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4.4. ''Analysis of n-dimensional nonlinear nonuniform grounded infinite grids,''

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Examples of Multidimensional Optimal Interpolative Functional Artificial Neural Networks

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Abstract – This paper presents examples of a class of optimal interpolative (OI) functional artificial neural networks (FANNs) which process continuous multidimensional signals. These networks embody for the present case the structure of OI networks, previously derived in the literature, which best approximate a nonlinear dynamical system in a Generalized Fock Space (GFS) under input-output training data constraints. Among other applications, these networks are useful in the modeling and identification of the degradation process of image signals occuring while propagating in nonlinear media.

1 Introduction

This paper presents examples of a class of optimal interpolative (OI) functional artificial neural networks (FANNs) which process continuous multidimensional signals. As shown in the generic case [1], to which we briefly allude below, the structure of these networks results from the best approximation of their input-output map in a Generalized Fock Space under data constraints. Sometimes, in the system theory literature, this type of best approximation has been called system identification [2][3].

FANNs are continuous-time and/or continuous-space versions of conventional artificial neural networks (heretofore referred to as ANNs). Conceptually at least, most results obtained from ANNs easily generalize to FANNs except for some phenomena, such as limit cycles, which genuinely depend on the continuous nature of the system.

A unified approach for the implementation of both FANNs and ANNs, based on a Generalized Fock Space (GFS) framework was presented by de Figueiredo and Dwyer in 1980 [4]. In this framework, the input u to the network is assumed to belong to a real abstract Hilbert space H, and the network's input-output map V is represented as an abstract Volterra series in elements of H, belonging to a Generalized Fock Space $F_s(H)$ over H weighted by a sequence s. The space $F_s(H)$ is a reproducing Kernel Hilbert space with a reproducing kernel $K(u,v)$ (see [1] for details). In the framework just mentioned, the implementation of a neural network map V is specified by a set of interpolative constraints $V(u_i) = y_i$, where (u_i, y_i), $i = 1, \ldots, m$ constitute the training data. This implementation is obtained by projecting V into the span of the representers of the point evaluation functionals $K(u_i, .)$ in $F_s(H)$ corresponding to the training points (in H) $u_1, \ldots, u_m$. For obvious reasons, the implementation V has been called an optimal interpolative (OI) neural network and can be explicitly written in the form

$$V(.) = \sum_{j=1}^{m} c_j K(u_j, .) \qquad (1)$$

where the coefficients c_j are obtained by requiring that (1) satisfy the interpolating training data constraints.

In the case where H is an Euclidean space E^n, the OI net realization V takes the form of a conventional feedforward ANN with two hidden layers. This OI net was presented in 1990 [5] and its theory and applications have been widely discussed in the literature [6]. In the case where H is $L^2(I)$, $I \subset R^1$, the OI net is a FANN which was analyzed by Zyla and de Figueiredo [7] and reconsidered recently by Newcomb and de Figueiredo [1][2]. In the present paper we consider the class of OI FANNs for which H is $L^2(I^n)$, $I^n \subset R^n$.

In what follows we first briefly recall the derivation of the explicit expression for the OI FANN obtained in [1][3]. Then we illustrate this result by some examples from multidimensional signal processing.

2 Multivariable OI FANNs

We assume that m pairs of representative input-output test functions (equivalent to exemplars in artificial neural networks), $u_j(.)$ and $y_j(.)$ for $j = 1, \ldots, m$, are available, with these functions, along with their K derivatives, being square integrable over I^n. We solve for an optimum operator V having the smallest norm in the following sense:

$$\min \|V_x^{(i)}\|_{F_r} \quad \forall x \in I^n \quad and \quad \forall \quad V_x^{(i)} \in F_r \qquad (2)$$

subject to the data constraints

$$V_x^{(i)}(u_j(.)) = y_j^{(i)}(x) \quad i = 0, \ldots, K \quad j = 1, \ldots, m \qquad (3)$$

We note that in order to have sufficient information to perform an identification, we select the m input functions

to be linearly independent over I^n. Following [7] and as developed over I^n in [8], the solution to this equivalent problem is outlined below:

1. Form the $m \times m$ Grammian matrix

$$G = [G_{ij}] = \left[\exp\left[\frac{1}{r}\langle u_i(.), u_j(.)\rangle_{L^2(I^n)}\right]\right] \quad (4)$$

where, for completeness, we recall that

$$\langle u_i(.), u_j(.)\rangle_{L^2(I^n)} = \int_{x \in I^n} u_i(x) u_j(x) dx \quad (5)$$

Note that G is nonsingular, since the test input functions are linearly independent.

2. Form the column m-vector of test outputs

$$y_{test}(.) = [y_j(.)] \quad (6)$$

3. Obtain a column m-vector of coefficients

$$c(x) = [c_j(x)] = G^{-1} y_{test}(x) \quad (7)$$

4. Determine the optimum estimate $\hat{V}_x(.)$ of $V_x(.)$

$$\hat{V}_x(.) = \sum_{j=1}^{m} c_j(x) \exp\left[\frac{1}{r}\langle u_j(.), .\rangle_{L^2(I^n)}\right] \quad (8)$$

which is the key equation [4] upon which we base our functional artificial neural network discussed next.

The schematic of the resulting functional neural network is depicted in Figure 1, showing a feed-forward two-layer architecture. The first layer consists of m input neurons, each one processing the same input function $u(.)$, presented to the network, and producing a nonlinear response of exponential form, i.e., $\exp[\langle u_i, u\rangle / r]$. The neural network design is carried out in a supervised manner, i.e., using m representative exemplar pairs $u_j(.)$ $y_j(.)$. The entries of $c(x) = G^{-1} y_{test}(x)$, formed with these pairs, correspond to x-varying synaptic weights, whereas the entries of the Grammian matrix G act in a linear manner as neuron nonlinearities, with the weighted neuron outputs added to give the overall output $y(x)$.

When presented with an arbitrary input (of the class allowed by the system), this neural network produces an output that is an approximation to the output of the dynamical system the neural network is modeling. The network uses information acquired during its training on the exemplars to give the desired output in terms of functionals. As a consequence, the network attempts to incorporate with a best fit the nonlinear dynamics of the system being modeled.

3 Examples

Here we illustrate the key theoretical and design ideas with examples from 2D signal processing.

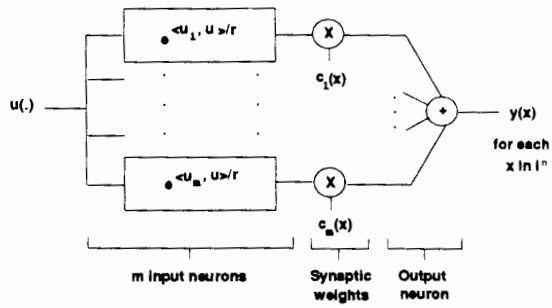

Figure 1: Multidimensional FANN.

3.1 Closed Curve Classification

For this case the exemplar inputs $U_1(x_1, x_2)$ and $U_2(x_1, x_2)$ are surfaces enclosed by closed curves consisting of a circle of radius 1 centered at $(0, 0)$ and a square of side 1, also centered at $(0, 0)$ (Figure 2). As part of the FANN design, a scheme is set up to detect the inside and the outside of the curves, quantify the result as a 1 or a 0 respectively, and assign the appropriate values to U_1 and U_2.

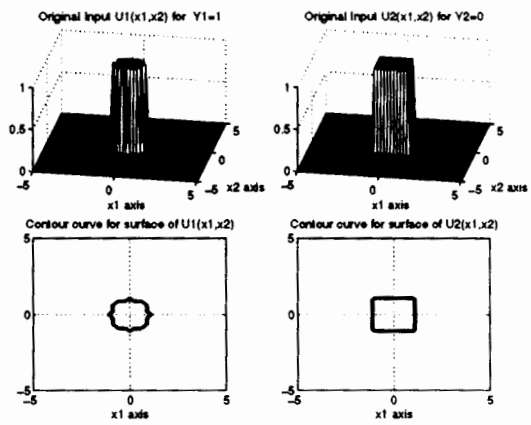

Figure 2: Exemplar inputs $U_1(x_1, x_2)$ and $U_2(x_1, x_2)$

The desired input-output mapping is achieved by forcing the network to associate an output $Y_1(x_1, x_2) = 1$ with the circle and $Y_1(x_1, x_2) = 0$ with the square during the training process, with the aim of enabling the network to generalize the classification to circles and squares of any size, centered anywhere in the plane. The latter is carried out by shifting the center of the curves to $(0, 0)$ and normalizing the dimensions to unity.

The generalization ability of the network is tested on a set of circles and squares which we refer to as large and small (see Figures 3 and 4). The associated FANN output values are, as anticipated, 1 for the circles and 0 for the squares.

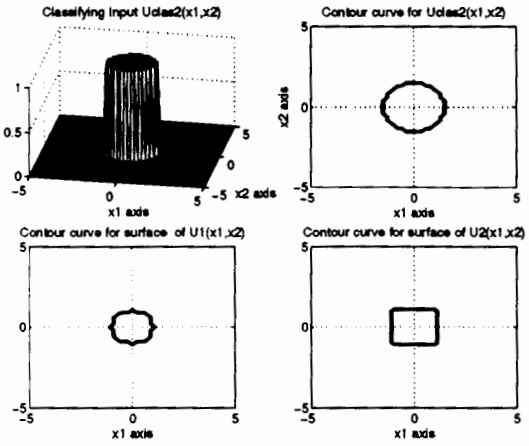

Figure 3: Classification of a large circle

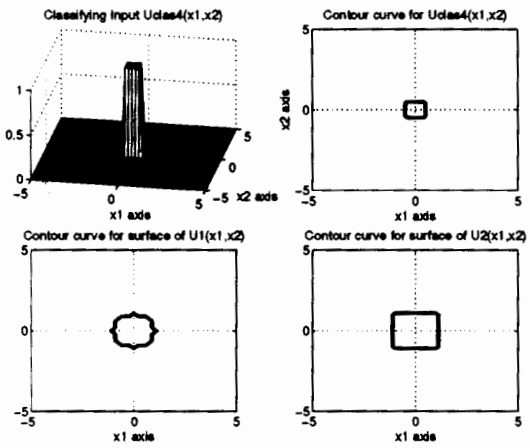

Figure 4: Classification of a small square.

3.2 Pattern Synthesis

Perhaps one of the most powerful and interesting application of the FANN in image processing is pattern synthesis from a database. For example, using either an a priori or a posteriori approach, one can design a FANN for image enhancement and restoration.

As a simple illustration, a FANN is trained to generate a unit circle, centered at $(0,0)$, for an input of 1 over the x_1-x_2 plane and a square of side 1, also centered at $(0,0)$, for an input of 0 over the plane x_1-x_2 (Figures 5 and 6). When the FANN is presented with non-exemplar inputs close to the constants 1 or 0, the FANN produces a circle or a square as expected. However, when presented with a different input, interestingly, as shown in Figures 7, 8, and 9, the FANN combines the circle and the square to create a new pattern.

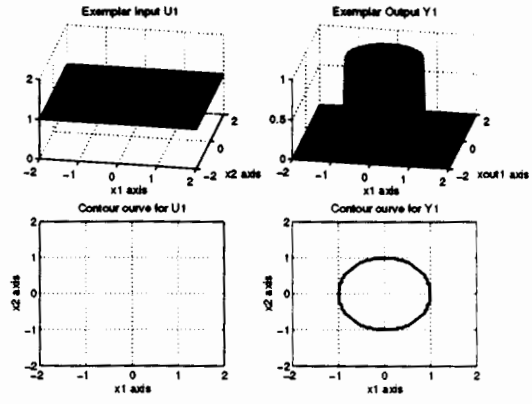

Figure 5: Exemplar input-output pair U_1-Y_1.

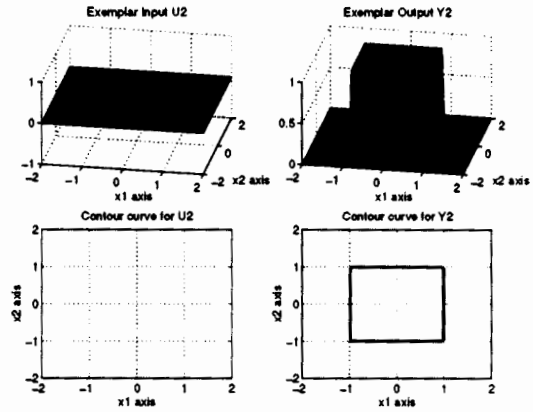

Figure 6: Exemplar input-output pair U_2-Y_2.

4 Discussion

The success of the one-dimensional network introduced in [1, 2] led us to consider the multi-variate case [8]. Thus, in this paper we proposed a neural network approach to the problem of identification of multi-variable nonlinear dynamical systems. The resulting neural network structure, called optimal interpolative multidimensional functional artificial neural network (OI FANN) leads to an optimum characterization of the system via a functional estimation approach.

The proposed approach employs the idea of the reproducing kernel within the mathematical framework of Fock and Hilbert space concepts to approximate nonlinear dynamical systems, specified by representative sets of input-output pairs. In so doing, the approach solves the minimum norm problem in a Bochner space. The use of the reproducing kernel allows the approximation problem to revert back to that of linear systems while still incorporating the nonlinearities for which the Volterra series is tailored. As such it is an attractive alternative to other system modeling techniques [9].

The design of the OI FANN is carried out through a supervised training of the network with exemplar input-output functional pairs and constructs a set of synaptic weights, which are also functionals. When non-exemplar inputs are presented to the network, the latter performs a system identification by associating a Volterra functional input-output map.

The key theoretical and design ideas were exploited in two applications from n-dimensional signal processing, these being closed curve classification and pattern synthesis, the details of which are covered in the full version of the paper.

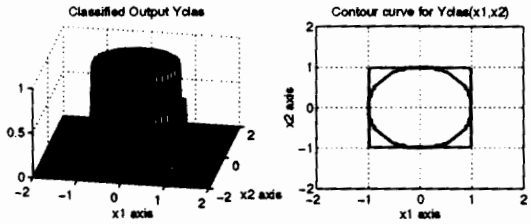

Figure 7: FANN output for $U = 3/4$.

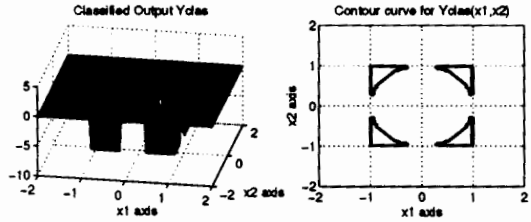

Figure 8: FANN output for $U = 2$

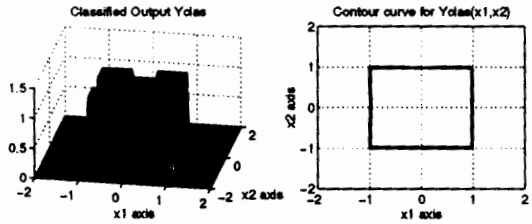

Figure 9: FANN output for $U = -1$.

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MYOPIC MAPS AND UNIFORM APPROXIMATION

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EXTENDED ABSTRACT

Our main result is a theorem which gives necessary and sufficient conditions under which multidimensional myopic input-output maps with vector-valued inputs drawn from a certain large set can be uniformly approximated arbitrarily well using a structure consisting of a linear preprocessing stage followed by a memoryless nonlinear network. Such structures were first considered in an important but very special context by Wiener. We consider causal as well as noncausal maps. Approximations for noncausal maps for which inputs and outputs are functions of more than one variable are of current interest in connection with, for example, image processing. Throughout the paper inputs and outputs are defined on the infinite m-dimensional interval $(-\infty, \infty)^m$ where m is an arbitrary positive integer.

Preliminaries

Throughout this abstract, $\mathbb{R}$ is the set of reals and $\mathbb{N}$ is the set of positive integers. Let n and m in $\mathbb{N}$ be arbitrary. $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ are the Euclidean norm and inner product on $\mathbb{R}^n$, respectively, and $|\cdot|$ is the Euclidean norm on $\mathbb{R}^m$. With $\mathbb{R}_- = (-\infty, 0]$, $\mathbb{R}_-^m$ denotes $(\mathbb{R}_-)^m$.

For any positive integer n_0, let $C(\mathbb{R}^{n_0}, \mathbb{R})$ denote the set of continuous maps from $\mathbb{R}^{n_0}$ to $\mathbb{R}$, and let D_{n_0} stand for any subset of $C(\mathbb{R}^{n_0}, \mathbb{R})$ that is dense on compact sets, in the usual sense that given $\epsilon > 0$ and $f \in C(\mathbb{R}^{n_0}, \mathbb{R})$, as well as a compact $V \subset \mathbb{R}^{n_0}$, there is a $q \in D_{n_0}$ such that $|f(v) - q(v)| < \epsilon$ for $v \in V$. The D_{n_0} can be chosen in many different ways, and may involve, for example, radial basis functions, polynomial functions, piecewise linear functions, sigmoids, or combinations of these functions.

Let w be a continuous $\mathbb{R}$-valued function defined on $\mathbb{R}^m$ such that $w(\alpha) \neq 0$ for all α and $\lim_{|\alpha|\to\infty} w(\alpha) = 0$. With $C(\mathbb{R}^m, \mathbb{R}^n)$ the set of continuous maps from $\mathbb{R}^m$ to $\mathbb{R}^n$, denote by X_w the normed linear space given by

$$X_w = \{x \in C(\mathbb{R}^m, \mathbb{R}^n) : \sup_{\alpha \in \mathbb{R}^m} \|w(\alpha)x(\alpha)\| < \infty\}$$

with the norm

$$\|x\|_w = \sup_{\alpha \in \mathbb{R}^m} \|w(\alpha)x(\alpha)\|.$$

Now let $\mathcal{C}(\mathbb{R}^m, \mathbb{R}^n)$ be the set of all bounded functions contained in $C(\mathbb{R}^m, \mathbb{R}^n)$, and let S be a nonempty subset of $\mathcal{C}(\mathbb{R}^m, \mathbb{R}^n)$. For each $\beta \in \mathbb{R}^m$, define T_β on S by

$$(T_\beta x)(\alpha) = x(\alpha - \beta), \quad \alpha \in \mathbb{R}^m.$$

The set S is said to be *closed under translation* if $T_\beta S = S$ for each $\beta \in \mathbb{R}^m$. Let G map S to the set of $\mathbb{R}$-valued functions on $\mathbb{R}^m$. Such a G is *shift-invariant* if S is closed under translation and

$$(Gx)(\alpha - \beta) = (GT_\beta x)(\alpha), \quad \alpha \in \mathbb{R}^m$$

for each $\beta \in \mathbb{R}^m$ and $x \in S$. The map G is *causal* if

$$x(\alpha) = y(\alpha) \text{ with } \alpha_j \leq \beta_j \ \forall j \quad \Longrightarrow \quad (Gx)(\beta) = (Gy)(\beta)$$

for each $\beta \in \mathbb{R}^m$ and every x and y in S.

We assume that G is shift-invariant. We say that G is *myopic* on S with respect to w if given an $\epsilon > 0$ there is a $\delta > 0$ with the property that x and y in S and

$$\sup_{\alpha \in \mathbb{R}^m} \|w(\alpha)[x(\alpha) - y(\alpha)]\| < \delta \quad \Longrightarrow \quad |(Gx)(0) - (Gy)(0)| < \epsilon. \tag{1}$$

Thus, and roughly speaking, G is myopic if the value of $(Gx)(\alpha)$ is always relatively independent of the values of x at points remote from α.

In our theorem to be presented we refer to certain sets $\mathcal{G}(w)$ and $\mathcal{G}_-(w)$. These sets concern integrals of the form

$$\int_D \langle g(\beta), x(\beta) \rangle \, d\beta \tag{2}$$

in which $x \in \mathcal{C}(\mathbb{R}^m, \mathbb{R}^n)$ and $D = \mathbb{R}^m$ or $\mathbb{R}^m_-$. Such integrals are well defined and finite for any (Lebesgue) measurable g from D to $\mathbb{R}^n$ such that

$$\int_D \left\| w(\beta)^{-1} g(\beta) \right\| d\beta < \infty \tag{3}$$

(because (3) implies the integrability of g). Let $D = \mathbb{R}^m$. By $\mathcal{G}(w)$ we mean any set of measurable functions g from D to $\mathbb{R}^n$ such that (3) is met for each g, and for each nonzero $x \in \mathcal{C}(\mathbb{R}^m, \mathbb{R}^n)$ there corresponds a g for which (2) is nonzero. Similarly, with $D = \mathbb{R}^m_-$, the set $\mathcal{G}_-(w)$ is any set of measurable functions g from D to $\mathbb{R}^n$ such that (3) is met for each g, and for each $x \in \mathcal{C}(\mathbb{R}^m, \mathbb{R}^n)$ whose restriction to D is nonzero there is a g for which (2) is nonzero. The sets $\mathcal{G}(w)$ and $\mathcal{G}_-(w)$ can be chosen in many ways. For example, we show that $\mathcal{G}(w)$ can be taken to be the set

of all continuous functions from $\mathbb{R}^m$ to $\mathbb{R}^n$ such that (3) is met with $D = \mathbb{R}^m$. A similar proposition holds for $\mathcal{G}_-(w)$.

Finally, let Q be the map from S to $C(\mathbb{R}^m, \mathbb{R}^n)$ defined by $(Qs)(\alpha) = s(\beta)$ for each s and α, where $\beta_j = -|\alpha_j|$ for all j.

Our Main Result

Our main result below gives a necessary and sufficient condition for the uniform approximation of myopic maps with vector-valued inputs of a finite number of variables. In stating this result, we use the fact that integrals of the form

$$\int_{\mathbb{R}^m} \langle p(\beta - \alpha), x(\beta) \rangle \, d\beta$$

and

$$\int_{(-\infty,\alpha]} \langle q(\beta - \alpha), x(\beta) \rangle \, d\beta$$

are well defined and finite for each $\alpha \in \mathbb{R}^m$ when x is an element of $C(\mathbb{R}^m, \mathbb{R}^n)$, $p \in \mathcal{G}(w)$, and $q \in \mathcal{G}_-(w)$. This follows from the observation that by (3) both p and q are integrable on their respective domains.

Theorem 1: Assume that S is uniformly bounded and equicontinuous.[1] (Recall that G is a shift-invariant map from S to the set of $\mathbb{R}$-valued functions defined on $\mathbb{R}^m$.) Then the following two statements are equivalent.

(i) G is myopic on S with respect to w.

(ii) For each $\epsilon > 0$, there are an $n_0 \in \mathbb{N}$, elements $g_1, \ldots, g_{n_0}$ of $\mathcal{G}(w)$, and an $N \in D_{n_0}$ such that

$$|(Gx)(\alpha) - N[(Lx)(\alpha)]| < \epsilon, \quad \alpha \in \mathbb{R}^m \tag{4}$$

for all $x \in S$, where L is given by

$$(Lx)_j(\alpha) = \int_{\mathbb{R}^m} \langle h_j(\alpha - \beta), x(\beta) \rangle \, d\beta \tag{5}$$

with $h_j(\beta) = g_j(-\beta)$ for all β and j.

Moreover, if G is causal and $QS \subseteq S$ then (ii) can be replaced with:

(ii′) For each $\epsilon > 0$, there are an $n_0 \in \mathbb{N}$, elements $g_1, \ldots, g_{n_0}$ of $\mathcal{G}_-(w)$, and an $N \in D_{n_0}$ such that $|(Gx)(\alpha) - N[(Lx)(\alpha)]| < \epsilon$ for all $\alpha \in \mathbb{R}^m$ and $x \in S$, where L is given by

$$(Lx)_j(\alpha) = \int_{(-\infty,\alpha]} \langle h_j(\alpha - \beta), x(\beta) \rangle \, d\beta \tag{6}$$

[1] S is *uniformly bounded* if there is a positive constant c for which $\|x(\alpha)\| \leq c$ for all $x \in S$ and all α, and S is *equicontinuous* if for each $\epsilon > 0$ there is a $\delta > 0$ such that $\|x(\alpha_a) - x(\alpha_b)\| < \epsilon$ whenever $x \in S$ and $|\alpha_a - \alpha_b| < \delta$.

with $h_j(\beta) = g_j(-\beta)$ for all β and j, and where $(-\infty, \alpha]$ means $(-\infty, \alpha_1] \times \cdots \times (-\infty, \alpha_m]$.

The full version of the paper includes extensive introductory material, a section on comments, and one on the specialization of our result to generalized finite Volterra series approximations.[2]

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[2]For recent related material concerning the approximation of system maps that are not shift invariant, see [8].

Does it Really Make Sense to Use Differentiable Manifolds in the Theory of Electrical Networks?

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In this paper we consider the nontrivial question of the usefulness of manifolds and especially differentiable manifolds in the theory of electrical networks. Of course, it is obvious that many concrete mathematical objects, e.g. the set of zeros of linear and nonlinear equations, a set of functions and operators and so on, include such mathematical structures but it is not clear at all that we have to extract these structures in order to clarify a certain systematic question in network theory. In other words, there is a danger that a more abstract description presents no deeper insight in a problem or emphasizes a not suitable mathematical structure. There are several examples of this kind in the history of electrical engineering science. A typical example is the so-called phasor method where complex voltages and currents are introduced in order to solve the linear time-invariant network equations with sinusoidal inputs quantities. The complexification which is common used introduced an additional mathematical structure into this problem that have to be extracted after the process of calculation be means of building the real (or imaginary) part of the final result. It can be shown that using complex voltages and currents is unnecessary (see Mathis, Marten [3], [4]). It can be shown that not a complexification is the useful way to abstract this problem but we have to use a complex structure in the sense of linear algebra in order to get a structure that is well-adapted to point of view of applications.

We consider some cases of network theory where differentiable manifolds can be helpful to understand or to solve problems in network theory. At first we consider the set of linear electrical problems where the network parameters and the network topology characterize the concrete mathematical object. Now it can be shown that Grassmannian manifolds are useful to study these networks. More detailed considerations can be found in Diepold and Pauli [1]; see also Pauli [8] [9].

Another subject of this paper is the discussion of linear and nonlinear dynamical RLC networks where differentiable manifolds can be used to characterize their descriptive equations and to understand the problems that are arise in circuit simulation. For this reason we separate the typical or generic behavior of these networks from the nongeneric behavior that forces a circuit simulator into problems; see e.g. Mathis [4]. Furthermore a geometric presentation of the theory of networks emphasizes a unified concept of these essential class of

physical systems; see Mathis [5] [6] and Hermann [2]. For example a unified theory of duality of linear and nonlinear networks can be set up Mathis and Marten [7].

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WEIGHTED NORMS AND NETWORK APPROXIMATION OF FUNCTIONALS

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EXTENDED ABSTRACT

One of the earliest results in the area of neural networks is the proposition that any continuous real function defined on a compact subset of $\mathbb{R}^k$ (k an arbitrary positive integer) can be approximated arbitrarily well using a single-hidden-layer network with sigmoidal nonlinearities (see, for example, [1]). Among other results in the literature concerning arbitrarily good approximation that concern more general types of "target" functionals, different network structures, other nonlinearities, and various measures of approximation errors is the proposition in [2], [3] that any continuous real nonlinear functional on a compact subset of a real normed linear space can be approximated arbitrarily well using a single-hidden-layer neural network with a linear functional input layer and exponential (or polynomial or sigmoidal or radial basis function) nonlinearities. This has applications concerning, for instance, the theory of classification of signals (see [3]).

In interesting papers [4], [5] by Chen and Chen related results are given concerning the approximation of nonlinear functionals. In [5] they show, for example, that for the case in which the nonlinear functional's domain is a compact subset of a Banach space with a basis, the linear functionals can be taken to be the coefficient maps associated with the basis (see their Theorem 3). Here we observe that this type of result follows from either [3, Theorem 1], or the variant of part of [3, Theorem 1] proved in the appendix[1], and the observation we state in the form of a proposition in the appendix.

We also give a tool theorem in the appendix that is useful in focusing attention on the range of applications of results concerning the approximation of nonlinear functionals. This theorem shows that certain sets of functions defined on unbounded domains are relatively compact in spaces with weighted norms. The theorem is useful because while the usual criteria for compactness concern signals defined on a compact subset of $\mathbb{R}^n$ (where typically $n = 1, 2$, or 3 in engineering and scientific applications), signals defined on infinite n-dimensional intervals are often of interest. A simple corollary of Theorem 2 is that for each positive α and β, the set of all real-valued continuous functions x defined on $\mathbb{R}^n$ such that

[1]Theorem 1 in the appendix is stated without proof in [6] which establishes a connection between [3, Theorem 1] and one of the main results in [4] involving linear functionals that have the special form of a finite sum of integrals of a certain type.

$\sup_\gamma |x(\gamma)| \leq \alpha$, and such that x satisfies a Lipschitz condition with Lipshitz constant β, is compact with respect to the metric $\rho(x,y) = \sup_\gamma |w(\gamma)[x(\gamma) - y(\gamma)]|$, where w is any continuous positive function on $\mathbb{R}^n$ such that $w(\gamma) \to 0$ as $||\gamma|| \to 0$.[2] This establishes, for example, a setting in which it is possible to use certain simple network structures (see [3]) to classify patterns represented by real-valued functions defined on $\mathbb{R}^n$. Further details and other types of applications of Theorem 2 are described in [8] and [9].

I. APPENDIX

A.1 Approximation Theorem

Let C be a nonempty compact subset of a real normed linear space X, and let X^* be the set of bounded linear functionals on X (i.e., the set of bounded linear maps from X to the reals $\mathbb{R}$). For each $\rho > 0$, let Y_ρ be any set of maps from X to $\mathbb{R}$ that is ρ-dense on C in the closed unit ball of X^* in the sense that given $\phi \in X^*$ with $||\phi|| \leq 1$ there is a $y \in Y_\rho$ such that $|\phi(x) - y(x)| < \rho$, $x \in C$.

Let U be any set of maps $u : \mathbb{R} \to \mathbb{R}$ such that given $\alpha \geq 1$ and $\sigma > 0$ and any bounded interval $(\beta_1, \beta_2) \subset \mathbb{R}$ there exists a finite number of elements $u_1, \ldots, u_\ell$ of U for which $|\exp(\alpha\beta) - \sum_j u_j(\beta)| < \sigma$ for $\beta \in (\beta_1, \beta_2)$.[3]

Theorem 1. Let f be a continuous map of C into $\mathbb{R}$. Then given $\epsilon > 0$ there are a positive integer k, real numbers $c_1, \ldots, c_k$, elements $u_1, \ldots, u_k$ of U, a positive number ρ and elements $y_1, \ldots, y_k$ of Y_ρ such that $|f(x) - \sum_j c_j u_j[y_j(x)]| < \epsilon$ for $x \in C$.

Proof

Let f be given, and notice that the set V of all functions $v : C \to \mathbb{R}$ of the form $v(x) = \sum_j a_j \exp[\phi_j(x)]$, in which the sum is finite and the a_j and the ϕ_j belong to $\mathbb{R}$ and X^*, respectively, is an algebra under the natural definition of addition and multiplication. By a consequence [10, p.198] of the Hahn-Banach theorem, given distinct x_a and x_b in C there is a ϕ in X^* such that $\exp[\phi(x_a)] \neq \exp[\phi(x_b)]$, showing that V separates the points of C. It is clear that $v(x) \neq 0$ for some $v \in V$ for each x. Thus, by a version of the Stone-Weierstrass Theorem [11, p.162], given $\epsilon > 0$ there are a positive integer p, real numbers $d_1, \ldots, d_p$, and elements $w_1, \ldots, w_p$ of X^* such that

$$|f(x) - \sum_j d_j \exp[w_j(x)]| < \varepsilon/3$$

for $x \in C$.[4] Select $\alpha \geq 1$ so that each $z_j := w_j/\alpha$ has norm at most unity.

We may assume that $\sum_j |d_j| \neq 0$. Choose $\gamma > 0$ such that $\gamma \sum_j |d_j| < \varepsilon/3$. Let $[a', b']$ be an interval in $\mathbb{R}$ that contains all of the sets $w_j(C)$, and let real a and b be such that $a < a', b > b'$. Select $\eta > 0$ such that $|\exp(\beta_1) - \exp(\beta_2)| < \gamma$ for $\beta_1, \beta_2 \in [a, b]$ with $|\beta_1 - \beta_2| < \eta$. With $\rho = \alpha^{-1} \min(\eta, a' - a, b - b')$, choose $y_j \in Y_\rho$ such that $|z_j(x) - y_j(x)| < \rho$, $x \in C$ for all j. This gives $|\exp[\alpha z_j(x)] - \exp[\alpha y_j(x)]| < \gamma$, $x \in C$ for each j (because

[2] A result similar to this for $n = 1$ is proved in [7].

[3] Of course we can take U to be the set $\{\exp(\alpha \cdot), \alpha \geq 1\}$, or the set $\{u : u(\beta) = (\alpha\beta)^n/n!, \alpha \geq 1, n \in \{0, 1, \ldots\}\}$. Another acceptable choice is $\{u : u(\beta) = cs(w\beta + \rho), c, w, \rho \in \mathbb{R}\}$, where s is a continuous function with $\lim_{\beta \to \infty} s(\beta) = 1$ and $\lim_{\beta \to -\infty} s(\beta) = 0$.

[4] Here we view C as a metric space with the metric derived in the usual way from the norm in X.

we have $\alpha y_j(C) \in [a,b]$ and $|\alpha z_j(x) - \alpha y_j(x)| < \eta$ for each j and x), and thus

$$|f(x) - \sum_j d_j \exp[\alpha y_j(x)]| \le |f(x) - \sum_j d_j \exp[\alpha z_j(x)]| + |\sum_j d_j \exp[\alpha z_j(x)] - \sum_j d_j \exp[\alpha y_j(x)]|$$

$$\le \varepsilon/3 + \sum_j |d_j| \cdot |\exp[\alpha z_j(x)] - \exp[\alpha y_j(x)]| \le (2\varepsilon)/3, \ x \in C.$$

Now let $[c,d] \subset \mathbb{R}$ be such that $\alpha y_j(C) \subset [c,d]$ for each j. Pick $u_1, \ldots, u_\ell \in U$ so that $|\exp(\alpha\beta) - \sum_i u_i(\beta)| \le \gamma_1, \ \beta \in [c,d]$ where $\gamma_1 \sum_j |d_j| < \varepsilon/3$. Then

$$|f(x) - \sum_j \sum_i d_j u_i[y_j(x)]| \le |f(x) - \sum_j d_j \exp[\alpha y_j(x)]| + |\sum_j d_j \exp[\alpha y_j(x)]$$

$$- \sum_j \sum_i d_j u_i[y_j(x)]| \le (2\varepsilon)/3 + \sum_j |d_j \exp[\alpha y_j(x)] - d_j \sum_i u_i[y_j(x)]|$$

$$\le (2\epsilon)/3 + \sum_j |d_j| \cdot |\exp[\alpha y_j(x)] - \sum_i u_i[y_j(x)]| \le (2\varepsilon)/3 + \gamma_1 \sum_j |d_j| < \varepsilon.$$

Since $\sum_j \sum_i d_j u_i[y_j(x)]$ can be written in the form $\sum_j c_j u_j[y_j(x)]$, with the c_j, u_j,and y_j in $\mathbb{R}, U$, and Y_ρ, respectively, we have proved the theorem.[5]

A.2 Approximation of Linear Functionals on Banach Spaces With a Basis

In the proposition below we consider the case in which X and C are as described earlier, and X is an infinite-dimensional Banach space with a (Schauder) basis $e_1, e_2, \ldots$. We use $g_j, j = 1, 2, \ldots$ to denote the functionals with the property that $x = \sum_{j=1}^{\infty} g_j(x) e_j$ for $x \in X$.

Proposition. Given $\rho > 0$ there is a positive integer ℓ such that $|\phi(x) - \sum_{j=1}^{\ell} \phi(e_j) g_j(x)| < \rho, \ \ x \in C$ for all $\phi \in X^*$ with $||\phi|| \le 1$.

Proof

The proposition follows directly from the fact that given ρ there is an ℓ such that $||\sum_{j=\ell+1}^{\infty} g_j(x) e_j|| < \rho, \ \ x \in C$ (see [12, p. 136]).

A.3 Weighted-Space Tool

Theorem 2. Let S be a subset of a complete metric space A with metric ρ, and let $T_1, T_2, \ldots$ be maps of A into itself such that

(*i*) $T_k(S)$ is a relatively compact subset of A for each k, and

(*ii*) $\rho(s, T_k s) \to 0$ as $k \to \infty$ uniformly for $s \in S$.

Then S is a relatively compact subset of A.

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[5] As mentioned earlier, Theorem 1 is a variant stated in [6] of part of [3, Theorem 1]. That part of [3, Theorem 1], which is provable using a direct modification of the proof above, asserts that Theorem 1 above remains true if "a positive number ρ and elements $y_1, \ldots, y_k$ of Y_ρ" is replaced with "and elements $y_1, \ldots, y_k$ of Y," where Y is any set of continuous maps from X to $\mathbb{R}$ that is dense in X^* on C, in the sense that for each $\phi \in X^*$ and any $\epsilon > 0$ there is a $y \in Y$ such that $|\phi(x) - y(x)| < \epsilon, \ x \in C$, and U is instead any set of continuous maps $u : \mathbb{R} \to \mathbb{R}$ such that given $\sigma > 0$ and any bounded interval $(\beta_1, \beta_2) \subset \mathbb{R}$ there exists a finite number of elements $u_1, \ldots, u_\ell$ of U for which $|\exp(\beta) - \sum_j u_j(\beta)| < \sigma$ for $\beta \in (\beta_1, \beta_2)$.

Proof

Let $\epsilon > 0$ be given. Select k so that $\rho(s, T_k s) < \epsilon/2$ for $s \in S$. Since $T_k(S)$ is relatively compact, A contains an $\epsilon/2-$net $a_1, a_2, \ldots, a_p$ for $T_k(S)$ (see p. 200 of [13])[6]. Now let $u \in S$ be given. Choose $j \in \{1, 2, \ldots, p\}$ so that $\rho(T_k u, a_j) < \epsilon/2$. Thus,

$$\rho(u, a_j) \leq \rho(u, T_k u) + \rho(T_k u, a_j) < \epsilon/2 + \epsilon/2 = \epsilon,$$

showing that $a_1, a_2, \ldots, a_p$ is an ϵ-net for S. Since A is complete and A contains a finite ϵ-net for S for every $\epsilon > 0$, S is relatively compact (by the theorem on p. 201 of [13]).

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[6] In [13], compact means what we call relatively compact.

MONOTONE RESISTIVE NETWORKS

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1. Introduction.

We revisit the work of Minty [6 and 7] who gave among other things an existence result for the branch voltages and currents in a finite network, which on each branch are related by a maximal monotone resistance function, and we extend it to an infinite network.

This contrasts with the papers in which existence is given for all current or voltage sources in some vector space. A feature of Minty's Theorem is the set of bounds for the solution, which is used to give an easier proof than the original one, for a finite network.

We call a function $f : \mathbf{R} \to 2^{\mathbf{R}}$ *monotone* to mean $(f(x)^* - f(y)^*)(x - y) \geq 0$ for $f(x)^*$ in $f(x)$ and $f(y)^*$ in $f(y)$. We may write $(x, y) \in f$ to mean $y \in f(x)$. We say $f : \mathbf{R} \to 2^{\mathbf{R}}$ is *maximal monotone* if $(x_1, y_1) \in \mathbf{R}^2$ and $(y - y_1)(x - x_1) \geq 0$ for all $(x, y) \in f$ implies $(x_1, y_1) \in f$.

We define with Minty a *resistor* in branch b of a locally finite digraph (B, N) to be a maximal monotone $M_b : \mathbf{R} \to 2^{\mathbf{R}}$. We write, given $f : \mathbf{R} \to 2^{\mathbf{R}}$, *domain* $(f) = \{x \in \mathbf{R} : \exists y, (x, y) \in f\}$ and *range* $(f) = \{y \in \mathbf{R} : \exists x, (x, y) \in f\}$.

We say $i : B \to \mathbf{R}$ *satisfies KCL* or is a *cycle*, where (B, N) is a locally finite digraph, if its boundary ∂i is zero [4] , where for any $n \in N, \partial i(n) = \Sigma\{i_b : \text{the head of } b \in n\} - \Sigma\{i_b : \text{the tail of } b \in n\}$.

We say $v : B \to \mathbf{R}$ *satisfies KVL* if it is a *coboundary* δe [4], ie. if there is $e : N \to \mathbf{R}$ with $v_b = e(n_t) - e(n_h)$ where n_t and n_n contain the head and tail of b respectively, equivalently

$$\sum_{g \in B} v_b i_b = 0$$

for i any finitely supported cycle.

We say, following Minty, that a *monotone network* M on a locally finite digraph (B, N) is a function that assigns to every branch b a resistor M_b. And a *solution* of M consists of i satisfying KCL and v satisfying KVL such that for all $b \in B$,

$$v_b \in M_b(i_b).$$

Given a closed nonempty convex subset A of $\mathbf{R}$, let A° denote the element of least norm, $a \in A$ such that $|a| \leq |b|$ for all $b \in A$. Given $f : \mathbf{R} \to 2^{\mathbf{R}}$, maximal monotone, and $(x, y) \in$ domain $(f) \times$ range (f), we let $f^{-1}(y) = \{z : (z, y) \in f\}$, and

$$\delta'(x, y, f) = |(f^{-1}(y) - x)^\circ|,$$

the distance from (x, y) to the intersection of the graph of f and the horizontal line through (x, y). And we denote

$$\delta''(x, y, f) = |(f(x) - y)^\circ|.$$

Given a monotone network M on (B, N) and $(i_d, v_d) \in$ domain $(M_d) \times$ range (M_d) for all $d \in B$, giving i and v from B to $\mathbf{R}$, we set

$S'_b(i, v, M) = \sup\{\sum_{d \in C} \delta'(i_d, v_d, M_d) : C$ a finite cutset of (B, N) containing $b\}$.

Similarly we set

$S''_b(i, v, M) = \sup\{\sum_{d \in C} \delta''(i_d, v_d, M_d) = C$ a finite loop or path containing $b\}$.

3. Existence and Bounds.

Theorem 1. (Minty's Theorem). Let M be a monotone network on the locally finite digraph (B, N). Suppose $i^1 : B \to \mathbf{R}$ and $v^1 : B \to \mathbf{R}$ satisfy KCL and KVL respectively, and for all $b \in B$, $i_b^1 \in \text{domain}(M_b), v_b^1 \in \text{range}(M_b)$, $S_b'(i^1, v^1, M) < \infty$ and $S_b''(i^1, v^1, M) < \infty$.

Then M has a solution (i^0, v^0) such that as functions from B to $\mathbf{R}$ we have

$$|i^0 - i^1| \leq S'(i^1, v^1, M) \tag{1'}$$

and $$|v^0 - v^1| \leq S''(i^1, v^1, M). \tag{1'}$$

Lemma 1. Let $f : \mathbf{R} \to 2^{\mathbf{R}}$ be maximal monotone, and let $i \in \text{domain}\ (f)$ and $v \in \text{range}\ (f)$.

Then there exists a sequence $\langle f_n \rangle$ of monotone homeomorphisms f_n from $\mathbf{R}$ onto $\mathbf{R}$ such that:

(a) if $(x_n, y_n) \in f_n$ for all n and $(x_n, y_n) \to (x, y)$ in $\mathbf{R}^2$ then $(x, y) \in f$,

(b) $\delta'(i, v, f_n) \nearrow \delta'(i, v, f)$, and

(c) $\delta''(i, v, f_n) \nearrow \delta''(i, v, f)$.

The following is a statement of the "no gain" property, as in Wolaver [10].

Lemma 2. Let M be a finite network, with every M_b a monotone bijection from $\mathbf{R}$ to $\mathbf{R}$. If i^1 and v^1 satisfy KCL and KVL, and (i^0, v^0) is a solution of M, we have for each branch b :

$$|i_b^0 - i_b^1| \leq \textstyle\sum_{d \in C} \delta'(i_d^1, v_d^1, M_d) \text{ for some cutset } C \text{ containing } b, \text{ and} \tag{2'}$$

$$|v_b^0 - v_b^1| \leq \sum_{d \in C} \delta''(i_d^1, v_d^1, M_d) \text{ for some loop } C \text{ containing } b. \tag{2'}$$

The finite dimensional case of Minty's Theorem is proved using the approximations of Lemma 1. Then the network is approximated by finite subnetworks. In both cases the bounds give compactness and hence existence of a solution.

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Networks with Distributed and Lumped Parameters

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ABSTRACT

For a network composed by transmission lines (described by telegraph equations) which interconnect grounded resistors and capacitors, two main results are obtained: existence of solutions and bounds for voltages in any point.

1. Introduction

The progressive miniaturization as well as the continuous increasing of clock frequency, gave emphasis to the interconnection wire as being responsible, to a great extent, for the delay time in VLSI circuits [1]. This is why the adequate modeling of interconnections and apriori estimations of the delay time, are research topics of major importance.

By using the lumped parameter approximation of interconnections, many papers provided bounds of the delay time (see, for instance, [2]–[5]). Other works use the one-dimensional telegraph equations to model the lines, treating them by Laplace transform [6], [7] or in time domain [8], [9].

Our present paper works in time domain and tries to cover the lack of generality in our previous attempts. Namely, we consider the network in Fig. 1, where the distributed parameters lines connect the RC grounded devices. If N is the set of nodes (excluding the ground) then we consider the set of lines $L \subset N \times N$ and each line (pq) is directed from the node p (where the space variable $x = 0$) to the node q (where $x = 1$). Neglecting the inductive effects, we have

$$\begin{cases} \dfrac{\partial u_{pq}}{\partial t} = \dfrac{1}{R_{pq}C_{pq}} \dfrac{\partial^2 u_{pq}}{\partial x^2} - \dfrac{G_{pq}}{C_{pq}} u_{pq}(t,x) \\ t \in (0,T), \quad x \in (0,1), \quad (pq) \in L \end{cases} \tag{1}$$

where $R_{pq} = r_{pq}d_{pq}$, $C_{pq} = c_{pq}d_{pq}$, $G_{pq} = g_{pq}d_{pq}$ are, respectively, the total resistance, capacitance and conductance and d_{pq} is the line length.

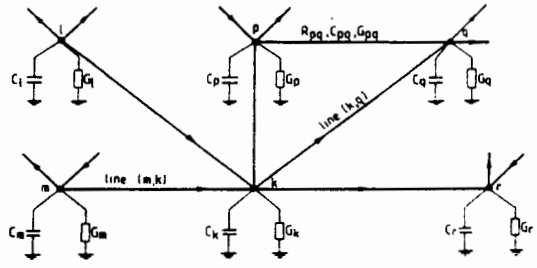

Fig. 1: The network under study

Let us denote by $\tilde{u}_k(t)$ the voltage of node k, and by $\mathcal{L}_k$ all nodes connected with k by a line which ends in k. Similarly, $\mathcal{R}_k$ are the nodes connected with k by a line which starts from k. Then, we have

$$u_{lk}(t,1) = u_{kr}(t,0) = \tilde{u}_k(t) \tag{2}$$

for all $l \in \mathcal{L}_k$, $r \in \mathcal{R}_k$, $t \in [0,T]$.

The KCL for the node k gives:

$$-\sum_{l\in\mathcal{L}_k} \frac{1}{R_{lk}} \frac{\partial u_{lk}}{\partial x}(t,1) + \sum_{r\in\mathcal{R}_{kr}} \frac{1}{R_{kr}} \frac{\partial u_{kr}}{\partial x}(t,0) = C_k \frac{d\tilde{u}_k}{dt} + G_k \tilde{u}_k(t) + j_k(t) \tag{3}$$

for all $t \in [0,T]$.

Also, the initial voltage at any point is given by

$$u_{pq}(0,x) = u^0_{pq}(x) \tag{4}$$

for all $x \in [0,1]$, $(pq) \in L$.

We have now a system of parabolic equation (1), coupled by boundary conditions (2) and (3). Let us denote the dynamic problem (1)+(2)+(3)+(4) by $DP(j(t), u^0)$ where $j(t) = (j_k(t))_{k\in N}$ and $u^0 = (u^0_{pq})_{(p,q)\in L}$. The time independent variant of it, consisting of (1)+(2)+(3) without time derivatives and with constant current sources, will be denoted by $SP(j)$.

2. Existence and Uniqueness of Solution – Case $C_k = 0$

For very smooth sources and initial conditions of square integrable type we have the following result.

Theorem 1. *Hypotheses:*

a) *For all $k \in N$, the functions $j_k : [0,T] \to \mathbf{R}$ are differentiable with Hölder continuous derivatives, $j_k' \in C^\nu(0,T;\mathbf{R})$.*

b) *For all $(pq) \in L$, $u^0_{pq} \in L^2(0,1;\mathbf{R})$.*

Then, for each $(pq) \in L$ there exists $u_{pq} \in C^1(0,T;L^2(0,1;\mathbf{R}))$ such that $DP(j(t),u^0)$ has an unique solution.

Here by "solution" we understand a vector function $u = (u_{pq})_{(pq)\in L}$ whose components have time derivatives with respect to $L^2(0,1;\mathbf{R})$ norm, and space derivatives in the distribution sense.

The main disadvantage of the previous result is the strong restriction imposed by assumption a). In order to manage the usual discontinuous signal sources we give the next theorem.

Theorem 2. *Hypotheses:*

a) *For all $k \in N$, $j_k \in L^1(0,T;\mathbf{R})$.*

b) *For all $(pq) \in L$, $u^0_{pq} \in L^2(0,1;\mathbf{R})$.*

Let $(j^n_k)_n$ be a sequence of step functions such that $j^n_k \to j_k$ when $n \to \infty$, in $L^1(0,T;\mathbf{R})$ for all $k \in N$. Let $(u^{0,n}_{pq})_n$ be a sequence of functions in $L^2(0,1;\mathbf{R})$ satisfying the boundary conditions (in distribution sense) such that $u^{0,n}_{pq} \to u^0_{pq}$ in $L^2(0,1;\mathbf{R})$. Then,

1) *For each n the $DP(j^n,u^{0,n})$ is satisfied a.e. in t and x by $u^n_{pq} \in C(0,T;L^2(0,1;\mathbf{R}))$, $(pq) \in L$.*

2) *$u^n_{pq} \to u_{pq}$ in $L^1(0,T;L^2(0,1;\mathbf{R}))$.*

This limit $(u_{pq})_{pq\in L}$ is the "weak" solution of our $DP(j(t),u^0)$.

Now we prove the existence of the steady-state solution.

Theorem 3. *If for each line (kl) without dielectric conductance (i.e. $G_{kl} = 0$) there exists at least one end which is resistively grounded (i.e. $G_k > 0$ or $G_l > 0$), then there exists $u^\infty = (u^\infty_{pq})_{(pq)\in L}$, $u^\infty_{pq} \in C^2(0,1;\mathbf{R})$ solving uniquely the $SP(j)$ problem.*

The next result assures that the transients will reach the steady state regardless of the values of the initial conditions.

Theorem 4. *If $j(t) = j = constant$, $u^0_{pq} \in L^2(0,1;\mathbf{R})$ for all $(pq) \in L$ and the assumption of Theorem 3 is fulfilled, then*

$$\lim_{t\to\infty} \|u_{pq}(t,\cdot) - u^\infty_{pq}(\cdot)\|_{L^2(0,1;\mathbf{R})} = 0$$

for all $(pq) \in L$.

The proofs are partially given in [10].

3. Bounds for the Solution – Case $j_k = 0$

We suppose now that the lumped capacitors are connected at some nodes, all the sources are zero, and all initial conditions have the same constant value, such that the solutions are of classical type (C^1 in time and C^2 in space). The following result regards the upper and lower bounds for the solution.

Theorem 5. *Hypotheses:*

a) $u^0_{pq}(x) = 1$ *for all* $(pq) \in L$, $x \in [0,1]$.
b) *There exists at least one node* $k \in N$ *such that* $G_k \neq 0$ *and* $C_k \neq 0$.
c) *Every node can be reached departing from any other one, i.e. the network is "horizontally connected".*

Then there exist the numbers $\alpha_k \in (-\frac{\pi}{2}, \frac{\pi}{2})$ *for all* $k \in N$, *and* $\overline{\lambda} > 0$ *such that*

$$\tan\alpha_k \sum_{m\in\mathcal{L}_k\cup\mathcal{R}_k} \frac{\alpha_k - \alpha_m}{R_{km}} \leq G_k - \overline{\lambda}C_k \quad \text{for all } k \in N, \tag{5}$$

and

$$\overline{\lambda} \leq \frac{(\alpha_k - \alpha_l)^2}{R_{kl}C_{kl}} + \frac{G_{kl}}{C_{kl}} \quad \text{for all } (kl) \in L. \tag{6}$$

With these numbers we denote

$$F_{pq}(x) = \frac{\cos[(1-x)\alpha_p + x\alpha_q]}{\min\{\cos\alpha_k;\ k \in N\}}$$

and obtain the upper bound:

$$u_{pq}(t,x) \leq F_{pq}(x)e^{-\overline{\lambda}t}.$$

A similar result holds for the lower bound case. The proof will appear in [11].

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NODE VOLTAGES IN NONLINEAR RESISTIVE TRANSFINITE NETWORKS*

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Abstract — Transfinite electrical networks need not have node voltages even when branch voltages and currents uniquely exist. The difficulty is that the sum of branch currents along a transfinite path from a selected ground to a particular node may diverge or may depend upon the choice of the path. Sufficient conditions are given herein under which node voltages exist and are unique at all nodes, whatever be the choice of ground node.

1 Introduction

The idea of a transfinite electrical network arises quite naturally when finite networks are expanded into infinite ones [3, Section 1.4]. In particular, a transfinite network is one wherein there are at least two nodes that are connected by an infinite path but not by any finite path. A theory for such networks was introduced in [1] and expanded in [3].

A peculiarity of such networks is that, even though branch voltages may exist throughout the network, node voltages may not. This is because the sum of branch voltages along a transfinite path from a preselected ground node to some other node may diverge or because that sum — though convergent — may depend upon the choice of path. Examples illustrating this are given in [3, Section 5.5]. Thus, additional conditions are needed to insure the existence and uniqueness of node voltages. Such have been devised for linear transfinite resistive networks in [2], but not for nonlinear ones. This paper presents a set of sufficient conditions for a class of nonlinear resistive networks that insures the existence and uniqueness of node voltages. In the following we freely use the definitions and results of [1] and [3]. The proofs of all the lemmas and theorems given below will appear elsewhere. Here we merely present our results along with an outline of how they are achieved.

2 A Class of Nonlinear Transfinite Networks.

Let N^ν $(0 \leq \nu \leq \omega)$ denote a nonlinear ν-network [1], every branch of which is in the Norton form, as shown in Figure 1. Thus, the jth branch is a parallel connection of a pure current source h_j and a nonlinear resistor $R_j(\cdot)$ carrying a current f_j. In accordance with the polarity conventions shown in Figure 1, the branch voltage v_j equals $R_j(f_j) = R_j(i_j + h_j)$, where i_j is the branch current. Thus, there is no coupling between branches. As always, we take the branch's orientation to be that indicated by the arrow for i_j in Figure 1. In the special case of a linear resistor, the constant branch resistance r_j is the slope of the straight line $f_j \mapsto R_j(f_j)$, i.e., $r_j = dR_j/df_j$.

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Conditions 2.1. N^ν *$(0 \le \nu \le \omega)$ is a nonlinear ν-network whose countably many branches are in the Norton form, shown in Figure 1. Every R_j is a continuous, strictly increasing, odd mapping of the real line R^1 into R^1 with $R_j(f) \to \infty$ as $f \to \infty$.*

For $f, v \in R^1$, we set $M_j(f) = \int_0^f R_j(x)\,dx$ and $M_j^*(v) = \int_0^v G_j(y)\,dy$, where $G_j : v \mapsto G_j(v) = f$ is the inverse function of $R_j : f \mapsto R_j(f) = v$. Given that N^ν satisfies Conditions 2.1, the modular sequence space l_{M} is defined as the set of all sequences $\mathbf{f} = (f_1, f_2, \ldots)$ of real numbers f_j such that $\sum M_j(f_j/t) < \infty$ for some $t > 0$, and c_{M} is the subspace of l_{M} for which $\sum M_j(f_j/t) < \infty$ for all $t > 0$. (Here, $\sum$ denotes a summation over all branch indices j.) l_{M} and c_{M} are Banach spaces with the norm

$$\|\mathbf{f}\| = \inf\{t : t > 0, \sum M_j\left(\frac{f_j}{t}\right) \le 1\}.$$

We assume henceforth

Conditions 2.2. *The nonlinear ν-network N^ν satisfies Conditions 2.1. Moreover, $\mathrm{h} = (h_1, h_2, \ldots) \in l_{\mathrm{M}}$. Furthermore, there are a positive integer j_0 and two positive numbers α and β, both greater than 1, such that, for every $j \ge j_0$,*

(i) *$fR_j(f) \le \alpha M_j(f)$ for $0 \le f \le a_j$, where a_j is the unique positive number for which $M_j(a_j) = 1$, and*

(ii) *$uG_j(u) \le \beta M_j^*(u)$ for $0 \le u \le d_j$, where d_j is the unique positive number for which $M_j^*(d_j) = 1$*

Let l_{M}^* be the dual of l_{M} and c_{M}^* be the dual of c_{M}. Also, let $\cong$ denote an isomorphism between Banach spaces. Under Conditions 2.2, we have the following relations [1, pages 129-130].

$$l_{\mathrm{M}} = c_{\mathrm{M}} \tag{1}$$

$$l_{\mathrm{M}}^* = c_{\mathrm{M}}^* \cong l_{\mathrm{M}^\bullet} \tag{2}$$

$$(l_{\mathrm{M}}^*)^* \cong (l_{\mathrm{M}^\bullet})^* \cong l_{(\mathrm{M}^\bullet)^\bullet} = l_{\mathrm{M}} \tag{3}$$

Thus, l_{M} is reflexive; that is, the dual of the dual of l_{M} is isomorphic to l_{M}.

A *basic current* in the transfinite network N^ν is defined on [1, page 154]. It is a countable superposition of (generally transfinite) loop currents satisfying certain conditions. $\mathcal{L}^0$ is the span of all basic currents in l_{M}, and $\mathcal{L}$ is the closure of $\mathcal{L}^0$ in l_{M}. $\mathcal{L}$ is a linear subspace of l_{M} and is a Banach space in itself when supplied with the norm of l_{M}.

In the next theorem R denotes the resistance operator $R(\mathbf{f}) = (R_1(f_1), R_2(f_2), \ldots)$, where $\mathbf{f} = (f_1, f_2, \ldots)$. R maps l_{M} into $l_{\mathrm{M}^\bullet}$ [1, Lemma 4.7-2]. According to the polarity conventions of Figure 1, the branch voltage vector $\mathbf{v} = (v_1, v_2, \ldots)$ is related to the branch current vector $\mathbf{i} = (i_1, i_2, \ldots)$ and the branch current source vector $\mathbf{h} = (h_1, h_2, \ldots)$ through the equation $\mathbf{v} = R(\mathbf{i} + \mathbf{h})$. Furthermore, for $\mathbf{v} \in l_{\mathrm{M}^\bullet}$ and $\mathbf{s} \in l_{\mathrm{M}}$, $\langle \mathbf{v}, \mathbf{s} \rangle$ will denote the pairing $\langle \mathbf{v}, \mathbf{s} \rangle = \sum v_j s_j$ [1, page 126].

Theorem 2.3. *Under Conditions 2.2, there exists a unique $\mathbf{i} \in \mathcal{L}$ such that*

$$\langle R(\mathbf{i} + \mathbf{h}), \mathbf{s} \rangle = 0 \tag{4}$$

for all $\mathbf{s} \in \mathcal{L}$. Furthermore, $\mathbf{i}$ satisfies Kirchhoff's current law at every finite maximal 0-node, and $\mathbf{v} = R(\mathbf{i} + \mathbf{h})$ satisfies Kirchhoff's voltage law around every 0-loop and also around every ζ-loop $(1 \le \zeta \le \nu)$ having a unit flow in $\mathcal{L}$.

3 Permissive Paths and Node Voltages.

Let p and q be real numbers such that $p^{-1} + q^{-1} = 1$ and $1 < p < \infty$. Thus, $\infty > q > 1$. The inequality (5) below is illustrated in Figure 2.

Lemma 3.1. *Assume that a branch resistance function R_j is bounded according to*

$$\left(\frac{f}{\gamma_j}\right)^{p-1} \leq R_j(f) \leq \rho_j f^{p-1} \tag{5}$$

for all $f > 0$, where γ_j and ρ_j are positive numbers with $1/\gamma_j^{p-1} \leq \rho_j$. Then, both restrictions (i) and (ii) of Conditions 2.2. are satisfied by that R_j.

Definition 3.2. Let P denote either a ζ-path or a ζ-loop ($\zeta \leq \nu$) with infinitely many branches [1, pages 144, 147, 148]. (Our conclusions hold trivially if P has only finitely many branches.) Also, let Θ be the index set for the branches embraced by P except for possibly finitely many of them. P is called *strongly permissive* if Θ can be so chosen that, for every $j \in \Theta$, there are two positive numbers γ_j and ρ_j such that the following hold:

(i) $c = \gamma_j^{p-1}\rho_j$, where c is independent of $j \in \Theta$ and $1 \leq c < \infty$.

(ii) The bounds (5) hold for all $f \geq 0$ and $j \in \Theta$.

(iii) $\sum_{j\in\Theta} \rho_j < \infty$.

If P is a representative of a ζ-tip t^ζ [1, pages 140, 148], then t^ζ is also called *strongly permissive*.

The *algebraic sum of the branch voltages* along the oriented ζ-path or ζ-loop P is

$$\sum_{j\in\Pi} \pm v_j \tag{6}$$

where Π is the index set for all the branches embraced by P and the plus (minus) sign is used if the orientations of P and the jth branch agree (respectively, disagree). Kirchhoff's voltage law, when it holds for a ζ-loop P, asserts that (6) equals 0.

Lemma 3.3. *Let P be a strongly permissive ζ-path or ζ-loop. Then, (6) converges absolutely.*

4 The existence and uniqueness of node voltages.

Theorem 4.1. *Under Conditions 2.2 and the voltage-current regime dictated by Theorem 2.3, Kirchhoff's voltage law is satisfied around every strongly permissive ζ-loop ($\zeta \leq \nu$), and for such a loop (6) converges absolutely.*

Let n and m be two totally disjoint nodes [1, pages 72, 141], whose ranks need not be the same. Also, let P be a ζ-path ($\zeta \leq \nu$) that meets n and m terminally and is oriented from n to m. If P is strongly permissive, we define(6) to be the *node voltage of n with respect to m along P*. The definition of "nondisconnectable tips" for tips of arbitrary ranks is given in [3, Section 3.3]. Also, a tip is called *strongly permissive* if one of its representatives is strongly permissive.

Condition 4.2. *If two tips are nondisconnectable, then either they are shorted together or at least one of them is open. If those tips are nondisconnectable and strongly permissive, then they are shorted together.*

Theorem 4.3. *Under the hypothesis of Theorem 4.1, assume that the tips of ranks no larger than $\bar{\omega}$ in the ν-network $\mathbf{N}^\nu$ ($\nu \leq \omega$) satisfy Condition 4.2. Let n_g and n_0 be two nodes (of possibly*

different ranks), and let there be at least one strongly permissive path connecting n_g and n_0. Then, n_0 has a unique node voltage with respect to n_g; that is, n_0 obtains the same node voltage with respect to n_g along all strongly permissive paths between n_g and n_0.

Corollary 4.4. *Under the hypothesis of Theorem 4.1, assume that the tips of all ranks no larger than $\vec{\omega}$ in the ν-network $\mathbf{N}^\nu$ ($\nu \leq \omega$) satisfy Condition 4.2. Also, assume that every two nodes of $\mathbf{N}^\nu$ are connected through at least one strongly permissive path. Choose a ground node n_g in $\mathbf{N}^\nu$ arbitrarly. Then, every node of $\mathbf{N}^\nu$ has a unique node voltage with respect to n_g.*

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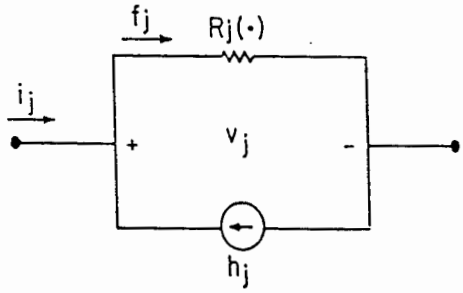

Figure 1.

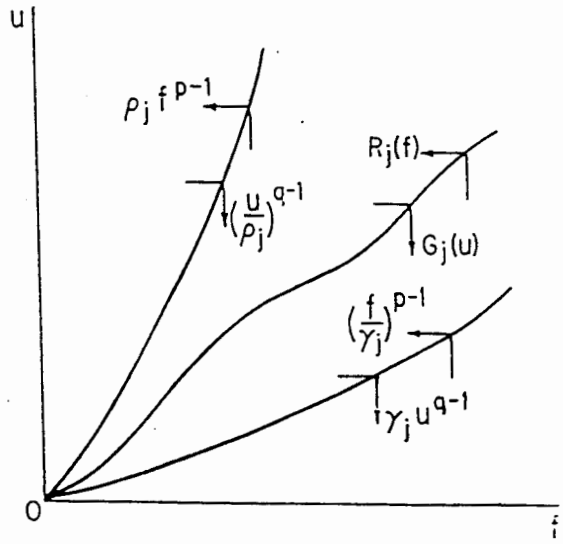

Figure 2.

BLIND STRUCTURE IDENTIFICATION FOR MULTI-LAYER MEDIUM

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ABSTRACT

There are a category of problems about internal structure identification for a multi-layer medium according to reflection wave measurement on the surface. The mathematic model is that the output signal, *i.e.*, the reflected wave series, is a convolution of the input signal, *i.e.*, the exploration wave and the sequence of reflectivity at the interface positions between every two layers, which describes the multi-layer structure. The difficulty for this problem is that the reflectivity sequence has to be identified with the input signal unknown, since the exploration wave is seldom obtained in many cases. The presented methods are all based on some additional assumptions and/or other information, so they can't guarantee to solve this problem in any case. To overcome this drawback, an essentially new approach is proposed in this paper based on blind identification theory. According to our method, reflection wave measurements are performed twice at every exploration point by different exploration waves so that the two exploration waves can be estimated from the two output signals, and then the reflectivity sequence can be obtained by some deterministic deconvolution algorithm. An example for seismic prospecting is given in this paper, which demonstrates the effectiveness of this new method.

1. INTRODUCTION

In practice, a category of problems are to identify multi-layer structure of medium from measurement data by reflection wave exploration on its surface. The principle of reflection wave exploration method is that the exploration wave, which is stimulated on the surface, can be reflected back to the surface again whenever it encounters an interface between two different layers in its way toward the medium interior. Thus, an reflected time series signal, *i.e.*, the reflected waves, can be received on the surface, which records information about every interface position under the exploration point. The collection of many such adjacent reflected signals constitutes a two-dimensional profile, from which the internal structure of the medium can be shown though not with high resolution and accurate position. For example, B mode ultrasonic scan technique is often used to study internal structure of a human body, especially for abdomen and breast, for tumor inspection. According to B mode ultrasonic scan technique, a very short duration interrogating ultrasonic pulse is launched into tissue as the exploration wave, then the reflected ultrasound is recorded as the measurement signal by a transducer, where abdomen and breast can be considered as layer-structure organs below their surfaces. As the transducer turns left and right, a two-dimensional image, which is usually in the shape of sector, is formed to display the tissue structure. Another typical example is seismic tomography, which is used to find out subsurface structure for the purpose of geologic survey and mine/oil prospecting. According to seismic tomography technique, explosion is conducted on the ground to make a very strong acoustic pulse, *i.e.*, seismic wavelet, as the exploration wave, then the reflected seismic waves, *i.e.*, seismic trace, are received as the measurement signal by the sensor. The subsurface is clearly one kind of multi-layer medium. In fact, there are huge amount of such seismic traces collected in the course of seismic data acquisition. They can even construct a three-dimensional data set to exhibit the subsurface formations.

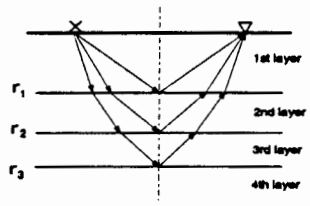

Fig.1 reflection wave exploration at local point

A linear mathematical model about reflection wave exploration with respect to one exploration point can be established by assuming: i). internal layers under the exploration point are locally horizontal and parallel; ii). exploration wave can be considered as a time limited signal and it is invariant along with its travel path; iii). multi-reflected signals are ignored in measurement, so received signal is only composed of first-reflected signals; iv). exploration wave and its first-reflected signals can penetrate layers without energy lost. Actually, above conditions can only be satisfied approximately in practice. Denoting $s(\cdot)$ as the exploration wave, $\{r_i\}$ as the reflectivity sequence, $n(\cdot)$ as the additive measurement noise, and $x(\cdot)$ as the received signal, we can describe the reflection exploration process as follows,

$$x(t) = \sum_{i=1}^{n} r_i s(t - \tau_i) + n(t) \qquad (1)$$

where, r_i is the reflection coefficient at the interface between ith and $i+1$th layer, which is determined by the difference of wave resistance between this two layers; τ_i is the delay for the exploration signal $s(\cdot)$ to travel from the stimulation point on the surface and pass through the reflection point at the ith interface then return to the receiver point on the

surface, $\tau_1 < \tau_2 < \cdots < \tau_n$; $n+1$ is the number of layers. As shown in Fig.1, it should be noticed that this mathematical model doesn't care if the stimulation point, where the exploration signal is sent out, and the receiver point, where the reflected waves are recorded, are located at the same position or not. If they are located at the same position, *e.g.*, in case of the B mode ultrasonic scan, the received signal shows the internal structure under this point. If not, *e.g.*, in case of seismic tomography, the received signal can represent the internal structure under the central point between them. In the later case, the exploration point refers to the central point between the stimulation point and the receiver point.

Replacing $\{r_i\}$ by $r(\cdot)$,

$$r(t) = \begin{cases} r_i\delta(t-\tau_i) & \text{for } t \in \{\tau_i\} \\ 0 & \text{for } t \notin \{\tau_i\} \end{cases} \tag{2}$$

we can find formula (1) actually represents a convolution model suffering from noise,

$$x(t) = r(t) \star s(t) + n(t) \tag{3}$$

Obviously, the purpose of internal structure identification is to get $\{r_i\}$ as accurate as possible. Our blind exploration method is designed to estimate $s(\cdot)$ based only on $x(\cdot)$, so that $r(\cdot)$ can be recovered easily with $s(\cdot)$ known.

According to formula (1), the measurement signal $x(\cdot)$ can not display internal multi-layer structure very clearly, since those interfaces shown by $x(\cdot)$ are represented by wave lobes of the exploration wave $s(\cdot)$, which are often relatively too wide to locate the interface positions accurately. In addition, they also introduce some illusions into the structure profile when thin layers are present because of wave interference. Hence, the conventional way for internal structure identification is to estimate $r(\cdot)$ based on $x(\cdot)$ and $s(\cdot)$, which is called as deconvolution. By the way, deconvolution can not recover the reflectivity sequence $r(\cdot)$ accurately if the exploration wave $s(\cdot)$ behaviors as a band-limited filter. In such case, this problem is usually named as resolution improvement. According to the convolution model, now that $x(\cdot)$ is already known, the next step is to measure $s(\cdot)$. Unfortunately, in many cases, $s(\cdot)$ can not be measured, or it can not be accurately measured. For example, such $s(\cdot)$ can not be measured in seismic data acquisition because it is conducted by explosion. In fact, when acoustic wave is adopted as the exploration wave, it would unavoidably be attenuated due to incomplete elastic media in the object and as a result $s(\cdot)$ varies with time actually. In this case, we have to introduce an average of $s(\cdot)$ into the convolution model, in which $s(\cdot)$ is supposed time-invariant, in order to reach an optimal approximation about $r(\cdot)$. Thus, even though we can measure the exploration wave accurately when it is stimulated, it is still useless because it is no more than the initial status of $s(\cdot)$ and it is usually far away from the average of $s(\cdot)$. It is due to these problems, various alternative methods have to be designed for resolution improvement for reflection wave exploration in all application field. For instance, very high frequency ultrasound have been received much attention to be used as the exploration wave, by which the wave lobe can be made very narrow so that the interface position can be located accurately enough based only on $x(\cdot)$. Nevertheless, this technique is severely limited by human safety and equipment property [5]. For another instance, various seismic deconvolution algorithms have been elaborated to improve resolution for seismic profile. However, because seismic wavelet $s(\cdot)$ is unknown, all presented methods for seismic deconvolution rely on imposing some "reasonable" assumptions on the reflectivity sequence and/or the seismic wavelet in order to get more constraints so as to make seismic deconvolution feasible. For example, the earliest canonical seismic deconvolution algorithm is based on two "reasonable" assumptions that the reflectivity sequence $r(\cdot)$ is a white noise and the seismic wavelet $s(\cdot)$ is minimum phase [6], which are later proved different from the reality in general. This kind of imposed additional assumptions can also be found in other famous seismic deconvolution methods, such as Maximum Likelihood Deconvolution (MLD) [2], Minimum Variance Deconvolution (MVD) [3], L1-norm Deconvolution [7], Lp-norm Deconvolution [1], and Minimum Entropy Deconvolution (MED) [9], which all assume that the reflectivity sequence is a sparse series more or less. It damages seriously the reliability of those seismic deconvolution methods because various assumptions all seem reasonable but they lead to different results. Moreover, they are taken for granted but it may not be consistent with the real situation.

In order to resolve above problems, we present an essentially new approach, the dual exploration method, for internal multi-layer structure identification, by which we can recover $r(\cdot)$ without any additional assumption on $r(\cdot)$ and $s(\cdot)$.

2. DUAL EXPLORATION METHOD

The idea about the dual exploration method for layered structure identification is as follows:

i). measure twice at every point on the surface by different exploration waves $s_1(\cdot)$ and $s_2(\cdot)$, and then record the corresponding reflected signals $x_1(\cdot)$ and $x_2(\cdot)$;

ii). estimate the two exploration waves $s_1(\cdot)$ and $s_2(\cdot)$ based only on the two reflected signals $x_1(\cdot)$ and $x_2(\cdot)$;

iii). recover the reflectivity sequence $r(\cdot)$ as clear as possible based on deterministic deconvolution algorithm.

The key to dual exploration method is in step 2, namely, to estimate $s_1(\cdot)$ and $s_2(\cdot)$ with only $x_1(\cdot)$ and $x_2(\cdot)$ known in the following equations,

$$\begin{cases} x_1(t) = r(t) \star s_1(t) + n_1(t) \\ x_2(t) = r(t) \star s_2(t) + n_2(t) \end{cases} \tag{4}$$

where $n_1(\cdot)$ and $n_2(\cdot)$ are random noise assumed as i.i.d. Gaussian white noise.

First let us not take the additive noises $n_1(\cdot)$ and $n_2(\cdot)$ into account. Thus, we can get rid of $r(\cdot)$ from (4) and get,

$$x_1(t) \star s_2(t) = x_2(t) \star s_1(t) \tag{5}$$

Considering that $s_1(\cdot)$ and $s_2(\cdot)$ are time-limited signals but $x_1(\cdot)$ and $x_2(\cdot)$ can be of any length, we can find (5) is actually a set of overdetermined linear equations if we rewrite it in discrete form. To avoid getting a singular solution to this equation set, we can solve it with the following constraint,

$$\|s_1(\cdot)\|^2 + \|s_2(\cdot)\|^2 = 1 \tag{6}$$

According to presented results about blind identification in the fields of communication and signal processing [8], the linear equation set (5) has unique solution under the constraint (6) if and only if the two FIR filters with $s_1(\cdot)$ and $s_2(\cdot)$ as their impulse responds have no common zeros. As far as the reflection wave exploration method is concerned, this sufficient and necessary condition can be almost always satisfied, since it takes at least several tens of samples to describe the waveform of $s_1(\cdot)$ and $s_2(\cdot)$. As we know, there

is little probability for two high-order FIR filters with different impulse responds to share common zeros.

Obviously, above method still works even if there are some noises. In this case, (5) is no longer satisfied accurately. We can pursue some solution with minimum error, such as the least square error solution under the constraint (6). There are already many effective algorithms to solve this problem [4] [8]. Limited by paper length, the concrete algorithm and its derivation are omitted here.

After we estimate $s_1(\cdot)$ and $s_2(\cdot)$ by blind identification method, it is then easy to recover $r(\cdot)$. There are already many deterministic deconvolution algorithms to solve this problem. How to do so is not the interest of this paper. However, we should point out that we can not expect the reflectivity sequence R to be recovered exactly as its actual form, even though the exploration waves $s_1(\cdot)$ and $s_2(\cdot)$ have been known. Because in practice the exploration wave is usually similar to a low-pass filter, it is almost impossible to restore the high-frequency information of the reflectivity sequence in noisy situation by inversing a low-pass filter. We can only get an estimation about reflectivity sequence without certainty of real high-frequency components.

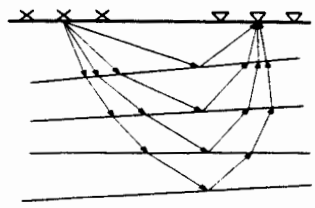

Fig.2(a) wavelet travel path

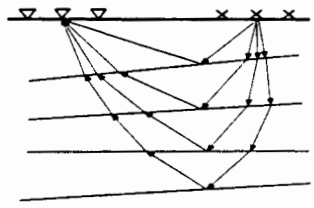

Fig.2(b) inverse wavelet travel path

3. AN APPLICATION EXAMPLE

Guided by above idea, we succeed in developing a blind seismic deconvolution algorithm for two-side CMP (Common Medium Point) prestack seismic data to improve its resolution, in which we take the advantage of current technology of seismic data acquisition skillfully so that twice measurements can be obtained with respect to every exploration position without additional acquisition work needed.

In seismic prospecting, the subsurface structure under every exploration point is described by a set of seismic traces, namely, a CMP gather. There are two reasons why so many seismic measurements are needed for one point exploration: one is due to low SNR in seismic data acquisition; the other is because the subsurface formations are not strictly horizontal. In case of slant subsurface formations, a seismic trace no longer represents the subsurface structure under the central point between the explosion point and the sensor point, as shown in Fig.2. However, the poststack seismic trace of a set of measurements, a CMP gather, with various distances between the explosion point and the sensor point is very close to do so when the subsurface formations are slightly slant with random directions.

As we know, two-side seismic CMP gather consists of many pairs of symmetrical seismic traces. In each pair, the explosion position of one trace is right the receiver position of the other trace, and *vice versa*, as shown in Fig.2. Obviously, no matter how irregular the subsurface structure is, if one wavelet can travel down from the explosion point and reflected up to the receiver point, see Fig.2(a), another wavelet is definitely able to travel down from the receiver point and reflected up to the explosion point through the same path, see Fig.2(b). This fact, not an imposed assumption, implies strongly that the reflectivity sequence represented by one trace should be *exactly* equal to those by its symmetrical trace in a CMP gather. In addition, we should note another fact that these two symmetrical traces are different to some extent because they are stimulated by different explosions and recorded by different receivers. That is to say, their wavelets are different to some extent. According to above discussion, our dual exploration model for blind seismic deconvolution is as follows: two symmetrical seismic traces are considered as two measurement signals $x_1(\cdot)$ and $x_2(\cdot)$, two different wavelets as two different exploration signals $s_1(\cdot)$ and $s_2(\cdot)$. Where, no additional measurement work is needed to construct the dual exploration model. Based on this model, two wavelets $s_1(\cdot)$ and $s_2(\cdot)$ can be estimated according only to a pair of symmetrical traces $x_1(\cdot)$ and $x_2(\cdot)$, and then deconvolution can be performed easily to get a pair of high-resolution seismic traces.

It should be emphasized that we can only get a high-resolution seismic trace instead of the reflectivity sequence even though seismic wavelets $s_1(\cdot)$ and $s_2(\cdot)$ are now known, because the effect of $s_1(\cdot)$ and $s_2(\cdot)$ are similar to that of two low-pass filters. In addition, considering low SNR and computation burden in prestack seismic data, we take deconvolution upon a whole CMP gather by an average zero-phase wavelet instead of doing it trace by trace by individual wavelets in practical seismic signal processing.

4. SIMULATIONS AND RESULTS

As shown in Fig.3(a), there are two synthetic seismic wavelets with a slight difference. One is represented by solid line and the other by dotted line. Convolving them with a random reflectivity sequence and adding 2% Gaussian white noise, two seismic traces are then obtained as shown in Fig.3(b), which are so similar that they can hardly be distinguished by eyes. In fact, there should be much difference between two symmetrical seismic traces in real CMP gather. Estimated wavelets are displayed in dotted line in Fig.3(c) and Fig.3(d) respectively. There is only a scalar difference and a very small phase shifting between synthetic wavelets and estimated ones, which don't matter for the result of seismic deconvolution. From the simulation results, It can be found that blind wavelet extraction is feasible even though there is only a slight difference between two wavelets.

Fig.4(a) shows an upper section of a *real* two-side CMP gather with rolling waves and first arriving breaks cut away. Fig.4(b) shows its high-resolution result processed by our blind prestack seismic deconvolution method. The effect of our new method for resolution enhancement is obviously good.

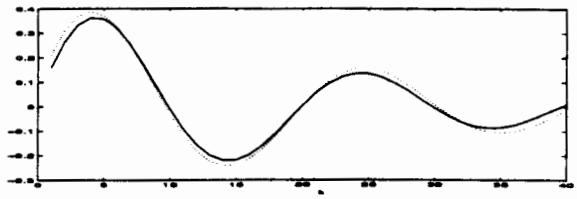

Fig.3(a) wavelet 1 (solid) and wavelet 2 (dotted)

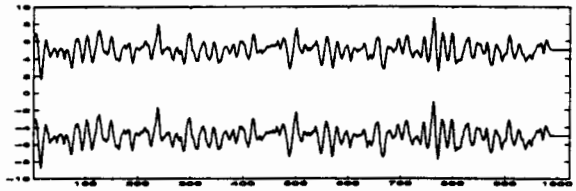

Fig.3(b) trace 1 (upper) and trace 2 (lower)

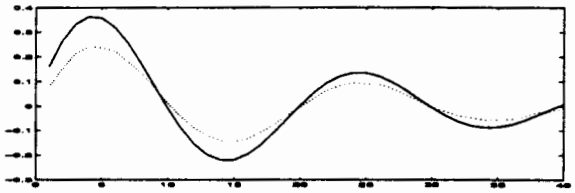

Fig.3(c) wavelet 1 (solid) and its estimation (dotted)

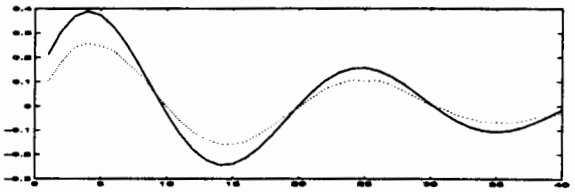

Fig.3(d) wavelet 2 (solid) and its estimation (dotted)

5. CONCLUSIONS

Simulations and results demonstrate our blind seismic deconvolution method is stable and effective. From this successful application example, it can be concluded that our dual exploration method for multi-layer structure identification is feasible in practice as a general way. The key to this method is to design twice measurements by different exploration waves at every point on the surface. If they are designed skillfully, measurement work need not be doubled.

ACKNOWLEDGEMENT

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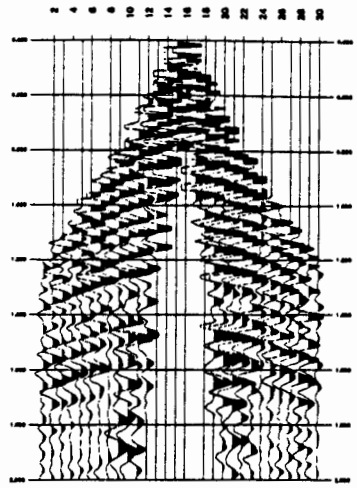

Fig.4(a) an low-resolution CMP

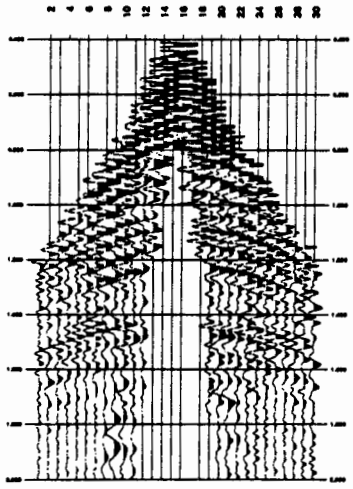

Fig.4(b) an high-resolution CMP

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Analysis of n-dimensional Nonlinear Nonuniform Grounded Infinite Grids *

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Abstract

An iterative method is devised for determining the voltage-current regime of an n-dimensional nonlinear nonuniform grounded infinite grid. This is accomplished under conditions restricting the nonlinearity and nonuniformity sufficiently to allow the operator arising from a nodal analysis to be decomposed into the sum of a Laurent operator and a nonlinear operator, which in turn can be rearranged into a contraction mapping.

1 Introduction

A structure appearing in early vision chips is a triangular grid of resistors whose nodes are excited by current sources due to an image falling upon an array of photosensitive devices (see [2] and the references therein). The grid is perforce finite and the resistors are in general nonlinear and nonidentical, that is, the grid's graph is uniform (i.e., automorphic) but its electrical elements have nonlinear characteristics which vary in general from place to place. An outstanding problem is the determination of the voltage-current regime for a given set of excitations. Since early-vision grids are large, standard nonlinear solution techniques are onerous and time-consuming for this purpose. Another approach is to replace the grid by an infinite linear uniform one, in which case the solution is easily obtained by using Laurent operators [3, Sections 7.1 to 7.3]. The use of an infinite grid is usually acceptable if the behavior at more central points of the grid is of primary interest. The assumptions of linear and uniform element characteristics are less acceptable. In this paper we present a new iterative method for solving an infinite grid whose nonlinearities and nonuniformities are sufficiently mild to allow a nodal analysis to be solved by means of a Laurent operator and a contraction mapping. The result will be a method of solution that is much more efficient than would be a standard technique for solving a large nonlinear network. Moreover, we establish herein a class of networks with possibly nonmonotone characteristics having unique operating points.

2 The general n-dimensional Grid

Consider an n-dimensional grid with nodes indexed by the integer n-tuple $j = (j_1, j_2, \ldots, j_n) \in \mathbf{Z}^n$ plus an additional ground node. Each node is connected to ground through a nonlinear conductance

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described by $i = g_j(v)$ and to a node displaced p units (where $p \in \mathbf{Z}^n$) by another nonlinear conductance described by $i = f_j^{(p)}(v)$.

Let $\mathcal{P} = \{p : p \in \mathbf{Z}^n\}$ be the finite set of integer n-tuples such that, for all j, nodes j and $j+p$ are connected by the nonzero conductance $f_j^{(p)}(\cdot)$. Since we are working with grids that have automorphic graphs, $\mathcal{P}$ does not depend on j. Also, $\mathcal{P}$ satisfies the following two conditions

1. $0^n = (0, 0, \ldots, 0) \notin \mathcal{P}$
2. if $p = (p_1, p_2, \ldots, p_n) \in \mathcal{P}$ then $-p = (-p_1, -p_2, \ldots, -p_n) \in \mathcal{P}$.

(The triangular grid shown in Figure 1 is a particular case of this structure with $n = 2$, $\mathcal{P} = \{(1,0), (0,1), (1,1), (-1,0), (0,-1), (-1,-1)\}$, and the node indices are as indicated). Of course, the conductance connected from j to $j+p$ is the same as the conductance connected from $j+p$ to j. However, since we are dealing with nonuniform grids, these conductances may vary as j and p vary. We also assume that, for each $j \in \mathbf{Z}^n$, there is a current source h_j connected from ground to each node j such that the h_j's comprise a vector $\mathbf{h} = \{h_j : j \in \mathbf{Z}^n\} \in l_{2r}^n$; l_{2r}^n denotes the real Hilbert coordinate space with coordinates indexed by $\mathbf{Z}^n$. Since Kirchhoff's current law is satisfied at each node j,

$$g_j(v_j) + \sum_{p \in \mathcal{P}} f_j^{(p)}(v_j - v_{j+p}) = h_j \tag{1}$$

for all j. (Here, $f_j^{(p)} = f_{j+p}^{(-p)}$).

3 The Conductance Operator

We now express each $f_j^{(p)}(\cdot)$ and $g_j(\cdot)$ as the sum of a linear component and a nonlinear one:

$$f_j^{(p)}(v_j - v_{j+p}) = m_{f(p)}(v_j - v_{j+p}) + \tilde{f}_j^{(p)}(v_j - v_{j+p}) \tag{2}$$

$$g_j(v_j) = m_g v_j + \hat{g}_j(v_j) \tag{3}$$

where $m_{f(p)}$ and m_g are constants independent of j. The $m_{f(p)}$'s may differ for different p, except that $m_{f(p)} = m_{f(-p)}$.

Since the linear part is chosen independent of j, it represents a uniform linear grid, and the nonlinear part represents in general a nonuniform nonlinear grid. Equation (1) can now be rewritten as

$$\sum_{p \in \mathcal{P}} [m_{f(p)}(v_j - v_{j+p}) + \tilde{f}_j^{(p)}(v_j - v_{j+p})] + m_g v_j + \hat{g}_j(v_j) = h_j. \tag{4}$$

Rearranging this, we obtain

$$(m_g + \sum_{p \in \mathcal{P}} m_{f(p)}) v_j - \sum_{p \in \mathcal{P}} m_{f(p)} v_{j+p} + \sum_{p \in \mathcal{P}} \tilde{f}_j^{(p)}(v_j - v_{j+p}) + \hat{g}_j(v_j) = h_j. \tag{5}$$

The first two terms of this equation are linear and combine into a Laurent matrix [1], [3, Section 7.1]. Let $\mathbf{y}_L$ be that matrix. The last two terms combine into a nonlinear operator $\mathbf{y}_N$. Then,

$$\mathbf{y}_L \cdot \mathbf{v} + \mathbf{y}_N(\mathbf{v}) = \mathbf{h}. \tag{6}$$

Here $\mathbf{v}, \mathbf{h} \in l_{2r}^n$ represent the node voltages and current sources respectively.

The 2n-dimensional Laurent matrix $\mathbf{y}_L = [y_{j,k}], (j, k \in \mathbf{Z}^n)$ is defined by

$$y_{j\,j+q} = \begin{cases} m_g + \sum_{p \in \mathcal{P}} m_{f(p)} & \text{if } q = 0, \\ -m_{f(q)} = -m_{f(-q)} & \text{if } q \in \mathcal{P}, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

The nonlinear operator $\mathbf{y}_N : l^n_{2r} \rightarrow l^n_{2r}$ is defined by

$$\mathbf{y}_N(\mathbf{v}) = [\hat{g}_j(v_j) + \sum_{p\in\mathcal{P}} \tilde{f}^{(p)}_j(v_j - v_{j+p})]_j, \tag{8}$$

where we denote a vector $\mathbf{x} \in l^n_{2r}$ as $\mathbf{x} = [x_j]_j, j \in \mathbf{Z}^n$, and x_j are the components of $\mathbf{x}$.

Let $[-\pi, \pi]^n$ denote the Cartesian product of n replicates of the real interval $[-\pi, \pi]$ and let $\theta \in [-\pi, \pi]^n$. Also, let multiplication by $Y_L(\theta)$ be the image of the operator $\mathbf{y}_L$ under the n-dimensional Fourier transformation of node-voltage vectors (See [3, Section 7.1]). Thus,

$$Y_L(\theta) = (m_g + \sum_{p\in\mathcal{P}} m_{f(p)}) - \sum_{p\in\mathcal{P}} m_{f(p)} cos(p \cdot \theta), \tag{9}$$

where $(p \cdot \theta)$ denotes the dot product.

Assume $m_{f(p)} > 0 \ \forall p$. We can define

$$\|\mathbf{y}_L^{-1}\| = \frac{1}{\inf_\theta Y_L(\theta)} = \frac{1}{m_g}. \tag{10}$$

We can then choose $m_g > 0$ so that $\mathbf{y}_L^{-1}$ exists. Equation (6) can be changed to

$$\mathbf{v} + \mathbf{y}_L^{-1}\mathbf{y}_N(\mathbf{v}) = \mathbf{y}_L^{-1}\mathbf{h}. \tag{11}$$

Therefore, we have

$$\mathbf{v} = \mathbf{y}_L^{-1}\mathbf{h} - \mathbf{y}_L^{-1}\mathbf{y}_N(\mathbf{v}) = \mathbf{d}_h(\mathbf{v}). \tag{12}$$

If $\mathbf{y}_L^{-1}\mathbf{y}_N$ is a contraction on $\mathbf{v}$ then so too is $\mathbf{d}_h$, and we can find a solution using the fixed point theorem and the usual iteration.

Now consider $\mathbf{y}_N$. Let $\mathbf{w}, \mathbf{x} \in l^n_{2r}$.

$$\|\mathbf{y}_N(\mathbf{w}) - \mathbf{y}_N(\mathbf{x})\| = \|[\hat{g}_j(w_j) + \sum_{p\in\mathcal{P}} \tilde{f}^{(p)}_j(w_j - w_{j+p})]_j - [\hat{g}_j(x_j) + \sum_{p\in\mathcal{P}} \tilde{f}^{(p)}_j(x_j - x_{j+p})]_j\|. \tag{13}$$

Applying the triangle inequality, we have

$$\|\mathbf{y}_N(\mathbf{w}) - \mathbf{y}_N(\mathbf{x})\| \leq \|[\hat{g}_j(w_j) - \hat{g}_j(x_j)]_j\| + \sum_{p\in\mathcal{P}} \|[\tilde{f}^{(p)}_j(w_j - w_{j+p}) - \tilde{f}^{(p)}_j(x_j - x_{j+p})]_j\|. \tag{14}$$

We assume a Lipschitz condition on $\hat{g}_j$ and $\tilde{f}^{(p)}_j$ uniformly for all j. In particular, we assume $\forall \alpha_j, \beta_j \in \mathbf{R}$ and $\forall j \in \mathbf{Z}^n$, that

$$|\hat{g}_j(\alpha_j) - \hat{g}_j(\beta_j)| \leq \gamma_{\hat{g}} |\alpha_j - \beta_j| \tag{15}$$

and

$$|\tilde{f}^{(p)}_j(\alpha_j) - \tilde{f}^{(p)}_j(\beta_j)| \leq \gamma_{f(p)} |\alpha_j - \beta_j|, \tag{16}$$

where $\gamma_{f(p)}$ and $\gamma_{\hat{g}}$ are positive real constants. Applying the triangle inequality again, we see that $\mathbf{y}_L^{-1}\mathbf{y}_N$ satisfies a Lipschitz condition:

$$\|\mathbf{y}_L^{-1}\mathbf{y}_N(\mathbf{w}) - \mathbf{y}_L^{-1}\mathbf{y}_N(\mathbf{x})\| \leq \frac{\gamma_{\hat{g}} + 2\sum_{p\in\mathcal{P}} \gamma_{f(p)}}{m_g} \|\mathbf{w} - \mathbf{x}\|. \tag{17}$$

If

$$k = \frac{\gamma_{\hat{g}} + 2\sum_{p\in\mathcal{P}} \gamma_{f(p)}}{m_g} \tag{18}$$

is less than 1, then both $\mathbf{y}_L^{-1}\mathbf{y}_N$ and $\mathbf{d}_h$ are contractions. This allows an iterative solution for the grid in the standard way.

4 Selecting the Linear Part of the Conductance Operator

Let $a_f^{(p)}$ and a_g be the infima of the slopes of all the chords of all the $f_j^{(p)}$ and all the g_j respectively. These are finite by our assumption of uniform Lipschitz conditions. Similarly, let $b_f^{(p)}$ and b_g be the corresponding suprema, also finite. For every $p \in \mathcal{P}$, set

$$m_{f(p)} = \frac{1}{2}(a_f^{(p)} + b_f^{(p)}), \tag{19}$$

$$m_g = \frac{1}{2}(a_g + b_g). \tag{20}$$

Then, we will have

$$\gamma_{f(p)} = \frac{1}{2}(b_f^{(p)} - a_f^{(p)}), \tag{21}$$

$$\gamma_g = \frac{1}{2}(b_g - a_g). \tag{22}$$

It can be shown that with these choices the contraction mapping constant k given by (18) assumes its lowest possible value and that value is less than 1.

References

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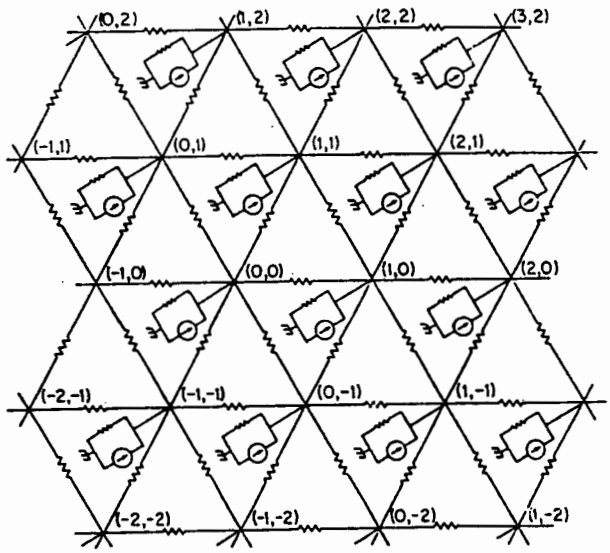

FIG. 1