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A NOTE ON DIFFUSION IN A STRONGLY UNSTABLE TURBULENT BOUNDARY LAYER

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Abstract

An entirely Lagrangian similarity analysis is used to determine the mean particle trajectory and mean ground level concentration resulting from continuous sources located at the origin in a strongly unstable turbulent boundary layer. The results obtained reduce to previous free convection results for $L \to -0$, where L is the Monin-Obukhov characteristic length. Specific predictions cannot be made since no data appears to be available that would enable the unknown universal constants in the equations to be evaluated.

A Note on Diffusion in a Strongly Unstable Turbulent Boundary Layer

1. Introduction

In this paper particle trajectory and mean ground level concentration results are obtained for diffusion in a strongly unstable turbulent boundary layer. Batchelor (1964) obtained trajectory and concentration results for the constant stress region of a neutrally stratified boundary layer and Mandell and O'Brien (1966) obtained the corresponding results for a slightly unstable boundary layer. The trajectory equations for thermal turbulence are given by Yaglom (1965).

The basic equations used herein, which were discussed by Mandell and O'Brien (loc cit) were obtained from dimensional reasoning by assuming Lagrangian similarity. The first order corrections to thermal turbulence given here depend upon the Monin-Obukhov characteristic length L. For $L \rightarrow -0$ the results reduce to those given by Yaglom.

2. Lagrangian Equations

The trajectory equations given by Mandell and O'Brien for the downwind and vertical distances $\bar{\bar{x}}$ and $\bar{\bar{z}}$ are, respectively:

$$\frac{d^2 \bar{\bar{x}}}{d t^2} = A \left(\frac{u_* t}{L} \right) \frac{u_*}{t} \qquad (1)$$

$$\frac{d^2 \bar{\bar{z}}}{d t^2} = \left(\frac{u_*^3}{t |L|} \right)^{\frac{1}{2}} f \left(\frac{u_* t}{L} \right) \qquad (2)$$

where A and f are two unknown universal functions, $u_* \left[= \left(\frac{\tau}{\rho_o} \right)^{\frac{1}{2}} \right]$ is the friction velocity, $L \left(= \frac{- u_*^3}{\frac{k\, q g}{c_p \rho_o T_o}} \right)$ is the Monin-Obukhov characteristic length, t is the time since the particle left the origin, τ is the constant shear stress; c_p, ρ_o and T_o are the reference specific heat, density and temperature respectively, k is von Kàrmàn's constant, g is the acceleration of gravity and q is the constant vertical heat flux, positive upward.

When $L \rightarrow -0$, $\frac{u_*}{L} \rightarrow -\infty$, $\frac{u_*^3}{L} \rightarrow$ a constant and equations (1) and (2) become identical to Yaglom's equations

$$\frac{d^2 \bar{x}}{d t^2} = 0$$

$$\frac{d^2 \bar{z}}{d t^2} = c \left(\frac{q g}{c_p \rho_o T_o} \right)^{\frac{1}{2}} \frac{1}{t^{1/2}}$$

where $c = \frac{4}{3} k^{\frac{1}{2}} f(-\infty)$ is a constant introduced by Yaglom.

2.1 Solution of Equations for Strongly Unstable Conditions

u_* has no physical significance in a strongly unstable boundary layer and therefore by using the definition of L, equations (1) and (2) can be rewritten in the following manner:

$$\frac{d^2\bar{x}}{dt^2} = \frac{-k^{\frac{1}{3}} L^{\frac{1}{3}}}{t} \left(\frac{q\,g}{c_p \rho_o T_o}\right)^{\frac{1}{3}} A_1 \left(\frac{L^{2/3}}{-k^{1/3} \left(\frac{q\,g}{c_p \rho_o T_o}\right)^{1/3} t} \right) \tag{3}$$

$$\frac{d^2\bar{z}}{dt^2} = k^{\frac{1}{2}} \left(\frac{q\,g}{c_p \rho_o T_o}\right)^{\frac{1}{2}} \frac{1}{t^{1/2}} \; f_1 \left(\frac{L^{2/3}}{-k^{1/3} \left(\frac{q\,g}{c_p \rho_o T_o}\right)^{1/3} t} \right) \tag{4}$$

where A_1 and f_1 are new unknown universal functions.

If it is assumed that A_1 and f_1 are analytic functions in a neighborhood about $\frac{L^{2/3}}{k^{1/3} \left(\frac{q\,g}{c_p \rho_o T_o}\right)^{1/3} t} = 0$, then A_1 and f_1 can be expanded in Taylor series about the zero value of their arguments. Equations (3) and (4) become

$$\frac{d^2\bar{x}}{dt^2} = -c_1 \left(\frac{q\,g}{c_p \rho_o T_o}\right)^{\frac{1}{3}} \frac{L^{\frac{1}{3}}}{t} + 0\left[\left\{ \frac{L^{\frac{2}{3}}}{\left(\frac{q\,g}{c_p \rho_o T_o}\right)^{\frac{1}{3}} t} \right\}^2 \right] \tag{5}$$

$$\frac{d^2\bar{z}}{dt^2} = c_2 \left(\frac{q\,g}{c_p \rho_o T_o}\right)^{\frac{1}{2}} \frac{1}{t^{1/2}} - c_3 \left(\frac{q\,g}{c_p \rho_o T_o}\right)^{\frac{1}{6}} \frac{L^{\frac{2}{3}}}{t^{\frac{3}{2}}} + 0\left[\left\{ \frac{L^{\frac{2}{3}}}{\left(\frac{q\,g}{c_p \rho_o T_o}\right)^{\frac{1}{3}} t} \right\}^2 \right] \tag{6}$$

where $c_1 = k^{\frac{1}{3}} A_1(0)$, $c_2 = k^{\frac{1}{2}} f_1(0)$ and $c_3 = k^{\frac{1}{6}} f_1'(0)$.

Integration of equations (5) and (6) twice with respect to t gives to $0\left[\frac{L^{2/3}}{\left(\frac{q\,g}{c_p \rho_o T_o}\right)^{1/3} t} \right]$

$$\bar{x}(t) = U_o t - c_1 \left(\frac{q g}{c_p \rho_o T_o} \right)^{\frac{1}{3}} L^{\frac{1}{3}} t \left(\log \frac{t}{t_o} - 1 \right) \qquad (7)$$

$$\bar{z}(t) = c \left(\frac{q g}{c_p \rho_o T_o} \right)^{\frac{1}{2}} t^{\frac{3}{2}} + 4 c_2 L^{\frac{2}{3}} \left(\frac{q g}{c_p \rho_o T_o} \right)^{\frac{1}{6}} t^{\frac{1}{2}} \qquad (8)$$

where the constants of integration were chosen so that the source height is zero, the $\bar{z}$ velocity is zero for $q = 0$, t_o is the time at which the downwind velocity equals U_o and c and U_o are universal constants introduced by Yaglom.

Equation (8) is a cubic equation in $t^{\frac{1}{2}}$ which has one real root

$$t(\bar{z}) = \left(\frac{c_p \rho_o T_o}{q g} \right)^{\frac{1}{3}} \left(\frac{\bar{z}}{c} \right)^{\frac{2}{3}} \left[1 - \frac{8}{3} \frac{c_2}{c^{1/3}} \left(\frac{L}{\bar{z}} \right)^{\frac{2}{3}} \right] \qquad (9)$$

Substitution of equation (9) into equation (7) gives for the mean particle trajectory

$$\bar{x}(\bar{z}) = U_o \left(\frac{c_p \rho_o T_o}{q g} \right)^{\frac{1}{3}} \left(\frac{\bar{z}}{c} \right)^{\frac{2}{3}} - \frac{c_1}{c^{2/3}} \left(\frac{L}{\bar{z}} \right)^{\frac{1}{3}} \bar{z} \left\{ \log \left[\left(\frac{c_p \rho_o T_o}{q g} \right)^{\frac{1}{3}} \frac{\bar{z}^{\frac{2}{3}}}{c^{\frac{2}{3}} t_o} \right] - 1 \right\}$$

$$+ 0 \left[\left(\frac{L}{\bar{z}} \right)^{\frac{2}{3}} \right] \qquad (10)$$

3. Mean Ground Level Concentration

The mean concentration for a point source is taken to be

$$C_p(x, y, z) = Q \int_0^\infty \frac{1}{\bar{z}^3} \psi\left(\frac{x - \bar{x}}{\bar{z}}, \frac{y}{\bar{z}}, \frac{z}{\bar{z}}, \frac{L^{\frac{2}{3}}}{\left(\frac{q\, g}{c_p \rho_o T_o}\right)^{1/3} t}\right) dt$$

where Q is the rate of release of particles from the source and ψ is the probability that a particle will be at position x, y, z at time t. If ψ is expanded in a Taylor series about $\frac{L^{2/3}}{\left(\frac{q\, g}{c_p \rho_o T_o}\right)^{1/3} t} = 0$ and t is replaced by using equation (9), then to $0[(\frac{L}{\bar{z}})^{\frac{1}{3}}]$, ψ is independent of $\frac{L}{\bar{z}}$. The concentration at ground level from a point source, by using Batchelor's method is

$$C_p(x, 0, 0) = \frac{Q}{\left[\bar{z}^2\left\{U_o - \left(\frac{q\, g}{c_p \rho_o T_o}\right)^{1/3} c_1 \left(\frac{L}{\bar{z}}\right)^{1/3} \bar{z}^{1/3} \log\left[\left(\frac{c_p \rho_o T_o}{q\, g}\right)^{1/3} \frac{\bar{z}^{2/3}}{c^{2/3} t_o}\right]\right\}\right]_{\bar{x}=x}}$$

and a similar expression for the concentration from a line source $C_\ell(x, 0)$ with $\bar{z}^2$ replaced by $\bar{z}$.

Discussion

The authors have found no data that would enable the unknown universal constants c, c_1 and c_2 to be evaluated. Thus no quantitative results can be obtained from the derived equations.

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