now adding $\lim\limits_{\epsilon\to 0^+} \frac{1}{a^{1-s}} \log\epsilon$ to each side and using $\{5\text{-}4\text{-}3\}$

$$\lim_{\epsilon\to 0^+}\left\{\int_{a+\epsilon}^{\infty} \frac{x^{s-1}dx}{(x-a)} + a^{s-1}\log\epsilon\right\} = a^{s-1}\lim_{\epsilon\to 0^+}\left\{\log(a+\epsilon)-\log\epsilon+\log\epsilon - \psi(1-s)-\gamma\right\}$$

$$-\infty < \operatorname{Re} s < 1$$

A comparison with $\{5\text{-}4\text{-}2\}$ yields,

$$\mathcal{M}\left\{P_f \frac{1_+(x-a)}{(x-a)}\right\} = a^{s-1}\left[\log\frac{a}{C} - \psi(1-s)\right] \qquad -\infty < \operatorname{Re} s < 1$$

Q.E.D.

(5-5) Derivation of formula 2-5

$$\mathcal{M}\left\{P_f \frac{1_+(x-a)1_+(b-x)}{(x-a)}\right\} = a^{s-1}\left[\log\frac{a}{C} - \psi(1-s) - \left(\frac{b}{a}\right)^{s-1}\Phi\left(\frac{a}{b},1,1-s\right)\right]$$

$$s \neq 1,2,3,\ldots$$

Formula 3-12 will be used. It is proven independently in section 6-12.

Thus,

$$\mathcal{M}\left\{1_+(x-a)1_+(b-x)\log(x-a)\right\} = \frac{a^s}{s}\quad \psi(-s)+\log\frac{C}{a}+\left(\frac{a}{b}\right)^{-s}\left(\Phi\left(\frac{a}{b},1,-s\right)+\log(b-a)\right)$$

$$= F_1(s) \qquad s \neq 0,1,2,\ldots \quad \{5\text{-}5\text{-}1\}$$

Now, $\left\langle \frac{d}{dx}\; 1_+(x-a)1_+(b-x)\log(x-a)\;, x^{s-1}\right\rangle = \left\langle 1_+(x-a)1_+(b-x)\log(x-a), -(s-1)x^{s-2}\right\rangle$

$$= \lim_{\epsilon\to 0^+}\left\{ -\log(x-a)x^{s-1}\Big|_{a+\epsilon}^{b} + \int_{a+\epsilon}^{b}\frac{x^{s-1}}{(x-a)}\,dx\right\}$$

$$= -\log(b-a)b^{s-1} + \lim_{\epsilon\to 0^+}\left\{\int_{a+\epsilon}^{b}\frac{x^{s-1}}{(x-a)}dx + \log\epsilon\; a^{s-1}\right\}, \text{ since } \lim_{\epsilon\to 0^+}(\log\epsilon)\epsilon^{\eta} = 0$$

$$\eta > 0$$

Using k = 1 in {5-4-1} in the above gives:

$$\mathcal{M}\left\{\frac{d}{dx}1_+(x-a)1_+(b-x)\log(x-a)\right\} = -\log(b-a)b^{s-1} + \mathcal{M}\left\{P_f\,\frac{1_+(x-a)1_+(b-x)}{(x-a)}\right\}$$

Now using formula 1-7, {5-5-1}, and rearranging,

$$\mathcal{M}\left\{P_f\,\frac{1_+(x-a)1_+(b-x)}{(x-a)}\right\} = b^{s-1}\log(b-a)-(s-1)F_1(s-1)$$

$$= b^{s-1}\log(b-a)-a^{s-1}\left[\psi(1-s)+\log\frac{e}{a}+\left(\frac{a}{b}\right)^{1-s}\left(\Phi\left(\frac{a}{b},1,1-s\right)+\log(b-a)\right)\right]$$

$$= a^{s-1}\left[\log\frac{a}{e}-\psi(1-s)-\left(\frac{b}{a}\right)^{s-1}\Phi\left(\frac{a}{b},1,1-s,\right)\right] \qquad \text{Q.E.D.}$$

(5-6) Proof of formula 2-6,

$$\mathcal{M}\left\{P_f\,\frac{1_+(x-a)}{(x-a)^k}\right\} = \frac{(-1)^{k+1}(1-s)_{k-1}\,a^{s-k}}{(k-1)!}\left[\log a-\psi(k-s)+\psi(k)\right]$$

$$-\infty < \operatorname{Re} s < k \qquad k = 1,2,3,\dots \qquad \{5\text{-}6\text{-}1\}$$

Proof by induction:

{5-6-1} is true for k = 1 by formula 2-4.

Assume {5-6-1} is true for k = n, then

$$\mathcal{M}\left\{P_f\,\frac{1_+(x-a)}{(x-a)^n}\right\} = \frac{(-1)^{n+1}(1-s)_{n-1}\,a^{s-n}}{(n-1)!}\left[\log a-\psi(n-s)+\psi(n)\right]$$

$$-\infty < \operatorname{Re} s < n \qquad \{5\text{-}6\text{-}2\}$$

Now, from [1] , section 2-5, equation (23),

$$\frac{d}{dx} P_f \frac{1_+(x-a)}{(x-a)^n} = -N\, P_f \frac{1_+(x-a)}{(x-a)^{N+1}} + \frac{(-1)^n\, \delta^{(n)}(x-a)}{n!} \qquad \{5\text{-}6\text{-}3\}$$

Using formula 1-7, {5-6-2} , {5-6-3} , and rearranging,

$$\mathcal{M}\left\{P_f \frac{1_+(x-a)}{(x-a)^{n+1}}\right\} = \frac{(-1)^n\,(1-s)_n\, a^{s-n-1}}{n\, n!} - \frac{(-1)^{n+1}(1-s)(2-s)_{n-1}\, a^{s-n-1}}{n\,(n-1)!}$$

$$\left[\log a - \psi(n+1-s) + \psi(n)\right]$$

$$= \frac{(-1)^{n+2}(1-s)_n\, a^{s-n-1}}{n!}\left[\log a - \psi(n+1-s) + \psi(n+1)\right]$$

$$-\infty < \mathrm{Re}\ s < n+1 \qquad \text{Q.E.D.}$$

(5-7) Proof of formula 2-7,

$$\mathcal{M}\left\{P_f \frac{1_+(x-a)1_+(b-x)}{(x-a)^k}\right\} = \frac{(-1)^{k+1}(1-s)_{k-1}\, a^{s-k}}{(k-1)!}$$

$$\left[\log a + \psi(k) - \psi(k-s) - \left(\tfrac{b}{a}\right)^{s-k} \Phi\left(\tfrac{a}{b}, 1, k-s\right)\right]$$

$$- \frac{b^{s-1}}{(b-a)^{k-1}} \sum_{\nu=0}^{k-3} \frac{(1-s)_\nu\,(k-2-\nu)!\,\left(\frac{a}{b}-1\right)^\nu}{(k-1)!}$$

$$- \frac{(-1)^{k+1}(1-s)_{k-2}\, b^{s-k+1}}{(b-a)(k-1)!} \qquad s \neq k, k+1, k+2, \ldots$$

$$\{5\text{-}7\text{-}1\}$$

Induction proof: $\{5\text{-}7\text{-}1\}$ is true for $k=1$ by formula 2-5

Assume $\{5\text{-}7\text{-}1\}$ is true for $k=n$, then:

$$\left\{ P_f \frac{1_+(x-a)1_+(b-x)}{(x-a)^n} \right\} = \frac{(-1)^{n+1}(1-s)_{n-1}\, a^{s-n}}{(n-1)!} \Big[\log a + \psi(n) - \psi(n-s)$$

$$-\left(\frac{b}{a}\right)^{s-n} \Phi\left(\frac{a}{b}, 1, n-s\right)\Big] - \frac{b^{s-1}}{(b-a)^{n-1}}$$

$$\sum_{\nu=0}^{n-3} \frac{(1-s)_\nu (n-2-\nu)!\, \left(\frac{a}{b}-1\right)^\nu}{(n-1)!} - \frac{(-1)^{n+1}(1-s)_{n-2}\, b^{s-n+1}}{(b-a)(n-1)!}$$

$$s \neq n, n+1, n+2, \ldots \qquad \{5\text{-}7\text{-}2\}$$

Using formula 1-7, $\{5\text{-}7\text{-}2\}$, and the relation proven in appendix 1, $\{A1\text{-}3\}$, gives:

$$\mathcal{M}\left\{ P_f \frac{1_+(x-a)1_+(b-x)}{(x-a)^{n+1}} \right\} = \frac{(-1)^n(1-s)_n\, a^{s-n-1}}{n \cdot n!} - \frac{b^{s-1}}{n(b-a)^n} - \frac{(1-s)(-1)^{n+1}(2-s)_{n-1} a^{s-n-1}}{n \qquad (n-1)!}$$

$$\left[\log a + \psi(n) - \psi(n+1-s) - \left(\frac{b}{a}\right)^{s-n-1} \Phi\left(\frac{a}{b}, 1, n+1-s\right)\right]$$

$$+ \frac{(1-s) b^{s-2}}{n(b-a)^{n-1}} \sum_{\nu=0}^{n-3} \frac{(2-s)_\nu\, (n-2-\nu)!\left(\frac{a}{b}-1\right)^\nu}{(n-1)!} + \frac{(1-s)(-1)^{n+1}(2-s)_{n-2}\, b^{s-n}}{n\,(b-a)(n-1)!}$$

$$= \frac{(-1)^{n+2}(1-s)_n a^{s-n-1}}{n!} \left[\log a + \psi(n+1) - \psi(n+1-s) - \left(\frac{b}{a}\right)^{s-n-1} \Phi\left(\frac{a}{b}, 1, n+1-s\right)\right]$$

$$- \frac{b^{s-1}}{(b-a)^n} \sum_{\nu=0}^{n-2} \frac{(1-s)_\nu (n-1-\nu)!\left(\frac{a}{b}-1\right)^\nu}{n!} - \frac{(-1)^{n+2}(1-s)_{n-1} b^{s-n}}{(b-a)\ n!}$$

$$s \neq n+1, n+2, n+3, \ldots$$

Q.E.D.

(5-8) Derivation of formula 2-8,

$$\mathcal{M}\left\{ P_f \frac{1_+(x-a)(a+b)}{(x-a)(x+b)} \right\} = a^{s-1}\left[\log \frac{a}{C} - \psi(1-s) - \Phi(-\frac{b}{a},1,1-s) \right]$$

$s \neq 1 \quad -\infty < \text{Re } s < 2 \quad b < a$

By a partial fraction expansion,

$$\mathcal{M}\left\{ P_f \frac{1_+(x-a)(a+b)}{(x-a)(x+b)} \right\} = \mathcal{M}\left\{ P_f \frac{1_+(x-a)}{(x-a)} \right\} - \mathcal{M}\left\{ \frac{1_+(x-a)}{(x+b)} \right\} \qquad \{5\text{-}8\text{-}1\}$$

Now,

$$\mathcal{M}\left\{ \frac{1_+(x-a)}{(x+b)} \right\} = \int_a^\infty x^{s-2}(1-(\frac{b}{x}))^{-1}dx = \int_a^\infty \sum_{\nu=0}^{\infty}(-1)^\nu b^\nu x^{s-2-\nu}dx \quad \text{since } \left|\frac{b}{x}\right| \leq \left|\frac{b}{a}\right| < 1$$

$$= \sum_{\nu=0}^{\infty} \frac{a^{s-1}(-\frac{b}{a})^\nu}{1-s+\nu} = a^{s-1}\Phi(-\frac{b}{a},1,1-s) \quad b < a \qquad \text{Re } s < 1$$

$\{5\text{-}8\text{-}2\}$

Combining formula 2-4, $\{5\text{-}8\text{-}1\}$, and $\{5\text{-}8\text{-}2\}$ gives the desired result.

(5-9) Derivation of formula 2-9,

$$\mathcal{M}\left\{ P_f \frac{1_+(x-a)2a}{(x-a)(x+a)} \right\} = a^{s-1}\left[\log \frac{a}{C} - \psi(1-s) - \frac{1}{2}G(1-s) \right] \quad -\infty < \text{Re } s < 2 \quad s \neq 1$$

From [3], page 114, equation(1),

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx \qquad \{5\text{-}9\text{-}1\}$$

$\text{Re } c > 0 \quad \text{Re } b > 0$

$|\arg(1-z)| < \pi$

Using this,

$$\mathcal{M}\left\{ \frac{1_+(x-a)}{(x+a)} \right\} = a^{s-1}\int_0^1 \frac{x^{-s}}{1+x}dx = \frac{a^{s-1}}{1-s}F(1,1-s;2-s;-1) \qquad \text{Re } s < 1$$

Now using a result proven in appendix 2,

$$\mathcal{M}\left\{\frac{1_+(x-a)}{(x+a)}\right\} = \frac{a^{s-1}}{2}\left[\psi\left(\frac{2-s}{2}\right) - \psi\left(\frac{1-s}{2}\right)\right] = \frac{a^{s-1}}{2} G(1-s) \qquad \text{Re } s < 1 \qquad \{5\text{-}9\text{-}2\}$$

Using formula 2-4 and $\{5\text{-}9\text{-}2\}$ in $\{5\text{-}8\text{-}1\}$ with a=b gives the desired result. Since the Mellin transform integral converges for all Re s < 2, the region of convergence for the expression can be extended by analytic continuation to include the interval $-\infty <$ Re s < 2 but $s \neq 1$.

(5-10) Derivation of formula 2-10,

$$\mathcal{M}\left\{P_f \frac{1_+(x-a)(a+b)}{(x-a)(x+b)}\right\} = a^{s-1}\left[\log\frac{a}{C} - \psi(1-s) - \left(\frac{a}{b}\right)^{1-s}\pi \csc \pi s + \frac{a}{b}\Phi\left(-\frac{a}{b}, 1, s\right)\right]$$

$$s \neq 1, 0, -1, -2, \ldots \qquad -\infty < \text{Re } s < 2$$

$$a < b$$

Now, using $\{5\text{-}9\text{-}2\}$ with a replaced by b, $\{5\text{-}9\text{-}1\}$, and the result proven in appendix 2,

$$\mathcal{M}\left\{\frac{1_+(x)}{x+b}\right\} = \int_0^b \frac{x^{s-1}dx}{x+b} + \int_b^\infty \frac{x^{s-1}dx}{x+b} = b^{s-1}\int_0^1 \frac{x^{s-1}dx}{1+x} + \mathcal{M}\left\{\frac{1_+(x-b)}{(x+b)}\right\}$$

$$= \frac{b^{s-1}}{s} F(1,s;s+1;-1) + \frac{b^{s-1}}{2}\left[\psi\left(\frac{2-s}{2}\right) - \psi\left(\frac{1-s}{2}\right)\right]$$

$$= \frac{b^{s-1}}{2}\left[\psi\left(1-\frac{s}{2}\right) - \psi\left(\frac{1}{2}-\frac{s}{2}\right) + \psi\left(\frac{1}{2}+\frac{s}{2}\right) - \psi\left(\frac{s}{2}\right)\right]$$

$$0 < \text{Re } s < 1$$

But from [3], page 16, equations (11),

$$\psi(z) - \psi(1-z) = -\pi \operatorname{ctn}(\pi z) \qquad \{5\text{-}10\text{-}1\}$$

$$\psi\left(\frac{1}{2}+z\right) - \psi\left(\frac{1}{2}-z\right) = \pi \tan(\pi z) \qquad \{5\text{-}10\text{-}2\}$$

Thus,

$$\mathcal{M}\left\{\frac{1}{x+b}\right\} = \frac{b^{s-1}}{2}\left[\pi\,\mathrm{ctn}\left(\frac{\pi s}{2}\right) + \pi\tan\left(\frac{\pi s}{2}\right)\right] = b^{s-1}\,\pi\csc\pi s \qquad \{5\text{-}10\text{-}3\}$$

$$0 < \mathrm{Re}\, s < 1$$

Also,

$$\int_0^a \frac{x^{s-1}}{x+b}\,dx = \frac{1}{b}\int_0^a \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}\, x^{s+\nu-1}}{b^{\nu}}\,dx = \frac{a^s}{b}\,\Phi\left(-\frac{a}{b},1,s\right) \quad \text{for } a < b$$

$$0 < \mathrm{Re}\, s < \infty \qquad \{5\text{-}10\text{-}4\}$$

Combining $\{5\text{-}10\text{-}3\}$ and $\{5\text{-}10\text{-}4\}$ gives:

$$\mathcal{M}\left\{\frac{1_+(x-a)}{x+b}\right\} = \mathcal{M}\left\{\frac{1}{x+b}\right\} - \int_0^a \frac{1}{x+b}\,dx = b^{s-1}\,\pi\csc\pi s - \frac{a^s}{b}\,\Phi\left(-\frac{a}{b},1,s\right)$$

$$a < b$$

$0 < \mathrm{Re}\, s < 1$ but with analytic continuation $-\infty < \mathrm{Re}\, s < 1$

$$s \neq 0,-1,-2,\ldots \qquad \{5\text{-}10\text{-}5\}$$

Finally putting formula 2-4 and $\{5\text{-}10\text{-}5\}$ in $\{5\text{-}8\text{-}1\}$ and extending the region of convergence by analytic continuation gives the desired result.

(5-11) Derivation of formula 2-11,

$$\mathcal{M}\left\{P_f\,\frac{1_+(x-a)(a+b)^2}{(x-a)^2(x+b)}\right\} = a^{s-1}\left\{\left[(s-1)\left(1+\frac{b}{a}\right)-1\right]\left[\log\frac{a}{C} - \psi(1-s)\right]\right.$$

$$\left. + s\left(1+\frac{b}{a}\right) + \Phi\left(-\frac{b}{a},1,1-s\right)\right\} \qquad -\infty < \mathrm{Re}\, s < 3$$

$$s \neq 1,2 \quad b < a$$

Expanding by partial fractions gives:

$$\mathcal{M}\left\{P_f\,\frac{1_+(x-a)(a+b)^2}{(x-a)^2(x+b)}\right\} = (a+b)\,\mathcal{M}\left\{P_f\,\frac{1_+(x-a)}{(x-a)^2}\right\} - \mathcal{M}\left\{P_f\,\frac{1_+(x-a)}{(x-a)}\right\}$$

$$+\,\mathcal{M}\left\{\frac{1_+(x-a)}{(x+b)}\right\} \qquad \{5\text{-}11\text{-}1\}$$

Using formula 2-6 with k = 2 and the difference formula for $\psi(z)$,

$$\mathfrak{M}\left\{ P_f \frac{1_+(x-a)}{(x-a)^2} \right\} = (s-1)a^{s-2} \left[\log \frac{a}{C} - \psi(1-s) + \frac{s}{s-1} \right] \qquad -\infty < \operatorname{Re} s < 2$$

$$s \neq 1 \qquad \{5\text{-}11\text{-}2\}$$

Putting formula 2-4, $\{5\text{-}8\text{-}2\}$, and $\{5\text{-}11\text{-}2\}$ in $\{5\text{-}11\text{-}1\}$ gives,

$$\mathfrak{M}\left\{ P_f \frac{1_+(x-a)(a+b)^2}{(x-a)^2(x+b)} \right\} = (s-1)(a+b)a^{s-2} \left[\log \frac{a}{C} - \psi(1-s) + \frac{s}{s-1} \right]$$

$$- a^{s-1} \left[\log \frac{a}{C} - \psi(1-s) \right] + a^{s-1}\, \Phi(-\tfrac{b}{a},1,1-s)$$

$$b < a \qquad \operatorname{Re} s < 1$$

$$= a^{s-1} \left\{ \left[(s-1)(1+\tfrac{b}{a}) - 1 \right] \cdot \left[\log \frac{a}{C} - \psi(1-s) \right] \right.$$

$$\left. + s(1+\tfrac{b}{a}) + \Phi(-\tfrac{b}{a},1,1-s) \right\}$$

and by analytic continuation $-\infty < \operatorname{Re} s < 3$

$$b < a \qquad s \neq 1,2$$

Q.E.D.

(5-12) Derivation of formula 2-12,

$$\mathfrak{M}\left\{ P_f \frac{1_+(x-a)4a^2}{(x-a)^2(x+a)} \right\} = a^{s-1} \left\{ \left[(s-1)(1+\tfrac{b}{a}) - 1 \right] \left[\log \frac{a}{C} - \psi(1-s) \right] \right.$$

$$\left. + s(1+\tfrac{b}{a}) + \tfrac{1}{2}G(1-s) \right\} \qquad -\infty < \operatorname{Re} s < 3 \qquad s \neq 1,2$$

This follows from inserting formula 2-4, $\{5\text{-}11\text{-}2\}$, and $\{5\text{-}9\text{-}2\}$ in $\{5\text{-}11\text{-}1\}$ with a = b. The region of convergence in s has been extended by analytic continuation.

(5-13) Derivation of formula 2-13,

$$\mathcal{M}\left\{P_f \frac{1_+(x-a)(a+b)^2}{(x-a)^2(x+b)}\right\} = a^{s-1} \left\{ \left[(s+1)(1+\frac{b}{a})-1\right]\left[\log \frac{a}{c} - \psi(1-s)\right] \right.$$

$$\left. + S(1+\frac{b}{a}) + (\frac{a}{b})^{1-s}\pi \csc \pi s - \frac{a}{b}\ \Phi(-\frac{a}{b},1,s)\right\}$$

$$a < b \quad s \neq 1,2 \quad -\infty < \text{Re } s < 3 \quad s \neq 0,-1,-2,\dots$$

This formula is obtained by combining formula 2-4, {5-11-2}, {5-10-5}, {5-11-1}, and using analytic continuation.

(5-14) Derivation of formula 2-14,

$$\mathcal{M}\left\{P_f \frac{1}{x-a}\right\} = -a^{s-1}\pi \operatorname{ctn} \pi s \qquad 0 < \text{Re } s < 1$$

From [ ], section 2-5, equation (21),

$$\left\langle P_f \frac{1_+(x-a)1_+(b-x)}{|x-b|^k}, \phi(x)\right\rangle = \lim_{\epsilon\to 0^+}\left\{\int_a^{b-\epsilon}\frac{\phi(x)}{|x-b|^k}\,dx - \sum_{\nu=0}^{k-2}\frac{(-1)^\nu\ \phi^{(\nu)}(b)}{(k-1-\nu)\,\epsilon^{k-1-\nu}}\right.$$

$$\left. \frac{(-1)^{k-1}\phi^{(k-1)}(b)\log\epsilon}{(k-1)!}\right\} \qquad \{5\text{-}14\text{-}1\}$$

With $k = 1$, $a = 0$, and $\phi(x) = x^{s-1}$ in {5-14-1},

$$\mathcal{M}\left\{P_f \frac{1_+(b-x)}{(b-x)}\right\} = \lim_{\epsilon\to 0^+}\left\{\int_0^{b-\epsilon}\frac{x^{s-1}dx}{(b-x)} + b^{s-1}\log\epsilon\right\}$$

$$= \lim_{\epsilon\to 0^+}\left\{\int_0^{b-\epsilon}\frac{x^{s-1}}{(b-x)}dx - b^{s-1}\int_0^{b-\epsilon}\frac{dx}{(b-x)} + b^{s-1}\log b\right\}$$

$$= \int_0^b \frac{x^{s-1}-b^{s-1}}{b-x}\,dx + b^{s-1}\log b$$

Making the change of variable, $x = bt$, using {5-4-4} with $z = s$ gives:

$$\mathcal{M}\left\{P_f \frac{1_+(b-x)}{(x-b)}\right\} = b^{s-1}\left[\psi(s)+\gamma\right] - b^{s-1}\log b = b^{s-1}\left[\psi(s)-\log\frac{b}{c}\right]$$

$\mathrm{Re}\, s > 0$ {5-14-2}

Now, using formula 2-4 and {5-14-2} with a in place of b,

$$\mathcal{M}\left\{P_f \frac{1}{x-a}\right\} = \mathcal{M}\left\{P_f \frac{1_+(a-x)}{(x-a)}\right\} + \mathcal{M}\left\{P_f \frac{1_+(x-a)}{x-a}\right\}$$

$$= a^{s-1}\left[\psi(s) - \psi(1-s)\right] = -a^{s-1}\pi \operatorname{ctn}\pi s \quad \text{using } \{5\text{-}10\text{-}1\}$$

$0 < \mathrm{Re}\, s < 1$

Q.E.D.

(5-15) Derivation of formula 2-15,

$$\mathcal{M}\left\{P_f \frac{1_+(x-a)(a-b)}{(x-a)(x-b)}\right\} = a^{s-1}\left[\log\frac{a}{c} - \psi(1-s) - \Phi\left(\frac{b}{a},1,1-s\right)\right]$$

$-\infty < \mathrm{Re}\, s < 2 \quad s \neq 1$

$a > b$

Now,

$$\mathcal{M}\left\{P_f \frac{1_+(x-a)(a-b)}{(x-a)(x-b)}\right\} = \mathcal{M}\left\{P_f \frac{1_+(x-a)}{(x-a)}\right\} - \mathcal{M}\left\{P_f \frac{1_+(x-a)}{(x-b)}\right\}$$

{5-15-1}

For a > b,

$$\mathcal{M}\left\{ P_f \frac{1_+(x-a)}{(x-b)} \right\} = \mathcal{M}\left\{ \frac{1_+(x-a)}{(x-b)} \right\} = \int_a^\infty \sum_{\nu=0}^\infty b^\nu x^{s-\nu-2}\, dx$$

$$= a^{s-1}\ \Phi\left(\frac{b}{a}, 1, 1-s\right) \qquad \text{Re } s < 1 \qquad \{5\text{-}15\text{-}2\}$$

Using analytic continuation, {5-15-1}, {5-15-2} and formula 2-4 gives the desired result.

(5-16) Derivation of formula 2-6,

$$\mathcal{M}\left\{ P_f \frac{1_+(x-a)(a-b)}{(x-a)(x-b)} \right\} = a^{s-1}\left[\log\frac{a}{C} - \psi(1-s) + \left(\frac{a}{b}\right)^{1-s} \pi \operatorname{ctn} \pi s - \frac{a}{b}\Phi\left(\frac{a}{b}, 1, s\right) \right] \qquad -\infty < \text{Re } s < 2$$

$$a < b \qquad s \neq 1, 0, -1, -2, \ldots$$

For $a < b$, $$\int_0^a \frac{x^{s-1}}{x-b}\, dx = -\frac{1}{b}\int_0^a \sum_{\nu=0}^\infty \frac{x^{\nu+s-1}}{b^\nu}\, dx = -\frac{a^s}{b}\ \Phi\left(\frac{a}{b}, 1, s\right) \qquad \{5\text{-}16\text{-}1\}$$

$$\text{Re } s > 0$$

Now, using formula 2-14, and the above,

$$\mathcal{M}\left\{ P_f \frac{1_+(x-a)}{(x-b)} \right\} = \mathcal{M}\left\{ \frac{1}{x-b} \right\} - \int_0^a \frac{x^{s-1}}{x-b}\, dx$$

$$= b^{s-1}\left[\left(\frac{a}{b}\right)^s \Phi\left(\frac{a}{b}, 1, s\right) - \pi \operatorname{ctn} \pi s \right]$$

$$-\infty < \text{Re } s < 1 \qquad s \neq 0, -1, -2, \ldots \qquad \{5\text{-}16\text{-}2\}$$

$$a < b$$

Using analytic continuation, {5-15-1}, {5-16-2}, and formula 2-4 gives the desired result.

(5-17) Derivation of formula 2-17,

$$\mathcal{M}\left\{P_f \frac{1_+(x-a)(a-b)^2}{(x-a)^2(x-b)}\right\} = a^{s-1}\left\{\left[(s-1)(1-\frac{b}{a})-1\right]\left[\log\frac{a}{C}-\psi(1-s)\right] + s(1-\frac{b}{a}) + \Phi(\frac{b}{a},1,1-s)\right\} \quad a > b$$

$$s \neq 1,2 \qquad -\infty < \mathrm{Re}\ s < 3$$

Now,

$$\mathcal{M}\left\{P_f \frac{1_+(x-a)(a-b)^2}{(x-a)^2(x-b)}\right\} = (a-b)\mathcal{M}\left\{P_f \frac{1_+(x-a)}{(x-a)^2}\right\} - \mathcal{M}\left\{P_f \frac{1_+(x-a)}{(x-a)}\right\} + \mathcal{M}\left\{P_f \frac{1_+(x-a)}{(x-b)}\right\} \qquad \{5\text{-}17\text{-}1\}$$

Using analytic continuation and {5-11-2} , {5-15-2} , and formula 2-4 gives the result.

(5-18) Derivation of formula 2-18,

$$\mathcal{M}\left\{P_f \frac{1_+(x-a)(a-b)^2}{(x-a)^2(x-b)}\right\} = a^{s-1}\left\{\left[(s-1)(1-\frac{b}{a})-1\right]\left[\log\frac{a}{C}-\psi(1-s)\right] + s(1-\frac{b}{a}) - (\frac{a}{b})^{1-s}\pi\,\mathrm{ctn}\,\pi s + \frac{a}{b}\Phi(\frac{a}{b},1,s)\right\}$$

$$a < b \qquad -\infty < \mathrm{Re}\ s < 3 \qquad s \neq 2,1,0,-1,-2,\ldots$$

Using analytic continuation and {5-11-2} , {5-16-2} , {5-17-1} ,and formula 2-4 gives the result.

(5-19) Derivation of formula 2-19,

$$\mathcal{M}\left\{P_f \frac{1_+(x-a)}{(x-b)}\right\} = b^{s-1}\left[(\frac{a}{b})^s \Phi(\frac{a}{b},1,s) - \pi\,\mathrm{ctn}\,\pi s\right]$$

$$-\infty < \mathrm{Re}\ s < 1 \qquad s \neq 0,-1,-2,\ldots \qquad a < b$$

This result is given as {5-16-2} .

(5-20) Derivation of formula 2-20,

$$\mathcal{M}\left\{\frac{1_+(x-a)}{x-b}\right\} = a^{s-1}\ \Phi\left(\frac{b}{a},1,1-s\right) \qquad -\infty < \operatorname{Re} s < 1 \qquad a > b$$

This result is from $\{5\text{-}15\text{-}2\}$.

(5-21) Derivation of formula 2-21,

$$\mathcal{M}\left\{\frac{1}{x+a}\right\} = a^{s-1}\ \pi \csc \pi s \qquad 0 < \operatorname{Re} s < 1$$

This result has been derived as $\{5\text{-}10\text{-}3\}$.

(5-22) Derivation of formula 2-22,

$$\mathcal{M}\left\{\frac{1_+(x-a)}{x+b}\right\} = a^{s-1}\ \Phi\left(-\frac{b}{a},1,1-s\right) \qquad a > b \qquad -\infty < \operatorname{Re} s < 1$$

This result is given as $\{5\text{-}8\text{-}2\}$.

(5-23) Derivation of formula 2-23,

$$\mathcal{M}\left\{\frac{1_+(x-a)}{x+a}\right\} = \frac{a^{s-1}}{2}\ G(1-s) \qquad -\infty < \operatorname{Re} s < 1$$

This Result has been derived as $\{5\text{-}9\text{-}2\}$.

(5-24) Derivation of formula 2-24,

$$\mathcal{M}\left\{\frac{1_+(x-a)}{x+b}\right\} = b^{s-1}\left[\pi \csc \pi s - \left(\frac{a}{b}\right)^s \Phi\left(-\frac{a}{b},1,s\right)\right] \qquad a < b$$

$$-\infty < \operatorname{Re} s < 1 \qquad s \neq 0,-1,-2,\ldots$$

This result has been derived as $\{5\text{-}10\text{-}5\}$.

(5-25) Derivation of formula 2-25,

$$\mathfrak{M}\left\{ P_f \frac{1_+(a-x)}{a-x} \right\} = a^{s-1}\left[\log \frac{a}{C} - \psi(s) \right] \qquad \infty > \operatorname{Re} s > 0$$

This result has been derived as $\{5\text{-}14\text{-}2\}$.

(5-26) Derivation of formula 2-26

$$\mathfrak{M}\left\{ P_f \frac{1_+(x-a)1_+(b-x)}{b-x} \right\} = b^{s-1}\left[\log \frac{b}{C} - \psi(s) + \left(\frac{a}{b}\right)^s \Phi\left(\frac{a}{b}, 1, s\right) \right]$$

$$a < b \qquad s \neq 0, -1, -2, \ldots$$

$$\mathfrak{M}\left\{ P_f \frac{1_+(x-a)1_+(b-x)}{b-x} \right\} = \mathfrak{M}\left\{ P_f \frac{1_+(b-x)}{b-x} \right\} + \mathfrak{M}\left\{ \frac{1_+(a-x)}{x-b} \right\} \qquad a < b$$

Using $\{5\text{-}16\text{-}1\}$ and formula 2-25, with analytic continuation gives the result.

(5-27) Derivation of formula 2-27,

$$\mathfrak{M}\left\{ P_f \frac{1_+(x-a)1_+(b-x)}{(b-x)^2} \right\} = (s-1)b^{s-2}\left[\psi(s-1) - \log \frac{b}{C} - 1 + \left(\frac{a}{b}\right)^{s-1} \left\{ \Phi\left(\frac{a}{b}, 1, s-1\right) - \frac{1}{(s-1)(1-\frac{a}{b})} \right\} \right]$$

$$a < b \qquad s \neq 1, 0, -1, \ldots$$

Now,

$$\mathfrak{M}\left\{ P_f \frac{1_+(x-a)1_+(b-x)}{(b-x)^2} \right\} = \mathfrak{M}\left\{ P_f \frac{1_+(b-x)}{(x-b)^2} \right\} - \mathfrak{M}\left\{ \frac{1_+(a-x)}{(x-b)^2} \right\}$$

$$a < b \qquad \{5\text{-}27\text{-}1\}$$

From [1], section 2-5, equation (24),

$$\frac{d}{dx} P_f \frac{1_+(b-x)}{(x-b)^k} = -k\, P_f \frac{1_+(b-x)}{(x-b)^{k+1}} - \frac{(-1)^k \delta^{(k)}(x-b)}{k!} \qquad \{5\text{-}27\text{-}2\}$$

Thus,

$$\mathfrak{M}\left\{P_f \frac{1_+(b-x)}{(x-b)^2}\right\} = \mathfrak{M}\left\{\frac{d\,\delta(x-b)}{dx}\right\} - \mathfrak{M}\left\{\frac{d}{dx} P_f \frac{1_+(b-x)}{(x-b)}\right\}$$

And using formulas 1-7 and 2-25 gives with analytic continuation:

$$\mathfrak{M}\left\{P_f \frac{1_+(b-x)}{(x-b)^2}\right\} = (s-1)b^{s-2}\left[-\log\frac{b}{C} + \psi(s-1) - 1\right]$$

$$s \neq 1 \qquad 0 < \mathrm{Re}\, s < \infty \qquad \{5\text{-}27\text{-}3\}$$

Now using formula 1-7, $\{5\text{-}16\text{-}1\}$, and analytic continuation,

$$\mathfrak{M}\left\{\frac{1_+(a-x)}{(x-b)^2}\right\} = \mathfrak{M}\left\{\frac{\delta(x-a)}{(b-x)}\right\} - \mathfrak{M}\left\{\frac{d}{dx}\frac{1_+(a-x)}{(x-b)}\right\} \qquad a < b$$

$$= a^{s-1}\left[\frac{1}{b-a} - \frac{(s-1)}{b}\,\Phi\left(\frac{a}{b},1,s-1\right)\right]$$

$$s \neq 1 \qquad 0 < \mathrm{Re}\, s < \infty \qquad \{5\text{-}27\text{-}4\}$$

Combining $\{5\text{-}27\text{-}1\}$, $\{5\text{-}27\text{-}3\}$, $\{5\text{-}27\text{-}4\}$ and applying analytic continuation,

$$\mathfrak{M}\left\{P_f \frac{1_+(x-a)1_+(b-x)}{(b-x)^2}\right\} = (s-1)b^{s-2}\left[\psi(s-1) - \log\frac{b}{C} - 1 + \left(\frac{a}{b}\right)^{s-1}\Phi\left(\frac{a}{b},1,s-1\right) - \frac{\left(\frac{a}{b}\right)^{s-1}}{(s-1)\left(1-\frac{a}{b}\right)}\right]$$

$$a < b \qquad s \neq 1,0,-1,-2,\ldots$$

Q.E.D.

(5-28) Derivation of formula 2-28,

$$\mathfrak{M}\left\{\frac{1_+(b-x)}{x-a}\right\} = \frac{-b^s}{a}\,\Phi\left(\frac{b}{a},1,s\right) \qquad b < a \qquad 0 < \operatorname{Re} s < \infty$$

This result is given as $\{5\text{-}16\text{-}1\}$.

(5-29) Derivation of formula 2-29,

$$\mathfrak{M}\left\{P_f\,\frac{1_+(b-x)}{(x-b)^2}\right\} = (s-1)b^{s-2}\left[\Psi(s-1) - \log\frac{b}{C} - 1\right] \quad s \neq 1 \quad 0 < \operatorname{Re} s < \infty$$

$\{5\text{-}27\text{-}3\}$ supplies this result.

(5-30) Derivation of formula 2-30,

$$\mathfrak{M}\left\{\frac{1_+(b-x)}{(x-a)^2}\right\} = b^{s-1}\left[\frac{1}{a-b} - \frac{(s-1)}{a}\,\Phi\left(\frac{b}{a},1,s-1\right)\right] \qquad b < a$$

$$s \neq 1 \qquad 0 < \operatorname{Re} s < \infty$$

This formula is expressed as $\{5\text{-}27\text{-}4\}$

6. Elementary Transcendental Functions

(6-1) Proof of formula 3-1,

$$\mathcal{M}\left\{1_+(x-a)\log(x-a)\right\} = \frac{a^s}{s}\left[\psi(-s) - \log\frac{a}{C}\right] \qquad -\infty < \operatorname{Re} s < 0$$

Let $F(s) = \mathcal{M}\left\{1_+(x-a)\log(x-a)\right\}$, and using the result of appendix 3, formula 1-7, and formula 2-4,

$$\mathcal{M}\left\{P_f\,\frac{1_+(x-a)}{x-a}\right\} = \mathcal{M}\left\{\frac{d}{dx}\,1_+(x-a)\log(x-a)\right\} = (1-s)F(s-1)$$

$$= a^{s-1}\left[\log\frac{a}{C} - \psi(1-s)\right] \qquad -\infty < \operatorname{Re} s < 1$$

thus $F(s) = \dfrac{a^s}{s}\left[\psi(-s) - \log\dfrac{a}{C}\right] \qquad -\infty < \operatorname{Re} s < 0$

Q.E.D.

(6-2) Derivation of formula 3-2,

$$\mathcal{M}\left\{1_+(x-a)\log(x+b)\right\} = \frac{-a^s}{s}\left[\Phi\left(\frac{-b}{a},1,-s\right) + \log(a+b)\right] \qquad -\infty < \operatorname{Re} s < 0, \quad a > b$$

Integrating by parts,

$$\mathcal{M}\left\{1_+(x-a)\log(x+b)\right\} = \frac{-1}{s}\left[a^s\log(a+b) + \left\langle \frac{1_+(x-a)}{x+b}, x^s \right\rangle\right]$$

$$\operatorname{Re} s < 0 \qquad \{6\text{-}2\text{-}1\}$$

With aid of formula 2-22, $\left\langle \frac{1_+(x-a)}{x+b}, x^{s-1} \right\rangle = a^{s-1} \Phi(-\frac{b}{a},1,1-s)$ Re $s < 1$,

the desired result is obtained.

(6-3) Derivation of formula 3-3,

$$\mathcal{M}\left\{1_+(x-a)\log(x+a)\right\} = \frac{-a^s}{s}\left[\frac{1}{2}G(-s)+\log(2a)\right] \qquad -\infty < \text{Re } s < 0$$

Letting a = b in $\{6\text{-}2\text{-}1\}$ and using formula 2-23 gives this result.

(6-4) Derivation of formula 3-4,

$$\mathcal{M}\left\{1_+(x-a)\log(x+b)\right\} = \frac{a^s}{s}\left[\frac{a}{b}\Phi(-\frac{a}{b},1,s+1)+(\frac{b}{a})^s \pi \csc \pi s - \log(a+b)\right] \qquad -\infty < \text{Re } s < 0$$

$$a < b \qquad s \neq -1,-2,\ldots$$

Combining $\{6\text{-}2\text{-}1\}$ and formula 2-24 gives this result since $\csc \pi(s+1) = -\csc \pi s$.

(6-5) Derivation of formula 3-5,

$$\mathcal{M}\left\{1_+(x-a)\log|x-b|\right\} = \frac{-a^s}{s}\left[\Phi(\frac{b}{a},1,-s)+\log(a-b)\right] \qquad -\infty < \text{Re } s < 0$$

$$a > b$$

Integrating by parts,

$$\mathcal{M}\left\{1_+(x-a)\log(x-b)\right\} = \frac{-1}{s}\left[a^s\log(a-b)+\left\langle \frac{1_+(x-a)}{x-b}, x^s \right\rangle\right]$$

$$a > b \qquad -\infty < \text{Re } s < 0 \qquad \{6\text{-}5\text{-}1\}$$

Combining $\{6\text{-}5\text{-}1\}$ and formula 2-20 gives the result.

(6-6) Derivation of formula 3-6,

$$\mathcal{M}\left\{1_+(x-a)\log|x-b|\right\} = \frac{-a^s}{s}\left[\left(\tfrac{b}{a}\right)^s\{\pi\,\mathrm{ctn}(-\pi s)\} + \log(b-a) + \frac{a}{b}\,\Phi\!\left(\tfrac{a}{b},1,s+1\right)\right] \qquad a < b$$

$$-\infty < \mathrm{Re}\, s < 0 \qquad s \neq -1,-2,\ldots$$

$$\mathcal{M}\left\{1_+(x-a)\log|x-b|\right\} = \mathcal{M}\left\{1_+(x-a)1_+(b-x)\log(b-x)\right\}$$
$$\mathcal{M}\left\{1_+(x-b)\log(x-b)\right\} \qquad a < b \qquad \{6\text{-}6\text{-}1\}$$

Now, with k=1, $\phi(x) = x^s$ in $\{5\text{-}14\text{-}1\}$,

$$\left\langle P_f\,\frac{1_+(x-a)1_+(b-x)}{b-x}, x^s\right\rangle = \lim_{\epsilon\to 0^+}\left\{\int_a^{b-\epsilon}\frac{x^s}{b-x}\,dx + b^s\log\epsilon\right\}$$
$$a < b \qquad \{6\text{-}6\text{-}2\}$$

From formula 2-26,

$$\left\langle P_f\,\frac{1_+(x-a)1_+(b-x)}{b-x}, x^s\right\rangle = b^s\left[\log\frac{b}{e} - \psi(s+1) - \left(\tfrac{a}{b}\right)^{s+1}\Phi\!\left(\tfrac{a}{b},1,s+1\right)\right] \quad s \neq -1,-2,-3,\ldots$$
$$\{6\text{-}6\text{-}3\}$$

Also, $$\lim_{\epsilon\to 0^+}\int_a^{b-\epsilon}\log(b-x)x^{s-1}\,dx = \lim_{\epsilon\to 0^+}\left\{\left.\frac{x^s\log(b-x)}{s}\right|_a^{b-\epsilon} + \frac{1}{s}\int_a^{b-\epsilon}\frac{x^s}{b-x}\,dx\right\}$$

$$= \frac{-a^s\log(b-a)}{s} + \frac{1}{s}\lim_{\epsilon\to 0^+}\left\{\int_a^{b-\epsilon}\frac{x^s}{b-x}\,dx + b^s\log\epsilon\right\}$$

Since $(b-\epsilon)^s\log\epsilon = b^s\log\epsilon + O(\epsilon)$ $\qquad \{6\text{-}6\text{-}4\}$

Therefore combining $\{6\text{-}6\text{-}2\}$, $\{6\text{-}6\text{-}3\}$, $\{6\text{-}6\text{-}4\}$, and removing limit operation,

$$\mathcal{M}\left\{1_+(x-a)1_+(b-x)\log(b-x)\right\} = \frac{1}{s}\left\{-a^s\log(b-a) + b^s\left[\log\frac{b}{C} - \psi(s+1) - \left(\frac{a}{b}\right)^{s+1}\Phi\left(\frac{a}{b},1,s+1\right)\right]\right\} \quad a < b$$

$$s \neq 0,-1,-2,-3. \qquad \{6\text{-}6\text{-}5\}$$

Putting $\{6\text{-}6\text{-}5\}$ and formula 3-1 into $\{6\text{-}6\text{-}1\}$ gives:

$$\mathcal{M}\left\{1_+(x-a)\log|x-b|\right\} = \frac{a^s}{s}\left[\left(\frac{b}{a}\right)^s\left\{\psi(-s)-\psi(s+1)\right\} - \log(b-a) - \frac{a}{b}\Phi\left(\frac{a}{b},1,s+1\right)\right]$$

$$a < b \qquad -\infty < \operatorname{Re} s < 0 \qquad s \neq -1,-2,\ldots$$

Using $\{5\text{-}10\text{-}1\}$ in this gives the desired result.

(6-7) Proof of formula 3-7,

$$\mathcal{M}\left\{1_+(x-a)\log|x^2-b^2|\right\} = \frac{-a^s}{s}\left[\log(a^2-b^2) + \Phi\left(\frac{b^2}{a^2},1,\frac{s}{2}\right)\right]$$

$$a > b \qquad -\infty < \operatorname{Re} s < 0$$

$$\mathcal{M}\left\{1_+(x-a)\log|x^2-b^2|\right\} = \mathcal{M}\left\{1_+(x-a)\log(x-b)\right\} + \mathcal{M}\left\{1_+(x-a)\log(x+b)\right\} \qquad \{6\text{-}7\text{-}1\}$$

Using formulas 3-5 and 3-2 in $\{6\text{-}7\text{-}1\}$

$$\mathcal{M}\left\{1_+(x-a)\log|x^2-b^2|\right\} = \frac{-a^s}{s}\left[\log(a^2-b^2) + \Phi\left(-\frac{b}{a},1,-s\right) + \Phi\left(\frac{b}{a},1,-s\right)\right] \quad a > b \qquad \{6\text{-}7\text{-}2\}$$

Now, $\Phi(\frac{b}{a},1,-s) + \Phi(-\frac{b}{a},1,-s) = \sum_{N=0}^{\infty} \frac{(\frac{b}{a})^n + (-\frac{b}{a})^n}{n-s} = \Phi(\frac{b^2}{a^2}, 1, \frac{s}{2})$

Thus $\mathcal{M}\left\{1_+(x-a)\log|x^2-b^2|\right\} = \frac{-a^s}{s}\left[\log(a^2-b^2) + \Phi(\frac{b^2}{a^2},1,\frac{s}{2})\right]$

$$\mathrm{Re}\ s < 0 \qquad a > b$$

Q.E.D.

(6-8) Derivation of formula 3-8,

$$\mathcal{M}\left\{1_+(x-a)\log|x^2-a^2|\right\} = \frac{a^s}{s}\left[\psi(-s) - \frac{1}{2}G(-s) - \log\frac{2a^2}{C}\right] \qquad -\infty < \mathrm{Re}\ s < 0$$

Combining formulas 3-1 and 3-2 with {6-7-1} when a = b gives the result.

(6-9) Proof of formula 3-9,

$$\left\{1_+(x-a)\log|x^2-b^2|\right\} = \frac{-a^s}{s}\left[\log(b^2-a^2) + (\frac{b}{a})^s \pi\left(\frac{1+\cos\pi s}{\sin\pi s}\right) + \frac{a^2}{b^2}\Phi\left(\frac{a^2}{b^2},1,1+\frac{s}{2}\right)\right]$$

$$-\infty < \mathrm{Re}\ s < 0 \quad s \neq -1,-2,\ldots$$

Putting formulas 3-4 and 3-6 into {6-7-1},

$$\mathcal{M}\left\{1_+(x-a)\log|x^2-b^2|\right\} = \frac{-a^s}{s}\left[\log(b^2-a^2) + (\frac{b}{a})^s \pi\left\{-\csc\pi s + \mathrm{ctn}(-\pi s)\right\} + \frac{a}{b}\left\{\Phi(\frac{a}{b},1,s+1) - \Phi(-\frac{a}{b},1,s+1)\right\}\right]$$

$$a < b \qquad \mathrm{Re}\ s < 0 \quad s \neq -1,-2,\ldots \qquad \{6\text{-}9\text{-}1\}$$

Now, $$-\csc\pi s + \operatorname{ctn}(-\pi s) = -\left(\frac{1+\cos\pi s}{\sin\pi s}\right) \qquad \{6\text{-}9\text{-}2\}$$

Also,

$$\Phi(\tfrac{a}{b},1,s+1) - \Phi(-\tfrac{a}{b},1,s+1) = \sum_{N=0}^{\infty} \frac{(\frac{a}{b})^n - (-\frac{a}{b})^n}{(n+s+1)} = \frac{a}{b}\,\Phi(\frac{a^2}{b^2},1,1+\frac{s}{2})$$
$$\{6\text{-}9\text{-}3\}$$

Thus $\{6\text{-}9\text{-}1\}$, $\{6\text{-}9\text{-}2\}$, and $\{6\text{-}9\text{-}3\}$ give:

$$\mathcal{M}\left\{1_+(x-a)\log|x^2-b^2|\right\} = \frac{-a^s}{s}\left[\log(b^2-a^2) + (\tfrac{b}{a})^s \pi\left(\frac{1+\cos\pi s}{\sin\pi s}\right) + \frac{a^2}{b^2}\,\Phi(\frac{a^2}{b^2},1,1+\frac{s}{2})\right]$$

$$a < b \qquad -\infty < \operatorname{Re} s < 0 \qquad s \neq -1,-2,\ldots$$

Q.E.D.

(6-10) Proof of formula 3-10,

$$\mathcal{M}\left\{1_+(x-a)\log^2(x-a)\right\} = \frac{a^s}{s}\left\{2\left(\psi(-s)+\gamma\right)(\log a - \tfrac{1}{s}) - \log^2 a - \frac{\pi^2}{6} - \psi'(1-s) - \left(\gamma + \psi(1-s)\right)^2\right\}$$

Rewriting $\qquad -\infty < \operatorname{Re} s < 0$

$$\mathcal{M}\left\{1_+(x-a)\log^2(x-a)\right\} = \int_a^\infty [\log(x-a)]^2\, x^{s-1}dx$$
$$= \int_a^\infty \left[\log a + \log(\tfrac{x}{a}-1)\right]^2 x^{s-1}\,dx$$
$$= \log a^2 \int_a^\infty x^{s-1}dx + 2\log a \int_a^\infty \left[\log\tfrac{1}{a}(x-a)\right] x^{s-1}\,dx + \int_a^\infty \log^2(\tfrac{x}{a}-1)\,x^{s-1}\,dx$$
$$= -\log a^2 \int_a^\infty x^{s-1}dx + 2\log a \int_a^\infty \log(x-a)x^{s-1}dx + \int_a^\infty \log^2(\tfrac{x}{a}-1)x^{s-1}dx$$
$$\{6\text{-}10\text{-}1\}$$

From [8], page 75, equation 324-8a,

$$\int_0^1 x^{p-1}(1-x)^{q-1}\log^2 x \, dx = \frac{\Gamma(\frac{p}{\alpha})\Gamma(q)}{\alpha^3\Gamma(\frac{p}{\alpha}+q)}\left[\psi'(\tfrac{p}{\alpha}) - \psi'(\tfrac{p}{\alpha}+q) + \left\{\psi(\tfrac{p}{\alpha}) - \psi(\tfrac{p}{\alpha}+q)\right\}^2\right]$$

$$\alpha, p, q, > 0 \qquad \{6\text{-}10\text{-}2\}$$

Now, letting $x = \frac{a}{y}$

$$\int_a^\infty \log^2(\tfrac{x}{a}-1)x^{s-1}\,dx = a^s \int_0^1 \left[\log \tfrac{1}{y}(1-y)\right]^2 y^{-1-s}dy$$

$$= a^s\left\{\int_0^1 \log^2(1-y)y^{-1-s}dy + \int_0^1 \log^2 y\, y^{-1-s}dy - 2\int_0^1 \log y \log(1-y)y^{-1-s}\,dy\right\} \qquad \{6\text{-}10\text{-}3\}$$

Putting $\alpha = 1$, $q = 1$, and $p = -s$ in $\{6\text{-}10\text{-}2\}$ gives:

$$\int_0^1 y^{-s-1}\log^2 y\,dy = \frac{\Gamma(-s)}{\Gamma(1-s)}\left[\psi'(-s) - \psi'(1-s) + \left\{\psi(-s) - \psi(1-s)\right\}^2\right]$$

$$\text{Re } s < 0 \qquad \{6\text{-}10\text{-}4\}$$

Also letting $y = 1-x$, and using $\alpha = 1$, $p = 1$, and $q = -s$ in $\{6\text{-}10\text{-}2\}$,

$$\int_0^1 \log^2(1-y)\, y^{-1-s}\,dx = \int_0^1 (1-x)^{-1-s}\log^2 x\,dx$$

$$= \frac{\Gamma(-s)}{\Gamma(1-s)}\left[\psi'(1) - \psi'(1-s) + \left\{\psi(1) - \psi(1-s)\right\}^2\right]$$

$$\text{Re } s < 0 \qquad \{6\text{-}10\text{-}5\}$$

From [3], page 44, equation (3),

$$\psi^{(1)}(z) - \psi^{(1)}(1+z) = \frac{1}{z^2} \qquad \{6\text{-}10\text{-}6\}$$

And from [10], page 212, and from [3] page 22, equation (10),

$$\psi'(z) = \zeta(2, z)$$

$$\text{or} \quad \psi'(1) = \zeta(2,1) = \frac{\pi^2}{6} \qquad \text{Since } \zeta(2,1) = \zeta(2) \qquad \{6\text{-}10\text{-}7\}$$

Combining {6-10-4} and {6-10-5}, and using {6-10-6}, {6-10-7}, the difference relation for $\psi(z)$, and $\psi(1) = -\gamma$ gives;

$$\int_0^1 \log^2(1-y)\, y^{-1-s}\, dy + \int_0^1 \log^2 y\, y^{-1-s}\, dx = \frac{\Gamma(-s)}{\Gamma(1-s)} \left[\frac{2}{s^2} + \frac{\pi^2}{6} - \psi'(1-s) + \{\gamma + \psi(1-s)\}^2 \right] \qquad \text{Re } s < 0 \qquad \{6\text{-}10\text{-}8\}$$

The following lemma is now needed:

Lemma: $$I = \int_0^1 x^{-s-1} \log x \log(1-x)\, dx = \frac{\psi(1-s) + \gamma}{s^2} + \frac{\psi'(1-s)}{s}$$

$$-\infty < \text{Re } s < 0 \qquad \{6\text{-}10\text{-}9\}$$

Proof:

Using {6-10-7},

$$I = \frac{\psi(1-s) + \gamma}{s^2} + \frac{\zeta(2, 1-s)}{s} \qquad \{6\text{-}10\text{-}10\}$$

Now from [3], page 25, equation (3),

$$\zeta(s, \nu) = \frac{1}{\Gamma(s)} \int_0^1 x^{\nu-1} (1-x)^{-1} (\log 1/x)^{s-1}\, dx \qquad \text{Re } s > 1, \ \text{Re } \nu > 0 \qquad \{6\text{-}10\text{-}11\}$$

Using $\{6\text{-}10\text{-}11\}$ with $s=2$ and $\nu = 1-s$, and $\{5\text{-}4\text{-}4\}$ with $z = 1-s$ in $\{6\text{-}10\text{-}10\}$,

$$I = \frac{1}{s^2} \int_0^1 (1-x^{-s})(1-x)^{-1}\, dx + \frac{1}{s} \int_0^1 x^{-s}(1-x)^{-1} \log \frac{1}{x}\, dx \qquad \text{Re } s < 1$$

$$= \int_0^1 (1-x)^{-1} \left[\frac{1-x^{-s}}{s^2} - \frac{x^{-s} \log x}{s} \right] dx$$

and integrating by parts with $u = \left[\frac{1-x^{-s}}{s^2} - \frac{x^{-s} \log x}{s} \right]$, $dv = (1-x)^{-1}\, dx$,

and, $du = x^{-s-1} \log x\, dx$, gives:

$$I = \left[-\log(1-x) \left(\frac{1-x^{s}}{s^2} - \frac{x^{-s} \log x}{s} \right) \right]_0^1 + \int_0^1 x^{-s-1} \log x \log (1-x)\, dx$$

$$\text{Re } s < 1$$

Now let $R(x) = -\log(1-x) \left(\frac{1-x^{-s}}{s^2} - \frac{x^{-s} \log x}{s} \right)$

and $S(x) = \dfrac{-\log (1-x)(1-x^{-s})}{s^2}$, $T(x) = \dfrac{\log(1-x) x^{-s} \log x}{s}$, Thus $R(x) = S(x) + T(x)$

The following can readily be shown by repeated application of L'hospital's Rule and using limit of products equal to product of limits.

$\lim\limits_{x \to 1^-} S(x) = 0$ $\qquad$ $\lim\limits_{x \to 1^-} T(x) = 0$

$\lim\limits_{x \to 0^+} S(x) = 0 \quad \text{Re } s < 1$ $\qquad$ $\lim\limits_{x \to 0^+} T(x) = 0 \quad \text{Re } s < 0$

Thus $R(x) \Big|_0^1 = 0$ and $I = \int_0^1 \log x \log (1-x) x^{-s-1}\, dx \qquad \text{Re } s < 1$

Q.E.D. for the lemma.

Combining $\{6\text{-}10\text{-}3\}$, $\{6\text{-}10\text{-}8\}$, and $\{6\text{-}10\text{-}9\}$,

$$\int_a^\infty \log^2\left(\frac{x}{a}-1\right)x^{s-1}\,dx = \frac{-a^s}{s}\left\{\frac{\pi^2}{6}+\left(\gamma+\psi(1-s)\right)^2+\psi'(1-s)\right.$$
$$\left. -2\,\frac{\psi(1-s)+\gamma}{s}+\frac{2}{s^2}\right\} \qquad \text{Re } s < 0 \qquad \{6\text{-}10\text{-}12\}$$

Using formula 3-1 and $\{6\text{-}10\text{-}12\}$ in $\{6\text{-}10\text{-}1\}$,

$$\mathfrak{M}\left\{1_+(x-a)\log^2(x-a)\right\} = \frac{a^s}{s}\log^2 a + 2\log a\,\frac{a^s}{s}\left[\psi(-s)-\log\frac{a}{c}\right]$$
$$-\frac{a^s}{s}\left\{\frac{\pi^2}{6}+\left(\gamma+\psi(1-s)\right)^2+\psi'(1-s)\right.$$
$$\left.-\frac{2\left(\psi(1-s)+\gamma\right)}{s}+\frac{2}{s^2}\right\} \qquad \text{Re}\, s < 0$$

$$= \frac{a^s}{s}\left\{2\log a\;\psi(-s)-\log^2 a+2\gamma\log a-\frac{\pi^2}{6}-\left(\gamma+\psi(1-s)\right)^2-\psi'(1-s)\right.$$
$$\left.-\frac{2}{s}\left(\psi(1-s)+\frac{1}{s}+\gamma\right)\right\}$$

$$= \frac{a^s}{s}\left\{2\log a\left(\psi(-s)+\gamma\right)-\log^2 a-\frac{\pi^2}{6}-\left(\gamma+\psi(1-s)\right)^2-\psi'(1-s)\right.$$
$$\left.-\frac{2}{s}\left(\psi(-s)+\gamma\right)\right\}$$

$$= \frac{a^s}{s}\left\{2\left(\psi(-s)+\gamma\right)\left(\log a-\frac{1}{s}\right)-\log^2 a-\frac{\pi^2}{6}-\left(\gamma+\psi(1-s)\right)^2-\psi'(1-s)\right\}$$
$$-\infty < \text{Re } s < 0$$

Q.E.D.

(6-11) Derivation of formula 3-11,

$$\mathcal{M}\left\{P_f\, \frac{1_+(x-a)\log(x-a)}{(x-a)}\right\} = a^{s-1}\left\{\left(\psi(1-s)+\gamma\right)\left(\frac{1}{s-1}-\log a\right) + \log^2\sqrt{a} + \frac{\pi^2}{12} + \frac{\psi'(2-s)}{2} + \frac{1}{2}\left(\gamma+\psi(2-s)\right)^2\right\}$$

$$-\infty < \mathrm{Re}\, s < 1$$

Now using the result proved in Appendix 4, formula 1-7, and formula 3-10,

$$\mathcal{M}\left\{P_f\, \frac{1_+(x-a)\log(x-a)}{(x-a)}\right\} = \frac{1}{2}\mathcal{M}\left\{\frac{d}{dx}\, 1_+(x-a)\log^2(x-a)\right\}$$

$$= \frac{-(s-1)}{2(s-1)}\, a^{s-1}\left\{2\left(\psi(1-s)+\gamma\right)\left(\log a - \frac{1}{s-1}\right) - \log^2 a - \frac{\pi^2}{6} - \psi'(2-s) - \left(\gamma+\psi(2-s)\right)^2\right\}$$

$$= -a^{s-1}\left\{\left(\psi(1-s)+\gamma\right)\left(\log a - \frac{1}{s-1}\right) - \log^2\sqrt{a} - \frac{\pi^2}{12} - \frac{\psi'(2-s)}{2} - \frac{1}{2}\left(\gamma+\psi(2-s)\right)^2\right\} \qquad -\infty < \mathrm{Re}\, s < 1$$

Q.E.D.

(6-12) Proof of formula 3-12,

$$\mathcal{M}\left\{1_+(x-a)1_+(b-x)\log(x-a)\right\} = \frac{a^s}{s}\left[\psi(-s) + \log\frac{c}{a} + \left(\frac{a}{b}\right)^{-s}\left(\Phi\left(\frac{a}{b},1,-s\right) + \log(b-a)\right)\right] \qquad s \neq 0,1,2,\ldots$$

From [3], page 27, equation (3),

$$\Phi(z,s,\nu) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}e^{-\nu t}}{1-ze^{-t}}\,dt$$

$\mathrm{Re}\,\nu > 0$ and either $|z| \le 1$, $z \neq 1$, $\mathrm{Re}\, s > 0$

or $z = 1$, $\mathrm{Re}\, s > 1$

$\{6\text{-}12\text{-}1\}$

Now let $x = e^{t}$ in $\{6\text{-}12\text{-}1\}$ and let $z = \frac{a}{b}$, $s = 1$, and $\nu = -s$;

Also let $z = -s$ and $x = \frac{a}{bt}$ in $\{5\text{-}4\text{-}4\}$, therefore, if

$$I = \frac{a^{s}}{s}\left[\psi(-s) + \gamma - \log a\right] + \frac{b^{s}}{s}\left[\Phi\left(\tfrac{a}{b}, 1, -s\right) + \log(b-a)\right]$$

then,

$$I = \frac{a^{s}}{s}\left[-\log a + \int_{0}^{1} (1-t^{-s-1})(1-t)^{-1}\,dt\right] + \frac{b^{s}}{s}\left[\log(b-a) + \int_{0}^{\infty} \frac{e^{st}}{1-\frac{a}{b}e^{-t}}\,dt\right]$$

Re $s < 0$

$$= \frac{a^{s}}{s}\left[-\log a + \int_{\frac{a}{b}}^{\infty} \frac{\left(1-\left(\frac{a}{bx}\right)^{-s-1}\right)}{1-\frac{a}{bx}} \left(\frac{a}{bx^{2}}\right) dx\right] + \frac{b^{s}}{s}\left[\log(b-a)\right.$$

$$\left. + \int_{1}^{\infty} \frac{x^{s}}{1-\frac{a}{bx}} \, \frac{dx}{x}\right]$$

$$= \frac{a^{s}}{s}\left[-\log a + \int_{\frac{a}{b}}^{\infty} \frac{\left(\frac{bx}{a}\right)^{-1} - \left(\frac{bx}{a}\right)^{s}}{x-\frac{a}{b}}\,dx\right] + \frac{b^{s}}{s}\left[\log(b-a) + \int_{1}^{\infty} \frac{x^{s}}{x-\frac{a}{b}}\,dx\right]$$

$$= \frac{1}{s}\left[-a^{s}\log a + \int_{\frac{a}{b}}^{\infty} \frac{a^{s}\left(\frac{bx}{a}\right)^{-1} - (bx)^{s}}{x-\frac{a}{b}}\,dx + b^{s}\log(b-a) + \int_{1}^{\infty} \frac{(xb)^{s}}{x-\frac{a}{b}}\,dx\right]$$

$$= \frac{1}{s}\left[b^s \log(b-a) - a^s \log a + \int_{\frac{a}{b}}^{1} \frac{a^s \left(\frac{bx}{a}\right)^{-1} - (b\,x)^s}{x - \frac{a}{b}}\, dx \right.$$

$$\left. + \frac{a^{s+1}}{b} \int_{1}^{\infty} \frac{dx}{x\left(x - \frac{a}{b}\right)} \right]$$

$$= \frac{1}{s}\left[b^s \log(b-a) - a^s \log a + \int_{a}^{b} \frac{a^{s+1}x^{-1} - x^s}{x - a}\, dx \right.$$

$$\left. + \frac{a^{s+1}}{b} \int_{1}^{\infty} \frac{b}{a}\left[\frac{1}{x - \frac{a}{b}} - \frac{1}{x} \right] dx \right]$$

$$= \frac{1}{s}\left[b^s \log(b-a) - a^s \log a + a^s \log\left(1 - \frac{a}{b}\right) \right.$$

$$\left. + \int_{a}^{b} \frac{a^{s+1}x^{-1} - x^s}{x - a}\, dx \right] \qquad \{6\text{-}12\text{-}2\}$$

Integrating the integral in $\{6\text{-}12\text{-}2\}$ by parts with $u = a^{s+1}x^{-1} - x^{s}$

and $dv = \frac{dx}{x-a}$ gives:

$$\int_a^b \frac{a^{s+1}x^{-1}-x^{s}}{x-a}\,dx = \frac{a^{s+1}}{b}\log(b-a) - b^{s}\log(b-a) + a^{s+1}\int_a^b \frac{\log(x-a)}{x^2}\,dx$$

$$+\ s\int_a^b \log(x-a)\,x^{s-1}\,dx \qquad \{6\text{-}12\text{-}3\}$$

Now from [9], page 113, equation 323-12C,

$$\int \frac{\log(ax+b)}{x^2}\,dx = \frac{a}{b}\log x - \left(\frac{1}{x}+\frac{a}{b}\right)\log(ax+b) \qquad \{6\text{-}12\text{-}4\}$$

Thus,

$$\int_a^b \frac{\log(x-a)}{x^2}\,dx = -\frac{1}{a}\log b - \left(\frac{1}{b}-\frac{1}{a}\right)\log(b-a) + \frac{1}{a}\log a \qquad \{6\text{-}12\text{-}5\}$$

Combining $\{6\text{-}12\text{-}3\}$ and $\{6\text{-}12\text{-}5\}$,

$$\int_a^b \frac{a^{s+1}x^{-1}-x^{s}}{x-a}\,dx = s\int_a^b \log(x-a)x^{s-1}dx - b^{s}\log(b-a)$$

$$-a^{s}\log b + a^{s}\log a + a^{s}\log(b-a) \qquad \{6\text{-}12\text{-}6\}$$

Putting $\{6\text{-}12\text{-}2\}$ in $\{6\text{-}12\text{-}2\}$

$$I = \int_a^b \log(x-a)\,x^{s-1}\,dx \qquad -\infty < \operatorname{Re} s < 0$$

The Result can now be extended by analytic continuation to include all s except $S = 0,1,2,\ldots$

Q.E.D.

(6-13) Proof of formula 3-13,

$$\mathcal{M}\left\{1_+(x-a)\, e^{-\alpha(x-a)}\right\} = e^{\alpha a}\alpha^{-s}\,\Gamma(s) - \frac{a^s}{s}\,\Phi(1, s+1; \alpha a)$$

$$\operatorname{Re}\alpha > 0 \qquad s \neq 0, -1, -2, \ldots$$

$$\mathcal{M}\left\{1_+(x-a)\, e^{-\alpha(x-a)}\right\} = e^{\alpha a}\left[\int_0^\infty e^{-\alpha x} x^{s-1}\,dx - \int_0^a e^{-\alpha x} x^{s-1}\,dx\right]$$

Letting $U = \alpha X$ in the first integral and $U = \frac{X}{a}$ in the second,

$$\mathcal{M}\left\{1_+(x-a)\, e^{-\alpha(x-a)}\right\} = e^{\alpha a}\left[\alpha^{-s}\int_0^\infty e^{-u} u^{s-1}\,du - \int_0^a e^{-\alpha x} x^{s-1}\,dx\right]$$

$$= e^{\alpha a}\alpha^{-s}\,\Gamma(s) - a^s e^{\alpha a}\int_0^1 e^{-\alpha a u} u^{s-1}\,du$$

$$\operatorname{Re} s > 0 \qquad \{6\text{-}13\text{-}1\}$$

From [3], page 255, equation (1), the following integral representation of the confluent hypergeometric function is obtained:

$$\Phi(a,c;x) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)}\int_0^1 e^{xu} u^{a-1}(1-u)^{c-a-1}\,du$$

$$\operatorname{Re} c > \operatorname{Re} a > 0 \qquad \{6\text{-}13\text{-}2\}$$

In $\{6\text{-}13\text{-}2\}$, let $X = -\alpha a$, $a = s$, and $c = s+1$. Rearranging gives,

$$\int_0^1 e^{-\alpha a u} u^{s-1}\,du = \frac{\Gamma(s)\,\Gamma(1)}{\Gamma(s+1)}\,\Phi(s, s+1; -\alpha a)$$

$$= \frac{1}{s}\,\Phi(s, s+1; -\alpha a)$$

$$\operatorname{Re} s > 0 \qquad \{6\text{-}13\text{-}3\}$$

Now using Kummer's first formula given in [7], page 125, Theorem 42,

$$\Phi(a,c;x) = e^{z}\,\Phi(c-a,c;\,-x) \qquad c \neq 0,-1,-2,\ldots \qquad \{6\text{-}13\text{-}4\}$$

Using Kummer's formula,

$$\Phi(s,\, s+1;\, -\alpha a) = e^{-\alpha a}\,\Phi(1, s+1;\, \alpha a) \qquad s \neq -1,-2,\ldots$$

Combining this result with $\{6\text{-}13\text{-}3\}$ and $\{6\text{-}13\text{-}1\}$,

$$\mathfrak{M}\left\{1_{+}(x-a)\,e^{-\alpha(x-a)}\right\} = e^{\alpha a}\alpha^{-s}\Gamma(s) - \frac{a^{s}}{s}\Phi(1, s+1;\alpha a)$$

and using analytic continuation from Re $s > 0$, the result can be extended to include all S except $s = 0,-1,-2,\ldots$

Q.E.D.

(6-14) Derivation of formula 3-14,

$$\mathfrak{M}\left\{1_{+}(x-a)1_{+}(b-x)\,e^{-\alpha(x-a)}\right\} = \frac{b^{s}e^{-\alpha(b-a)}}{s}\Phi(1,S+1;\alpha b)$$

$$-\frac{a^{s}}{s}\Phi(1,\, s+1;\, \alpha a) \qquad S \neq 0,-1,-2,\ldots$$

Integrating term by term the uniformly convergent series of the exponential function in the interval $a \leq x \leq b$,

$$\mathfrak{M}\left\{1_{+}(x-a)1_{+}(b-x)\,e^{-\alpha(x-a)}\right\} = e^{\alpha a}\sum_{n=0}^{\infty}\frac{(-\alpha)^{n}}{n!}\int_{a}^{b}x^{n+s-1}\,dx$$

$$= e^{\alpha a}\left\{\sum_{n=0}^{\infty}\frac{(-\alpha)^{n}b^{n+s}}{n!\,(n+s)} - \sum_{n=0}^{\infty}\frac{(-\alpha)^{n}a^{n+s}}{n!\,(n+s)}\right\} \qquad s \neq -1,-2,\ldots$$

Since $\frac{1}{n+s} = \frac{1}{s} \frac{(s)_n}{(s+1)_n}$,

$$\mathcal{M}\left\{1_+(x-a)\ 1_+(b-x)\ e^{-\alpha(x-a)}\right\} = \frac{e^{\alpha a}\, b^s}{s}\ \Phi(s,s+1;-\alpha b)$$

$$-\ \frac{e^{\alpha a}\, a^s}{s}\ \Phi(s,s+1;\ -\alpha a) \qquad\qquad s \neq 0,-1,-2,\ldots$$

Employing $\{6-13-4\}$ twice gives the desired result.

(6-15) Proof of formula 3-15,

$$\mathcal{M}\left\{Pf\ 1_+(x-a)(x-a)^{\eta}\ e^{-\alpha(x-a)}\right\} = \Gamma(\eta+1)a^{s+\eta}\ \Psi\ (\eta+1,\eta+1+s;a\alpha)$$

$$\mathrm{Re}\,\alpha > 0,\ \eta \neq -1,-2,\ldots$$

First consider $\mathcal{M}\left\{1_+(x-a)(x-a)^{\eta}\ e^{-\alpha(x-a)}\right\}$ for $\mathrm{Re}\ \eta > -1$ and $\mathrm{Re}\,\alpha > 0$. From [5] , page 139, equation (22),

$$\int_b^\infty (t+a)^{2\mu-1}\ (t-b)^{2\nu-1}\ e^{-pt}\ dt$$

$$= \Gamma(2\nu)(a+b)^{\mu+\nu-1}\ p^{-\mu-\nu}\ e^{\frac{1}{2}p(a-b)}\, W_{\mu-\nu,\,\mu+\nu-\frac{1}{2}}\ (ap+bp)$$

$\mathrm{Re}\ \nu > 0$, $|\arg(a+b)| < \pi$, $\mathrm{Re}\ p > 0$. $\{6-15-1\}$

In $\{6-15-1\}$, let $a = 0$, $b = a$, $\mu = \frac{s}{2}$, $\nu = (\eta+1)/2$, $p = \alpha$, and $t = x$ then,

$$\int_a^\infty x^{s-1}\ (x-a)^{\eta}\ e^{-\alpha x}\ dx$$

$$= \Gamma(\eta+1)\ a^{\frac{s+\eta-1}{2}}\ \alpha^{-\left(\frac{s+\eta+1}{2}\right)}\ e^{-\frac{a\alpha}{2}}\ W_{\frac{s-\eta-1}{2},\,\frac{s+\eta}{2}}\ (a\alpha)$$

$$\mathrm{Re}\ \eta > -1,\ a(\text{real}) > 0,\ \mathrm{Re}\,\alpha > 0,\ -\infty < \mathrm{Re}\ s < \infty$$

thus,

$$\mathcal{M}\left\{1_+(x-a)(x-a)^\eta e^{-\alpha(x-a)}\right\}$$

$$= e^{\frac{a\alpha}{2}}\Gamma(\eta+1)\ a^{\frac{s+\eta-1}{2}}\ \alpha^{-\left(\frac{s+\eta+1}{2}\right)} W_{\frac{s-\eta-1}{2},\frac{s+\eta}{2}}(a\alpha)$$

$$\operatorname{Re}\eta > -1,\ \operatorname{Re}\alpha > 0,\ -\infty < \operatorname{Re} s. < \infty \qquad \{6\text{-}15\text{-}2\}$$

By definition from [5], page 264, equation (2),

$$W_{k,\mu}(x) = e^{-\frac{x}{2}} x^{\frac{c}{2}} \Psi(a_1, c; x) \qquad \{6\text{-}15\text{-}3\}$$

$$a_1 = 1/2 - k + \mu,\quad c = 2\mu - 1$$

If $k=(s-\eta-1)/2$, $\mu=(s+\eta)/2$, and $x=a\alpha$, then $a_1=\eta+1$, $c=s+\eta+1$ and,

$$W_{\frac{s-\eta-1}{2},\frac{s+\eta}{2}}(a\alpha) = e^{-\frac{a\alpha}{2}} a^{\frac{s+\eta+1}{2}} \alpha^{\frac{s+\eta+1}{2}} \Psi(\eta+1, s+\eta+1; a\alpha) \qquad \{6\text{-}15\text{-}4\}$$

Combining {6-15-4} and {6-15-2}

$$\mathcal{M}\left\{1_+(x-a)(x-a)^\eta e^{-\alpha(x-a)}\right\} = \Gamma(\eta+1)\ a^{s+\eta}\ \Psi(\eta+1, \eta+1+s;\ a\alpha)$$

$$\operatorname{Re}\eta > -1,\ \operatorname{Re}\alpha > 0, \text{ and } -\infty < \operatorname{Re} s < \infty \qquad \{6\text{-}15\text{-}5\}$$

Now consider $\mathcal{M}\left\{P_f\ 1_+(x-a)(x-a)^\eta e^{-\alpha(x-a)}\right\}$ for $\eta \neq -1, -2, \ldots$

$$\operatorname{Re}\alpha > 0, \text{ and } -\infty < \operatorname{Re} s < \infty.$$

To prove this more general result first consider k = 0 in {A5-1} in Appendix 5, rearranging this equation gives (with $-1 < \operatorname{Re}\beta < 0$):

$$\mathcal{M}\left\{P_f\ 1_+(x-a)(x-a)^{\beta-1}\ e^{-\alpha(x-a)}\right\} = \frac{1}{\beta}\left\{\mathcal{M}\left\{\frac{d}{dx}P_f\ 1_+(x-a)(x-a)^{\beta}\ e^{-\alpha(x-a)}\right\}\right.$$

$$\left. + \alpha\ \mathcal{M}\left\{P_f\ 1_+(x-a)(x-a)^{\beta}\ e^{-\alpha(x-a)}\right\}\right\} \qquad \{6\text{-}15\text{-}6\}$$

Since $P_f\, 1_+(x-a)(x-a)^{-\beta}\, e^{-\alpha(x-a)} = 1_+(x-a)(x-a)^{-\beta}\, e^{-\alpha(x-a)}$,

using formula 1-7 and $\{6\text{-}15\text{-}5\}$ twice gives,

$$\mathcal{M}\left\{P_f\, 1_+(x-a)(x-a)^{\beta-1} e^{-\alpha(x-a)}\right\} = \frac{1}{\beta}\Big\{-(s-1)\Gamma(\beta+1)a^{s+\beta-1}$$
$$\Psi(\beta+1, s+\beta; a\alpha) + \alpha\Gamma(\beta+1)a^{s+\beta}\,\Psi(\beta+1, s+\beta+1; a\alpha)\Big\}$$
$$= \Gamma(\beta)a^{s+\beta-1}\Big\{a\alpha\,\Psi(\beta+1, s+\beta+1; a\alpha) - (s-1)\Psi(\beta+1, s+\beta; a\alpha)$$
$$-\infty < \text{Re } s < \infty \qquad \{6\text{-}15\text{-}7\}$$

Now the following difference equation is obtained from [3], page 258, equation (7),

$$(c-a)\Psi(a,c;x) - x\Psi(a,c+1;x) + \Psi(a-1,c;x) = 0 \qquad \{6\text{-}15\text{-}8\}$$

Let $x = a\alpha$, $c = s+\beta$, $a = 1+\beta$ in $\{6\text{-}15\text{-}8\}$ to obtain after rearranging,

$$\Psi(+\beta, s+\beta; a\alpha) = a\alpha\,\Psi(\beta+1, s+\beta+1; a\alpha) - (s-1)\Psi(\beta+1, s+\beta; a\alpha)$$
$$\{6\text{-}15\text{-}9\}$$

Putting $\{6\text{-}15\text{-}9\}$ into $\{6\text{-}15\text{-}7\}$ gives:

$$\mathcal{M}\left\{P_f\, 1_+(x-a)(x-a)^{\beta-1} e^{-\alpha(x-a)}\right\} = \Gamma(\beta)a^{s+\beta-1}\Psi(\beta, s+\beta; a\alpha)$$

Thus $\mathcal{M}\left\{P_f\, 1_+(x-a)(x-a)^{\eta}\, e^{-\alpha(x-a)}\right\} = \Gamma(\eta+1)a^{s+\eta}\,\Psi(\eta+1; s+\eta+1; a\alpha)$

is now true for Re $\eta > -2$ and $\eta \neq -1$. Re $\alpha > 0$ $-\infty <$ Re $s < \infty$.

$$\{6\text{-}15\text{-}10\}$$

If $k=1$ is now used in $\{A5\text{-}1\}$ (alternatively this now means $-2 < \beta < -1$ in $\{6\text{-}15\text{-}6\}$ and the equations following), $\{6\text{-}15\text{-}10\}$ would be shown to be true for Re $\eta > -3$ $\eta \neq -1$ and -2. Continuing by induction the result is true for all η except $\eta = -1, -2, -3, \ldots$

Q.E.D.

(6-16) Proof of formula 3-16,

$$\mathcal{M}\left\{1_+(x-a)\sin b(x-a)\right\} = \frac{b^{-s}}{2i}\left\{e^{i\left[\frac{\pi s}{2}-ab\right]}\Gamma(s,-iab) - e^{-i\left[\frac{\pi s}{2}-ab\right]}(s,iab)\right\}$$

$$-\infty < \operatorname{Re} s < 1$$

First consider,

$$\mathcal{M}\left\{1_+(x-a)\sin bx\right\} \text{ and } \mathcal{M}\left\{1_+(x-a)\cos bx\right\}$$

$$\mathcal{M}\left\{1_+(x-a)\sin bx\right\} = \frac{1}{2i}\left\{\int_a^\infty x^{s-1}e^{ibx}\,dx - \int_a^\infty x^{s-1}e^{-ibx}\,dx\right\} \quad \{6\text{-}16\text{-}1\}$$

Letting $u = -ibx$ in the first integral and $u = ibx$ in the second of $\{6\text{-}16\text{-}1\}$,

$$\mathcal{M}\left\{1_+(x-a)\sin bx\right\} = \frac{b^{-s}}{2}\left\{\int_{-iab}^{-i\infty}\left(\frac{-u}{i}\right)^{s-1}e^{-u}\,du + \int_{iab}^{i\infty}\left(\frac{u}{i}\right)^{s-1}e^{-u}\,du\right\}$$

$$= \frac{b^{-s}}{2}\left\{e^{\frac{(s-1)\pi i}{2}}\int_{-iab}^{-i\infty}u^{s-1}e^{-u}\,du + e^{\frac{-(s-1)\pi i}{2}}\int_{iab}^{i\infty}u^{s-1}e^{-u}\,du\right\}$$

Using equations $\{A6\text{-}1\}$

$$\mathcal{M}\left\{1_+(x-a)\sin bx\right\} = \frac{b^{-s}}{2}\left\{e^{\frac{(s-1)\pi i}{2}}\int_{-iab}^{\infty}u^{s-1}e^{-u}\,du + e^{\frac{-(s-1)\pi i}{2}}\int_{iab}^{\infty}u^{s-1}e^{-u}\,du\right\}$$

$$= \frac{b^{-s}}{2}\left\{e^{\frac{(s-1)\pi i}{2}}\Gamma(s,-iab) + e^{\frac{(1-s)\pi i}{2}}\Gamma(s,iab)\right\}$$

$$\{6\text{-}16\text{-}2\}$$

and using analytic continuation from $\operatorname{Re} s < 0$ extends the result to include $\operatorname{Re} s < 1$.

Similarly,

$$\mathcal{M}\left\{1_+(x-a)\cos bx\right\} = \frac{1}{2}\left\{\int_a^\infty x^{s-1}e^{ibx}\,dx + \int_a^\infty x^{s-1}e^{ibx}\,dx\right\}$$

$$= \frac{b^{s}}{2}\left\{\left(\frac{-1}{i}\right)^s\int_{-iab}^{-i\infty}u^{s-1}e^{-u}\,du + \left(\frac{1}{i}\right)^s\int_{iab}^{i\infty}u^{s-1}e^{-u}\,du\right.$$

Again with the aid of {A6-1}

$$\mathcal{M}\left\{1_+(x-a)\cos bx\right\} = \frac{b^{-s}}{2}\left\{e^{\frac{i\pi s}{2}}\int_{-iab}^{\infty} u^{s-1} e^{-u}\,du + e^{\frac{-i\pi s}{2}}\int_{iab}^{\infty} u^{s-1} e^{-u}\,du\right\}$$

$$= \frac{b^{-s}}{2}\left\{e^{\frac{i\pi s}{2}}\Gamma(s,-iab) + e^{\frac{-i\pi s}{2}}\Gamma(s,iab)\right\} \qquad \{6\text{-}16\text{-}3\}$$

Re $s < 1$ by analytic continuation from Re $s < 0$.

Now finally using {6-16-2} and {6-16-3},

$$\mathcal{M}\left\{1_+(x-a)\sin b(x-a)\right\} = \cos ab\ \mathcal{M}\left\{1_+(x-a)\sin bx\right\}$$

$$- \sin ab\ \mathcal{M}\left\{1_+(x-a)\cos bx\right\}$$

$$= \frac{b^{-s}}{2}\cos ab\left\{e^{\frac{(s-1)\pi i}{2}}\Gamma(s,-iab) + e^{\frac{(1-s)\pi i}{2}}\Gamma(s,iab)\right\}$$

$$- \frac{b^{-s}}{2}\sin ab\left\{e^{\frac{i\pi s}{2}}\Gamma(s,-iab) + e^{\frac{-i\pi s}{2}}\Gamma(s,iab)\right\}$$

$$= \frac{b^{-s}}{2}\left\{e^{\frac{i\pi s}{2}}\Gamma(s,-iab)\left[-\sin ab - i\cos ab\right]\right.$$

$$\left. + e^{\frac{-i\pi s}{2}}\Gamma(s,iab)\left[-\sin ab + i\cos ab\right]\right\}$$

$$= \frac{b^{-s}}{2i}\left\{e^{\frac{i\pi s}{2}}\Gamma(s,-iab)\left[\cos ab - i\sin ab\right]\right.$$

$$\left. - e^{\frac{-i\pi s}{2}}\Gamma(s,iab)\left[\cos ab + i\sin ab\right]\right\}$$

$$= \frac{b^{-s}}{2i}\left\{e^{i\left[\frac{\pi s}{2}-ab\right]}\Gamma(s,-iab)\right.$$

$$\left. - e^{-i\left[\frac{\pi s}{2}-ab\right]}\Gamma(s,iab)\right\}$$

Re $s < 1$

Q. E. D.

(6-17) Proof of formula 3-17,

$$\mathcal{M}\left\{1_+(x-a)\cos b(x-a)\right\} = \frac{b^{-s}}{2}\left\{e^{i\left[\frac{\pi s}{2}-ab\right]}\Gamma(s,-iab)+e^{-i\left[\frac{\pi s}{2}-ab\right]}\Gamma(s,iab)\right\}$$

$$-\infty < \text{Re } s < 1$$

Using $\{6\text{-}16\text{-}2\}$ and $\{6\text{-}16\text{-}3\}$,

$$\mathcal{M}\left\{1_+(x-a)\cos b(x-a)\right\} = \cos a\,b\;\mathcal{M}\left\{1_+(x-a)\cos b\,x\right\} + \sin a\,b\;\mathcal{M}\left\{1_+(x-a)\sin b\,x\right\}$$

$$= \frac{\cos(a\,b)\,b^{-s}}{2}\left\{e^{\frac{i\pi s}{2}}\Gamma(s,-iab)+e^{\frac{-i\pi s}{2}}\Gamma(s,iab)\right\}$$

$$+\ \frac{\sin(a\,b)\,b^{-s}}{2}\left\{e^{(s-1)\pi i/2}\Gamma(s,iab)+e^{(1-s)\pi i/2}\Gamma(s,iab)\right\}$$

$$= \frac{b^{-s}}{2}\left\{e^{i\pi s/2}\Gamma(s,-iab)\left[\cos(a\,b)-i\sin(a\,b)\right]+e^{-i\pi s/2}\Gamma(s,iab)\right.$$

$$\left.\left[\cos(a\,b)+i\sin(a\,b)\right]\right.$$

$$= \frac{b^{-s}}{2}\left\{e^{i\left[\frac{\pi s}{2}-ab\right]}\Gamma(s,-iab)+e^{-i\left[\frac{\pi s}{2}-ab\right]}\Gamma(s,iab)\right\} \qquad \text{Re } s < 1$$

Q.E.D.

(6-18) Derivation of formula 3-18,

$$\mathcal{M}\left\{1_+(x-a)1_+(b-x)\sinh\alpha(x-a)\right\} = \frac{\alpha\,a^{s+1}}{s(s+1)}\,{}_1F_2\left(1;\frac{s+2}{2};\frac{s+3}{2};\frac{\alpha^2a^2}{4}\right)$$

$$+\ \frac{b^s}{2s}\left[e^{-\alpha a}\,\Phi(s,s+1;\alpha b)-e^{\alpha a}\,\Phi(s,s+1;-\alpha b)\right] \qquad a<b;\quad s\neq 0,-1,-2,\ldots$$

Using formula 3-14 twice

$$\mathcal{M}\left\{1_+(x-a)1_+(b-x)\sinh\alpha(x-a)\right\} = \frac{1}{2}\left\{\mathcal{M}\left\{1_+(x-a)1_+(b-x)\,e^{\alpha(x-a)}\right\}\right.$$

$$\left.-\mathcal{M}\left\{1_+(x-a)1_+(b-x)\,e^{-\alpha(x-a)}\right\}\right\}$$

$$= \frac{1}{2}\left\{\frac{e^{\alpha(b-a)}b^s}{s}\,\Phi(1,s+1;-\alpha b)-\frac{a^s}{s}\,\Phi(1,s+1;-\alpha a)-\frac{e^{-\alpha(b-a)}b^s}{s}\right.$$

$$\left.\Phi(1,s+1;\alpha b)+\frac{a^s}{s}\,\Phi(1,s+1;\alpha a)\right\} \qquad s\neq 0,-1,-2,\ldots \qquad \{6\text{-}18\text{-}1\}$$

Now using $\{6\text{-}13\text{-}4\}$,

$$\Phi(1, s+1; -\alpha b) = e^{-\alpha b}\, \Phi(s, s+1; \alpha b) \qquad \{6\text{-}18\text{-}2\}$$

$$\Phi(1, s+1; \alpha b) = e^{\alpha b}\, \Phi(s, s+1; -\alpha b)$$

From [7] , page 22, Lemma 5, $(\alpha)_{2n} = 2^{2n} \left(\frac{\alpha}{2}\right)_n \left(\frac{\alpha+1}{2}\right)_n$ $\{6\text{-}18\text{-}3\}$

With the aid of this,

$$\Phi(1, s+1; \alpha a) - \Phi(1, s+1; -\alpha a) = \sum_{n=0}^{\infty} \frac{(\alpha a)^n - (-\alpha a)^n}{(s+1)_n} = \frac{2\alpha a}{s+1} \sum_{n=0}^{\infty} \frac{(\alpha a)^{2n}}{(s+2)_{2n}}$$

$$= \frac{2\alpha a}{s+1} \sum_{n=0}^{\infty} \frac{\left(\frac{\alpha^2 a^2}{4}\right)^n}{\left(\frac{s+2}{2}\right)_n \left(\frac{s+3}{2}\right)_n} = \frac{2\alpha a}{s+1}\ {}_1F_2\left(1; \frac{s+2}{2}, \frac{s+3}{2}; \frac{\alpha^2 a^2}{4}\right)$$

$\{6\text{-}18\text{-}4\}$

Combining $\{6\text{-}18\text{-}4\}$, $\{6\text{-}18\text{-}2\}$, and $\{6\text{-}18\text{-}1\}$,

$$\mathcal{M}\left\{1_+(x-a)1_+(b-x)\ \sinh \alpha(x-a)\right\} = \frac{\alpha a^{s+1}}{s(s+1)}\ {}_1F_2\left(1; \frac{s+2}{2}, \frac{s+3}{2}; \frac{\alpha^2 a^2}{4}\right)$$

$$+ \frac{b^s}{2s}\left[e^{-\alpha a}\ \Phi(s, s+1; \alpha b) - e^{\alpha a}\ \Phi(s, s+1; -\alpha b)\right]$$

$$s \neq 0, -1, -2, \ldots$$

Q.E.D.

(6-19) Derivation of formula 3-19,

$$\mathcal{M}\left\{1_+(x-a)1_+(b-x)\cosh \alpha(x-a)\right\} = \frac{-a^s}{s}\ {}_1F_2\left(1; \frac{s+1}{2}, \frac{s+2}{2}; \frac{\alpha^2 a^2}{4}\right)$$

$$+ \frac{b^s}{2s}\left\{e^{-a\alpha}\ \Phi(s, s+1; \alpha b) + e^{a\alpha}\ \Phi(s, s+1; -\alpha b)\right\}$$

$$a < b \qquad s \neq 0, -1, -2, \ldots$$

Using formula 3-14 twice

$$\mathfrak{M}\left\{1_+(x-a)1_+(b-x)\cosh\alpha(x-a)\right\} = \frac{1}{2}\left[\mathfrak{M}\left\{1_+(x-a)1_+(b-x)e^{\alpha(x-a)}\right\}\right.$$

$$\left.\mathfrak{M}\left\{1_+(x-a)1_+(b-x)e^{-\alpha(x-a)}\right\}\right.$$

$$= \frac{1}{2}\left[\frac{e^{\alpha(b-a)}\,b^s}{s}\,\Phi(1,s+1;-\alpha b) - \frac{a^s}{s}\,\Phi(1,s+1;-\alpha a)\right.$$

$$\left.+ \frac{e^{-\alpha(b-a)}\,b^s}{s}\,\Phi(1,s+1;\alpha b) - \frac{a^s}{s}\,\Phi(1,s+1;a\alpha)\right]$$

$$s \neq 0,-1,-2,\ldots \qquad \{6\text{-}19\text{-}1\}$$

Now with the aid of $\{6\text{-}18\text{-}3\}$,

$$\Phi(1,s+1;-\alpha a) + \Phi(1,s+1;a\alpha) = \sum_{n=0}^{\infty}\frac{\left[(-\alpha a)^n + (a\alpha)^n\right]}{(s+1)_n} = 2\sum_{n=0}^{\infty}\frac{(a\alpha)^{2n}}{(s+1)_{2n}}$$

$$= 2\sum_{n=0}^{\infty}\frac{\left(\frac{a^2\alpha^2}{4}\right)^n}{\left(\frac{s+1}{2}\right)_n\left(\frac{s+2}{2}\right)_n}$$

$$= 2\,{}_1F_2\left(1;\frac{s+1}{2},\frac{s+2}{2};\frac{a^2\alpha^2}{4}\right) \qquad \{6\text{-}19\text{-}2\}$$

Combining $\{6\text{-}18\text{-}2\}$, $\{6\text{-}19\text{-}2\}$, and $\{6\text{-}19\text{-}1\}$,

$$\mathfrak{M}\left\{1_+(x-a)1_+(b-x)\cosh\alpha(x-a)\right\} = \frac{-a^s}{s}\,{}_1F_2\left(1;\frac{s+1}{2},\frac{s+2}{2};\frac{a^2\alpha^2}{4}\right)$$

$$+ \frac{b^s}{2s}\left[e^{-a\alpha}\,\Phi(s,s+1;\alpha b) + e^{a\alpha}\,\Phi(s,s+1;-\alpha b)\right]$$

$$s \neq 0,-1,-2,\ldots$$

Q.E.D.

(6-20) Derivation of formula 3-20,

$$\mathcal{M}\left\{1_+(x-a)1_+(b-x)\log(b-x)\right\} = \frac{-b^s}{s}\left[\Psi(s+1) + \log\frac{C}{b} + \left(\frac{a}{b}\right)^s \left(\log(b-a) + \left(\frac{a}{b}\right) \Phi\left(\frac{a}{b}, 1, s+1\right)\right)\right] \qquad s \neq 0, -1, -2, \ldots$$

This result is obtained through rearranging $\{6\text{-}5\text{-}5\}$.

7. Step and Delta Functionals

(7-1) Derivation of formula 4-1,

$$\mathfrak{M}\left\{\delta^{(k)}(x-a)\right\} = (1-s)_k\, a^{s-1-k} \qquad -\infty < \text{Re } s < \infty$$

$$k = 0, 1, 2, \ldots$$

This result is simply an application of formula 1-7 to

$$\int_0^\infty \delta\,(x-a)\,x^{s-1}dx = a^{s-1}.$$

(7-2) Derivation of formula 4-2,

$$\mathfrak{M}\left\{\sum_{\nu=1}^{\infty}\delta\,(x-a\nu)\right\} = a^{s-1}\,\zeta(1-s,1) \qquad -\infty < \text{Re } s < 0$$

$$\mathfrak{M}\left\{\sum_{\nu=1}^{\infty}\delta\,(x-a\nu)\right\} = \sum_{\nu=1}^{\infty}\mathfrak{M}\left\{\delta\,(x-a\nu)\right\} = \sum_{\nu=1}^{\infty}(a\nu)^{s-1}$$

$$= a^{s-1}\sum_{\nu=1}^{\infty}\nu^{s-1} = a^{s-1}\sum_{\nu=0}^{\infty}\frac{1}{(1+\nu)^{1-s}} = a^{s-1}\,\zeta(1-s,1)$$

$$\text{Re } s < 0$$

Q.E.D.

(7-3) Derivation of formula 4-3,

$$\mathfrak{M}\left\{\sum_{\nu=1}^{\infty}(-1)^{\nu}\,\delta\,(x-a\nu)\right\} = -a^{s-1}\,\Phi(-1,1-s,1) \qquad -\infty < \text{Re } s < 1$$

$$\mathfrak{M}\left\{\sum_{\nu=1}^{\infty}(-1)^{\nu}\,\delta\,(x-a\nu)\right\} = \sum_{\nu=1}^{\infty}(-1)^{\nu}\,\mathfrak{M}\left\{\delta\,(x-a\nu)\right\} = \sum_{\nu=1}^{\infty}(-1)^{\nu}(a\nu)^{s-1}$$

$$= -a^{s-1}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{(\nu+1)^{1-s}} = -a^{s-1}\,\Phi(-1,1-s,1) \qquad \text{Re } s < 1$$

Q.E.D.

(7-4) Derivation of formula 4-4,

$$\mathfrak{M}\left\{1_{+}(x-a)\right\} = -\frac{a^{s}}{s} \qquad -\infty < \operatorname{Re} s < 0.$$

This follows immediately from setting $\alpha = 0$ in formula 2-1.

(7-5) Derivation of formula 4-5,

$$\mathfrak{M}\left\{\sum_{\nu=1}^{\infty} 1_{+}(x-a\nu)\right\} = \frac{-a^{s}}{s}\,\zeta(-s,1) \qquad \infty < \operatorname{Re} s < -1$$

$$\mathfrak{M}\left\{\sum_{\nu=1}^{\infty} 1_{+}(x-a\nu)\right\} = \sum_{\nu=1}^{\infty} \mathfrak{M}\left\{1_{+}(x-a\nu)\right\} = \sum_{\nu=1}^{\infty} -\frac{(a\nu)^{s}}{s}$$

$$= \frac{-a^{s}}{s}\sum_{\nu=0}^{\infty}\frac{1}{(\nu+1)^{-s}} = \frac{-a^{s}}{s}\,\zeta(-s,1)$$

$\operatorname{Re} s < -1$

Q.E.D.

(7-6) Derivation of formula 4-6,

$$\mathfrak{M}\left\{\sum_{\nu=1}^{\infty}(-1)^{\nu}\, 1_{+}(x-a\nu)\right\} = \frac{a^{s}}{s}\,\Phi(-1,-s,1) \qquad -\infty < \operatorname{Re} s < 0$$

$$\mathfrak{M}\left\{\sum_{\nu=0}^{\infty}(-1)^{\nu}\, 1_{+}(x-a\nu)\right\} = \sum_{\nu=1}^{\infty}(-1)^{\nu}\mathfrak{M}\left\{1_{+}(x-a\nu)\right\} = \sum_{\nu=1}^{\infty}\left[(-1)^{\nu}-\frac{(a\nu)^{s}}{s}\right]$$

$$= \frac{a^{s}}{s}\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{(1+\nu)^{-s}} = \frac{a^{s}}{s}\,\Phi(-1,-s,1)$$

$\operatorname{Re} s < 0$

Q.E.D.

Appendix 1

Derivation of equation {A1-3}

Using $\varphi(x)$ in place of x^{s-1} in {5-4-1},

$$\left\langle P_f \frac{1_+(x-a)1_+(b-x)}{(x-a)^k}, \varphi(x)\right\rangle = \lim_{\epsilon\to 0^+}\left\{\int_{a+\epsilon}^{b} \frac{\varphi(x)dx}{(x-a)^k} - \sum_{\nu=0}^{k-2}\frac{\varphi^{(\nu)}(a)}{\nu!(k-1-\nu)\epsilon^{k-1-\nu}} + \frac{\varphi^{(k-1)}(a)\log\epsilon}{(k-1)!}\right\} \qquad \{A1\text{-}1\}$$

Using this equation,

$$\left\langle \frac{d}{dx} P_f \frac{1(x-a)1(b-x)}{(x-a)^k}, \varphi(x)\right\rangle = -\left\langle P_f \frac{1(x-a)1(b-x)}{(x-a)^k}, \varphi'(x)\right\rangle$$

$$= -\lim_{\epsilon\to 0^+}\left\{\int_{a+\epsilon}^{b} \frac{\varphi'(x)\,dx}{(x-a)^k} - \sum_{\nu=0}^{k-2}\frac{\varphi^{(\nu+1)}(a)}{\nu!(k-1-\nu)\epsilon^{k-1-\nu}} + \frac{\varphi^{(k)}(a)\log\epsilon}{(k-1)!}\right\}$$

$$= \frac{-\varphi(b)}{(b-a)^k} - \lim_{\epsilon\to 0^+}\left\{\frac{-\varphi(a+\epsilon)}{\epsilon^k} - k\int_{a+\epsilon}^{b}\frac{\varphi(x)}{(x-a)^{k+1}} - \sum_{\nu=0}^{k-2}\frac{\varphi^{(\nu+1)}(a)}{\nu!(k-1-\nu)^{k-1-\nu}} + \frac{\varphi^{(k)}(a)\log\epsilon}{(k-1)!}\right\}$$

Since, $\lim_{\epsilon\to 0^+}\frac{\varphi(a+\epsilon)}{\epsilon^k} = \lim_{\epsilon\to 0^+}\left[\sum_{\nu=0}^{k-1}\frac{\varphi^{(\nu)}(a)}{\nu!\,\epsilon^{k-\nu}} + \frac{\varphi^{(k)}(a)}{k!}\right]$, the above can be written as :

$$\left\langle \frac{d}{dx} P_f \frac{1_+(x-a)1_+(b-x)}{(x-a)^k}, \varphi(x) \right\rangle = \frac{\varphi^{(k)}(a)}{k!} - \frac{\varphi(b)}{(b-a)^k} - \lim_{\epsilon\to 0^+} \left\{ -k \sum_{\nu=0}^{k-1} \frac{\varphi^{(\nu)}(a)}{\nu!\,(k-\nu)\epsilon^{k-\nu}} \right.$$

$$\left. + k \int_{a+\epsilon}^{b} \frac{\varphi(x)dx}{(x-a)^{k+1}} + \frac{\varphi^{(k)}(a)\log\epsilon}{(k-1)!} \right\}$$

$$= \frac{\varphi^{(k)}(a)}{k!} - \frac{\varphi(b)}{(b-a)^k} - k\left\langle P_f \frac{1_+(x-a)1_+(b-x)}{(x-a)^{k+1}}, \varphi(x) \right\rangle,$$

Using $\{A1\text{-}1\}$. $\{A1\text{-}2\}$

This can be expressed as:

$$\mathfrak{M}\left\{ \frac{d}{dx} P_f \frac{1_+(x-a)1_+(b-x)}{(x-a)^k} \right\} = (-1)^k \frac{(1-s)_k\, a^{s-k-1}}{k!} - \frac{b^{s-1}}{(b-a)^k}$$

$$-k\,\mathfrak{M}\left\{ P_f \frac{1_+(x-a)1_+(b-x)}{(x-a)^{k+1}} \right\} \qquad \{A1\text{-}3\}$$

Appendix 2

<u>Prove:</u> $F(1,a;a+1;-1) = \frac{a}{2}\left[\Psi\left(\frac{1}{2}+\frac{a}{2}\right) - \Psi\left(\frac{a}{2}\right)\right]$

<u>Proof:</u> From [7], page 31, $\Psi(z) = -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right)$ {A2-1}

Now,

$$\Psi\left(\frac{a+1}{2}\right) - \Psi\left(\frac{a}{2}\right) = \frac{2}{a} - \frac{2}{a+1} + \sum_{n=1}^{\infty}\left(\frac{1}{\frac{a}{2}+n} - \frac{1}{\frac{a+1}{2}-n}\right)$$

$$= 2\sum_{n=0}^{\infty}\left(\frac{1}{a+2n} - \frac{1}{a+2n+1}\right) = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{a+n} = \frac{2}{a}F(1,a;a+1;-1)$$

Q.E.D.

Appendix 3

Prove: $$\mathfrak{M}\left\{\frac{d}{dx} 1_+(x-a)\log(x-a)\right\} = \mathfrak{M}\left\{P_f \frac{1_+(x-a)}{(x-a)}\right\}$$

Proof:

$$\left\langle \frac{d}{dx}\ 1_+(x-a)\ \log(x-a), x^{s-1} \right\rangle = \left\langle 1_+(x-a)\log(x-a),\ -\frac{dx^{s-1}}{dx} \right\rangle$$, and now

integrating by parts,

$$= \lim_{\epsilon \to 0^+} \left\{ \log \epsilon\ (a+\epsilon)^{s-1} + \int_{a+\epsilon}^{\infty} \frac{x^{s-1}}{x-a}\, dx \right\} \qquad \text{Re } s < 1$$

$$= \lim_{\epsilon \to 0^+} \left\{ a^{s-1} \log \epsilon + \int_{a+\epsilon}^{\infty} \frac{x^{s-1}}{x-a}\, dx \right\}$$ which is by $\{5\text{-}4\text{-}2\}$,

$$= \mathfrak{M}\left\{P_f \frac{1_+(x-a)}{x-a}\right\}$$

Q.E.D.

Appendix 4

Prove: $$\left\langle \frac{1}{2}\frac{d}{dx}\, 1_{+}(x-a)\log^2(x-a),\ \chi^{s-1} \right\rangle = \left\langle P_f\, \frac{1_{}(x-a)\log(x-a)}{x-a},\ \chi^{s-1} \right\rangle$$

Proof:
First determine Hadamard's Finite Part of $\displaystyle\int_a^\infty \frac{\log(x-a)}{(x-a)}\chi^{s-1}\,dx$

$$\int_a^\infty \frac{\log(x-a)\chi^{s-1}}{x-a}dx = \lim_{\epsilon\to 0^+}\left\{\int_{a+\epsilon}^{a+1} \frac{\log(x-a)\chi^{s-1}}{x-a}dx + \int_{a+1}^{\infty}\frac{\log(x-a)\chi^{s-1}}{x-a}dx\right\} \qquad \{A4\text{-}1\}$$

Let $\chi^{s-1} = a^{s-1} + (x-a)\,\Psi(x)$.

Then

$$\lim_{\epsilon\to 0^+}\int_{a+\epsilon}^{a+1}\frac{\log(x-a)\chi^{s-1}}{x-a}dx = \lim_{\epsilon\to 0^+}\left\{a^{s-1}\int_{a+\epsilon}^{a+1}\frac{\log(x-a)dx}{(x-a)} + \int_{a+\epsilon}^{a+1}\Psi(x)\log(x-a)\chi^{s-1}dx\right. \qquad \{A4\text{-}2\}$$

Integrating by parts,

$$\lim_{\epsilon\to 0^+}\int_{a+\epsilon}^{\infty}\frac{\log(x-a)}{x-a}dx = \lim_{\epsilon\to 0^+}\left\{-\log^2\epsilon - \int_{a+\epsilon}^{\infty}\frac{\log(x-a)}{x-a}dx\right\}$$

Thus, $$\int_{a+\epsilon}^{a+1}\frac{\log(x-a)}{(x-a)}dx = \frac{-1}{2}\log^2\epsilon\,;$$

and then, discarding $\dfrac{a^{s-1}}{2}\log^2\epsilon$ from $\{A4\text{-}2\}$

$$F_p\int_a^{a+1}\frac{\log(x-a)\chi^{s-1}}{x-a}dx = \lim_{\epsilon\to 0^+}\left\{\int_{a+\epsilon}^{a+1}\chi^{s-1}\frac{\log(x-a)}{x-a}dx + \frac{a^{s-1}}{2}\log^2\epsilon\right\}$$

From A4-1, it then follows:

$$F_p\int_a^\infty\frac{\log(x-a)}{x-a}\chi^{s-1}dx = \lim_{\epsilon\to 0^+}\left\{\int_{a+\epsilon}^{\infty}\frac{\chi^{s-1}\log(x-a)}{x-a}dx + \frac{a^{s-1}}{2}\log^2\epsilon\right\} \qquad \{A4\text{-}3\}$$

Now, $\left\langle \frac{d}{dx} 1_+(x-a)\log^2(x-a), x^{s-1} \right\rangle = \left\langle 1_+(x-a)\log^2(x-a), -(s-1)x^{s-2} \right\rangle$

and integrating by parts:

$$\left\langle \frac{d}{dx} 1_+(x-a)\log^2(x-a), x^{s-1} \right\rangle = \lim_{\epsilon \to 0^+} \left\{ -x^{s-1}\log^2(x-a) \Big|_{a+\epsilon}^{\infty} + 2\int_{a+\epsilon}^{\infty} \frac{\log(x-a)}{(x-a)} x^{s-1}\, dx \right\}$$

Now $\lim\limits_{x \to \infty} x^{s-1}\log^2(x-a) = 0 \qquad \operatorname{Re} s < 1$

Also $(a+\epsilon)^{s-1} = a^{s-1} + O(\epsilon)$, Thus $(a+\epsilon)^{s-1}\log^2\epsilon = a^{s-1}\log^2\epsilon$ as $\epsilon \to 0^+$.

This now gives:

$$\frac{1}{2}\left\langle \frac{d}{dx} 1_+(x-a)\log^2(x-a), x^{s-1} \right\rangle = \lim_{\epsilon \to 0^+} \left\{ \frac{1a^{s-1}\log^2\epsilon}{2} + \int_{a+\epsilon}^{\infty} \frac{\log(x-a)}{x-a} x^{s-1}\, dx \right\}$$

$$= \left\langle P_f \frac{1(x-a)\log(x-a)}{x-a}, x^{s-1} \right\rangle \quad \text{using } \{A4\text{-}3\}.$$

Appendix 5

<u>Prove:</u>

$$\frac{d}{dx}\left\{P_f\, 1_+(x-a)(x-a)^{\beta-k}e^{-\alpha(x-a)}\right\} = (\beta-k)P_f\, 1_+(x-a)(x-a)^{\beta-k-1}e^{-\alpha(x-a)}$$

$$-\alpha P_f\, 1_+(x-a)(x-a)^{\beta-k}e^{-\alpha(x-a)}$$

$$k=0,1,2,\ldots \qquad -1 < \operatorname{Re}\beta < 0 \qquad \operatorname{Re}\alpha < 0 \qquad \{A5\text{-}1\}$$

<u>Proof:</u>

Expanding in a series and differentiating term by term would give:

$$\frac{d}{dx}\left\{P_f\, 1_+(x-a)(x-a)^{\beta-k}e^{-\alpha(x-a)}\right\} = \sum_{j=0}^{k}\frac{(-\alpha)^j}{j!}\frac{d}{dx}P_f\, 1_+(x-a)(x-a)^{\beta-k+j}$$

$$+\sum_{j=k+1}^{\infty}\frac{(-\alpha)^j}{j!}\frac{d}{dx}1_+(x-a)(x-a)^{\beta-k+j} \qquad \operatorname{Re}\alpha > 0$$

Using $\{5\text{-}2\text{-}2\}$ gives:

$$\frac{d}{dx}\left\{P_f\, 1_+(x-a)(x-a)^{\beta-k}e^{-\alpha(x-a)}\right\} = \sum_{j=0}^{k}\frac{(-\alpha)^j}{j!}(\beta-k+j)P_f\, 1_+(x-a)(x-a)^{\beta-k+j-1}$$

$$+\sum_{j=k+1}^{\infty}\frac{(-\alpha)^j}{j!}(\beta-k+j)1_+(x-a)(x-a)^{\beta-k+j-1}$$

$$= (\beta-k)\sum_{j=0}^{\infty}P_f\frac{(-\alpha)^j}{j!}1_+(x-a)(x-a)^{\beta-k+j-1} + \sum_{j=0}^{\infty}P_f\frac{(-\alpha)^j}{j!}j\,1_+(x-a)(x-a)^{\beta-k+j-1}$$

$$= (\beta-k)P_f\, 1_+(x-a)(x-a)^{\beta-k-1}\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}(x-a)^j - \alpha P_f\, 1_+(x-a)(x-a)^{\beta-k}$$

$$\sum_{j=1}^{\infty}\frac{(-\alpha)^{j-1}}{(j-1)!}(x-a)^{j-1}$$

$$= (\beta-k)P_f\, 1_+(x-a)(x-a)^{\beta-k-1}e^{-\alpha(x-a)} - \alpha P_f\, 1_+(x-a)(x-a)^{\beta-k}e^{-\alpha(x-a)}$$

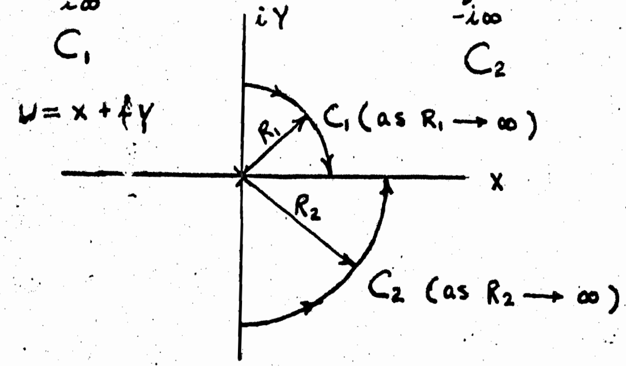
Q.E.D.

Appendix 6

Prove:

$$\underset{C_1}{\int_{i\infty}^{\infty}} u^{s-1}e^{-u}\,du = 0 \quad \text{and} \quad \underset{C_2}{\int_{-i\infty}^{\infty}} u^{s-1}e^{-u}\,du = 0 \qquad \{A6\text{-}1\}$$

for Re s < 0



Proof:

$$|u| = R_1 \text{ or } R_2 \qquad |e^{-u}| \leq 1 \text{ for right half plane}$$

Now,

$$\left|\int_{C_1} u^{s-1}e^{-u}\,du\right| \leq \int_{C_1} |u|^{s-1}\,|e^{-u}|\;|du| \leq \frac{R_1^{s-1}\,\pi R_1}{2} = \frac{\pi R_1^{s}}{2}$$

Therefore

$$\int_{i\infty}^{\infty} u^{s-1}e^{-u}\,du = \lim_{R_1 \to \infty} \int_{C} u^{s-1}e^{-u}\,du = 0 \qquad \text{for Re } s < 0$$

Similarly

$$\int_{-i\infty}^{\infty} u^{s-1}e^{-}\,du = \lim_{R_2 \to \infty} \int_{C_2} u^{s-1}e^{-u}\,du = 0 \qquad \text{for Re } s < 0$$

Q.E.D.

Since the only singularity in the integrand of the results in {A6-1} is at the orig the Cauchy-Goursat theorem ([11] , page 111 , section 49) shows the results are true for all contours in the right half plane.

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A list of distributional Mellin transforms is derived. Singular distributions that are considered include delta functionals and pseudo-functions obtained from Hadamard's Finite Part. The ordinary distributions include algebraic and elementary transcendental functions. Proofs or derivations are provided for all results.

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