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Spatial Traffic Intensity in One and Two Dimenionsal Networks

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Abstract

Means to calculate the spatial intensity of network traffic in one and two dimensional networks under shortest path routing are presented. Both uniformly geographically distributed traffic and traffic exhibiting geographic preference are considered. A surprising result is that traffic intensity in a network covering a circular area under geographic traffic uniformity and shortest path routing is a simple spatial quadratic function.

Keywords

Traffic intensity, spatial distribution, locality, preference, networks.

I. Introduction

Network traffic intensity can vary as a function of time and space. The statistical characterization of variations in time has received an increasing amount of attention in recent years [2]. In this paper a different aspect of traffic intensity variation, spatial variation, is examined. We think this is important as there are, to our knowledge, no baseline analytical studies of this topic. It has implications for our understanding of sizing and dimensioning networks. Using generic and canonical assumptions of topology, traffic uniformity and locality, and shortest path routing, we describe means to calculate network traffic intensity as a function of space. Because of the generic nature of the assumptions, this work is applicable to both circuit switched and packet switched networks.

A surprising result of this work is that in spite of a fairly involved derivation, traffic intensity in a network covering a circular area under assumptions of uniformity and shortest path routing, is a quadratic function of spatial position.

This article is organized as follows. Under a traffic uniformity assumption, one dimensional linear network are discussed in section II and two dimensional circular networks are discussed in section III. Calculating spatial traffic intensity for discrete and continuous network models with geographic traffic locality and preference is considered in section IV. A specific case study of the effect of geographic locality on a linear, discrete network is presented in section V. The conclusion appears in section VI.

II. Linear Network Case

A. Discrete Case

It is assumed that the distances between adjacent nodes are identical and traffic follows a uniform distribution. That is, every pair of nodes has the same amount of traffic flowing between the two nodes. Then:

$$\begin{aligned} & Traffic\ \ intensity\ in\ link\ x \\ = \ & 2(\#\ of\ nodes\ to\ the\ left\ of\ x)(\#\ of\ nodes\ to\ the\ right\ of\ x) \end{aligned} \tag{1}$$

Here bidirectional traffic is assumed (accounting for the "two" in (1)).

B. Continuous Case

Similarly, if a linear network is defined on the interval (0,L), then:

$$Traffic\ \ intensity\ at\ point\ x = x \cdot (L - x) = Lx - x^2 \tag{2}$$

This quadratic function is maximized at:

$$\begin{aligned} \frac{d}{dx}(Lx - x^2) &= L - 2x = 0 \\ x &= \frac{L}{2} \end{aligned} \tag{3}$$

III. Two Dimensional Circular Network Case

It is assumed that the network covers a circular region with radius R. There are two equations for calculation of network traffic density which are the line $y = ax + b$ and the circular network boundary equation $x^2 + y^2 = R^2$. Traffic is uniformly distributed between all pairs of points in the circular region. Traffic always follows a shortest path (straight line) route. To calculate the traffic "intensity" at an arbitrary point, Z, in the circular region, one can place a line through the point. Then there is an one dimensional problem involving traffic generated between pairs of points on either side of Z on the line and passing through Z. The line is rotated 180^o about Z and the intensity at Z is integrated. Fig. 1 can be referred to for the following steps.

As already mentioned, for calculating the amount of traffic at an arbitrary point (x, y), the distances of d_1 and d_2 must be found. By multiplying these two distances, one obtains

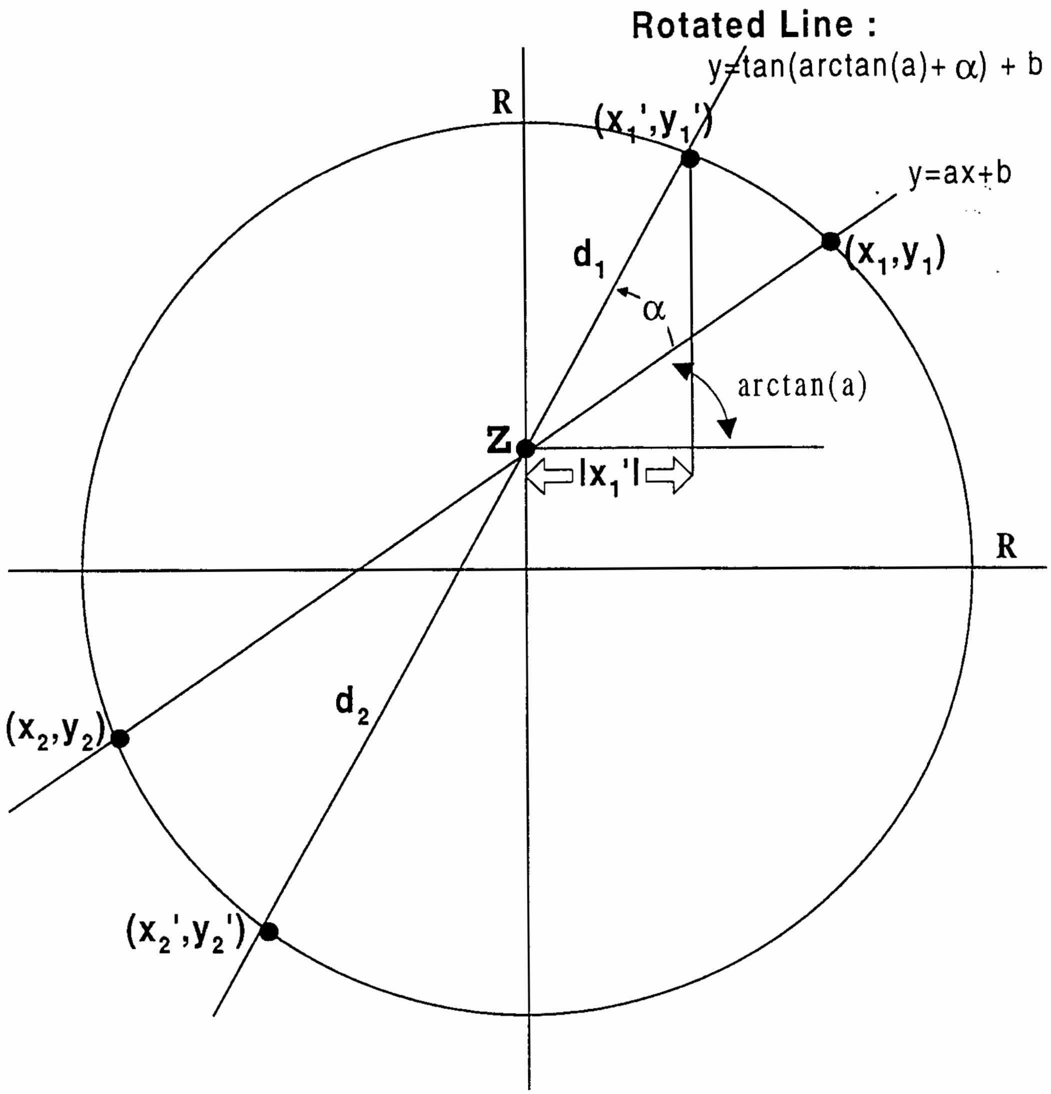

Fig. 1. Network topology

an amount of traffic intensity along the linear network passing through Z. This varies depending on the line position. Without loss of generality, it is assumed that $Z = (0, b)$ is on the line segment from (0,0) to (0,R).

$$d_1 = \frac{|x_1'|}{\cos(\alpha + \arctan(a))}, \quad d_2 = \frac{|x_2'|}{\cos(\alpha + \arctan(a))} \tag{4}$$

The rotated line which has a rotational angle α is found as (see Fig. 1),

$$y = \tan(\arctan a + \alpha)x + b \tag{5}$$

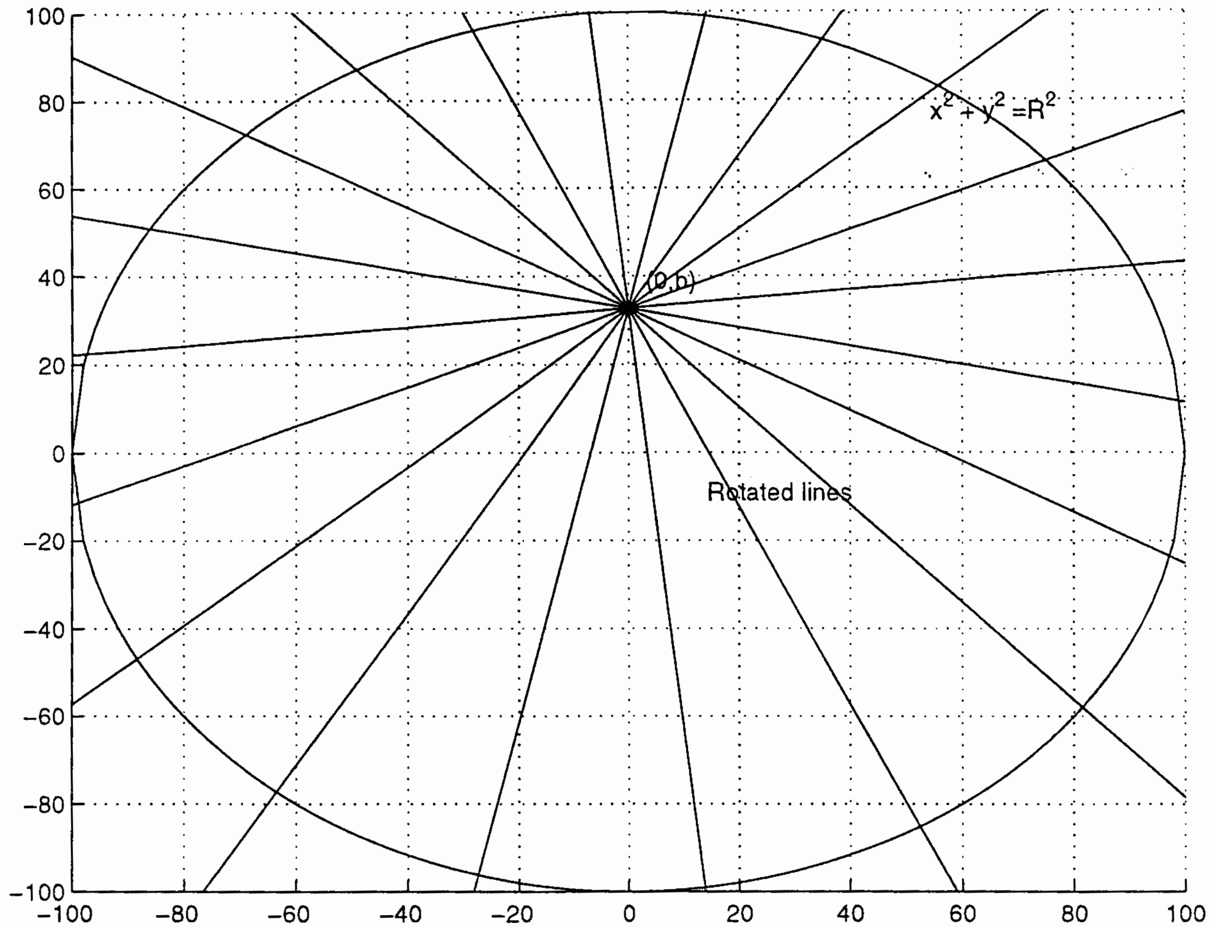

Fig. 2. Rotated lines on the circular network

In order to get the crossing points between the rotated line and the circle, substitute (5) into $x^2 + y^2 = R^2$. Then,

$$\begin{aligned} x'_{1,2} &= \frac{-b\tan(\arctan a + \alpha) \pm \sqrt{b^2\tan^2(\arctan a + \alpha) - (1 + \tan^2(\arctan a + \alpha))(b^2 - R^2)}}{1 + \tan^2(\arctan a + \alpha)} \\ &= \frac{-b\tan(\arctan a + \alpha) \pm \sqrt{R^2\tan^2(\arctan a + \alpha) + R^2 - b^2}}{1 + \tan^2(\arctan a + \alpha)} \end{aligned} \quad (6)$$

Note that Z is assumed to be at $(0, b)$. Then the traffic intensity I_{linear} at $(0, b)$ along the linear component of the network is,

$$\begin{aligned} I_{linear} &= 2 \times d_1 \times d_2 \quad (7) \\ &= 2\frac{1}{\cos^2(\arctan a + \alpha)}|x'_1||x'_2| \end{aligned}$$

$$
\begin{aligned}
&= 2\frac{1}{\cos^2(\arctan a+\alpha)}\left|\frac{b^2\tan^2(\arctan a+\alpha)-R^2\tan^2(\arctan a+\alpha)+b^2-R^2}{[1+\tan^2(\arctan a+\alpha)]^2}\right| \\
&= 2\frac{1}{\cos^2(\arctan a+\alpha)}\left|\frac{(b^2-R^2)\tan^2(\arctan a+\alpha)+b^2-R^2}{[1+\tan^2(\arctan a+\alpha)]^2}\right| \\
&= 2\frac{1}{\cos^2(\arctan a+\alpha)}\left|\frac{b^2-R^2}{\frac{1}{\cos^2(\arctan a+\alpha)}}\right| \\
&= 2\left|b^2-R^2\right| \\
&= 2(R^2-b^2) \quad , since \;\; R\geq b \qquad (8)
\end{aligned}
$$

This implies that the traffic intensity along the line is independent of its rotational angle. The total traffic intensity I_{total} at $Z=(0,b)$ is the integration of the above equation by α which is varying from 0 to π centered at $(0,b)$, that is,

$$
\begin{aligned}
I_{total} &= \int_0^{\pi} 2(R^2-b^2)d\alpha \\
&= 2(R^2-b^2)\cdot\pi \qquad (9)
\end{aligned}
$$

The density of traffic inside the circular network is a quadratic function with a maximum of πR^2 at b=0 (network center) and zero intensity at the boundary.

Fig. 3 shows the quadratic traffic intensity as one moves from the boundary through the center of the circular network and out toward the opposite boundary. Also, the simulated result from [1] is consistent with the traffic distribution of (9).

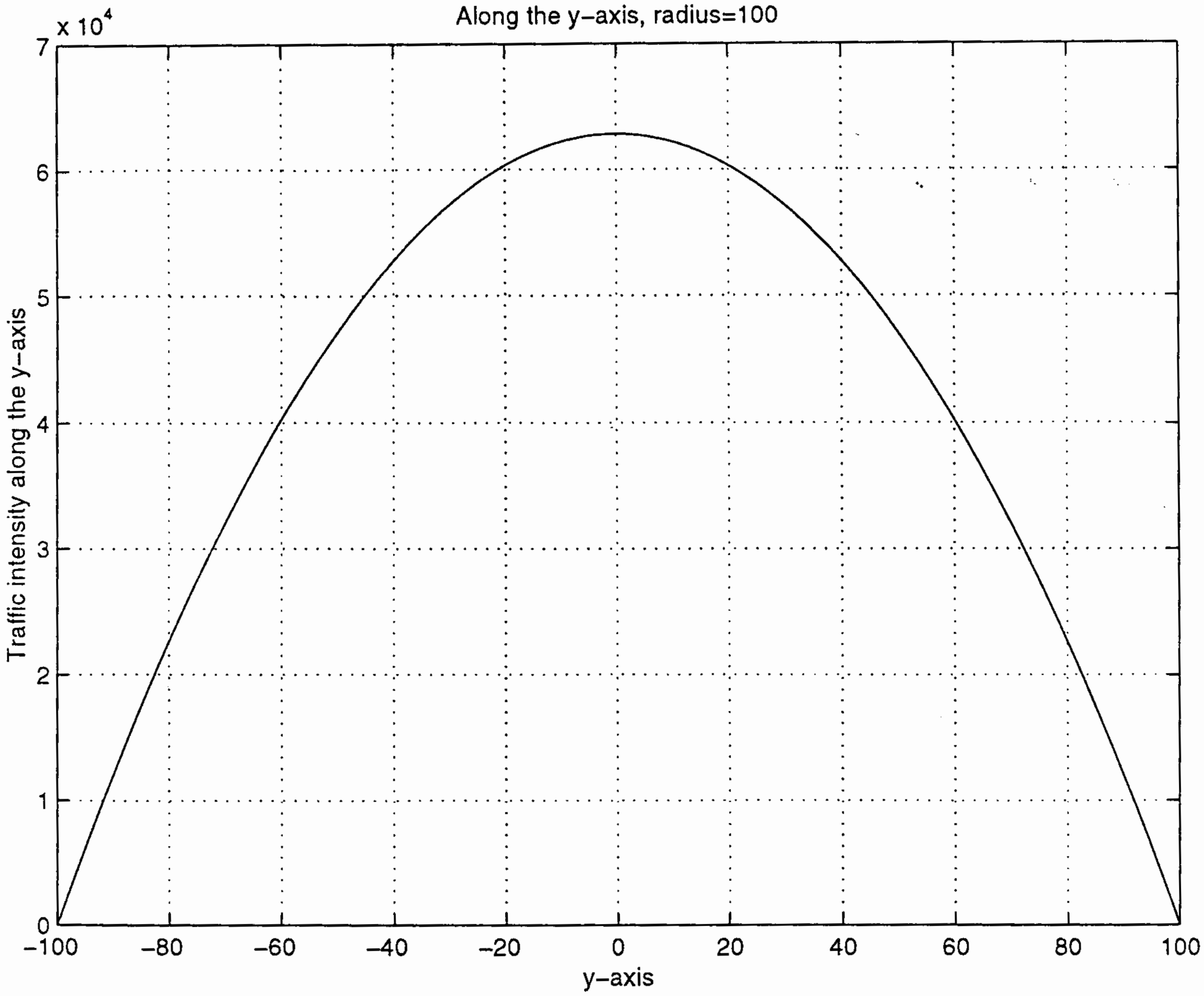

Fig. 3. Traffic distribution along the y-axis

IV. Locality and Preference

Traffic intensity may vary spatially because of various reasons. A basic model is that a node has a higher probability of communication with nearby nodes than with nodes far from that node. We call this "locality". However the analytical machinery developed model below can model any type of traffic preference (or attractivity) as well as basic geographic traffic locality.

A. Linear Network Case

We will start with the simple linear network case.

A.1 Discrete Case

There are N nodes in the linear network. Then,

$$
\begin{aligned}
&Total\ amount\ of\ traffic\ at\ link\ x \\
&\quad = amount\ of\ traffic\ from\ node\ 1 + amount\ of\ traffic\ from\ node\ 2 \\
&\quad + \cdots + amount\ of\ traffic\ from\ node\ N
\end{aligned} \tag{10}
$$

Let's define the preference probability mass function (p.m.f.) for node j, $p_j(i)$, which indicates the probabilities of communication from node j to all other node i's and,

$$\sum_{i=1}^{N} p_j(i) = 1 \tag{11}$$

Also, $P_j(i)$ indicates the amount of traffic from node j to node i instead of probability. If node j generates a total of N_j units of traffic, then its preference vector $\mathbf{P_j} = N_j\ \mathbf{p_j}$ or $P_j(i) = N_j p_j(i)$ for node i and,

$$\sum_{i=1}^{N} P_j(i) = \sum_{i=1}^{N} N_j p_j(i) = K_j \tag{12}$$

If $K_j = N - 1$, then all nodes generate traffic uniformly and there is no self traffic.

Let's define $T_l(x)$ to be the traffic intensity in link x with preference. Also, $T_l(x)^{left}$ is the amount of traffic from the nodes located to the left of the link x, and calculated by summing the traffic from each node to the left of link x that is going to nodes to the right of link x (see Fig. 4). That is,

$$
\begin{aligned}
T_l(x)^{left} &= \sum_{i=\lceil x \rceil}^{N} P_1(i) + \sum_{i=\lceil x \rceil}^{N} P_2(i) + \cdots + \sum_{i=\lceil x \rceil}^{N} P_{\lceil x \rceil - 1}(i) \\
&= \sum_{j=1}^{\lceil x \rceil - 1} \sum_{i=\lceil x \rceil}^{N} P_j(i)
\end{aligned} \tag{13}
$$

where node $(\lceil x \rceil - 1)$ is the node located to the left of the link x and $\lceil x \rceil$ is the node located to the right of the link x. This equation yields the amount of traffic at link x. Also, traffic passing through link x from node j's which are located to the right of link x, $T_l(x)^{right}$, is,

$$T_l(x)^{right} = \sum_{j=\lceil x \rceil}^{N} \sum_{i=1}^{\lceil x \rceil - 1} P_j(i) \tag{14}$$

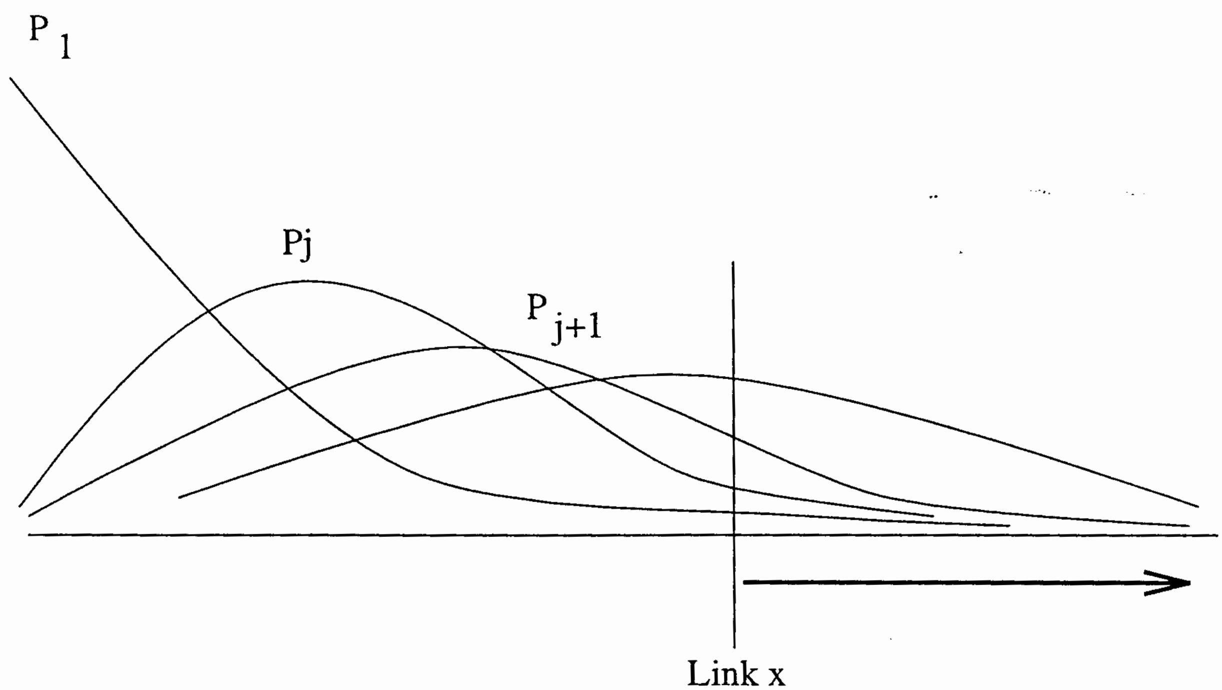

Fig. 4. Traffic passing through the link x from the nodes located to the left.

Thus, total amount of traffic in link x, $T(x)$,

$$\begin{aligned} T(x) &= T_l(x)^{left} + T_l(x)^{right} \\ &= \sum_{j=1}^{\lceil x \rceil - 1} \sum_{i=\lceil x \rceil}^{N} P_j(i) + \sum_{j=\lceil x \rceil}^{N} \sum_{i=1}^{\lceil x \rceil - 1} P_j(i) \\ &= \sum_{m=1}^{\lceil x \rceil - 1} \Big(\sum_{i=\lceil x \rceil}^{N} P_m(i) + \sum_{j=\lceil x \rceil}^{N} P_j(m) \Big) \\ &= \sum_{m=1}^{\lceil x \rceil - 1} \sum_{n=\lceil x \rceil}^{N} (P_m(n) + P_n(m)) \end{aligned} \tag{15}$$

If $P_m(n) = P_n(m)$ (i.e. traffic flows from node m to node n is equal to the traffic from node n to node m), then:

$$T(x) = 2 \sum_{m=1}^{\lceil x \rceil - 1} \sum_{n=\lceil x \rceil}^{N} P_m(n) \tag{16}$$

A.2 Continuous case

If a linear network is defined on the interval (0,L), then the preference probability density function (pdf) for point x is p_x and,

$$\int_o^L p_x(l)dl = 1 \tag{17}$$

Similar to the discrete case, the preference function for point x has the following property,

$$\int_0^L P_x(l)dl = L(x) \tag{18}$$

If each node generates uniform traffic, $L(x) = L$. The amount of traffic at link x (or point x) from the nodes to the left of link x (from 0 to x-ϵ) of the link x, $T_l(x)^{left}$, is,

$$T_l(x)^{left} = \int_0^{x-\epsilon} \int_x^L P_{x'}(l)dldx', \quad \epsilon > 0 \tag{19}$$

where ϵ is an extremely small distance. The amount of traffic at link x from the nodes to the right of the link x, $T_l(x)^{right}$, is

$$T_l(x)^{right} = \int_{x+\epsilon}^L \int_0^x P_{x'}(l)dldx', \quad \epsilon > 0 \tag{20}$$

The total amount of traffic passing thru link x, $T_l(x)$, is,

$$\begin{aligned} T_l(x) &= T_l(x)^{left} + T_l(x)^{right} \\ &= \int_0^{x-\epsilon} \int_x^L P_{x'}(l)dldx' + \int_{x+\epsilon}^L \int_0^x P_{x'}(l)dldx' \end{aligned} \tag{21}$$

B. Two Dimensional Case

B.1 Discrete case

If there are a total of M $\times$ N nodes, then the preference probability mass function or locality matrix $\mathbf{p_{ij}}$ for node (i,j) is

$$\sum_{m=1}^{M} \sum_{n=1}^{N} p_{ij}(m,n) = 1 \tag{22}$$

Here $p_{ij}(m,n)$ is the probability of communication from node (i,j) to node (m,n). For example, the preference matrix showing the distribution of locality for node (1,1) is

$$\mathbf{p_{11}} = \begin{bmatrix} 0 & p_{11}(1,2) & p_{11}(1,3) & \cdots & p_{11}(1,N) \\ p_{11}(2,1) & p_{11}(2,2) & p_{11}(2,3) & \cdots & p_{11}(2,N) \\ \vdots & & & \ddots & \vdots \\ p_{11}(M,1) & p_{11}(M,2) & p_{11}(M,3) & \cdots & p_{11}(M,N) \end{bmatrix} \tag{23}$$

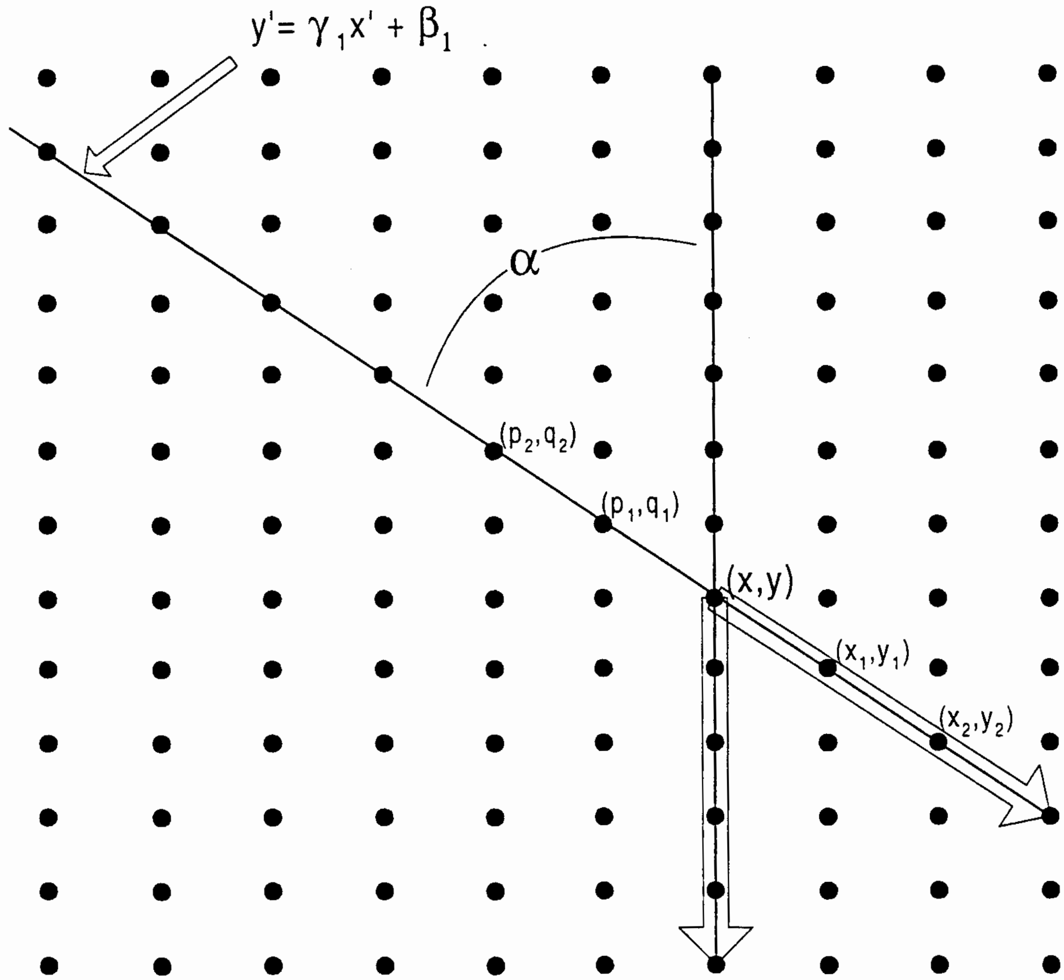

Fig. 5. Traffic to link (x,y) along the linear paths in a discrete network

Here it is assumed that there is no self-traffic. The preference function, $P_{ij}(m,n)$, indicates the amount of traffic from node (i,j) to node (m,n).

$$\sum_{m=1}^{M}\sum_{n=1}^{N} P_{ij}(m,n) = K_{ij} \tag{24}$$

where K_{ij} is the total amount of traffic generated by node (i,j). It is assumed that the routing algorithm uses shortest paths. To find the amount of traffic at link (x,y) from node (p,q), a line equation from (p,q) to (x,y) is required, that is,

$$y' = \frac{y-q}{x-p}x' + \frac{xq-yp}{x-p} \quad (25)$$

$$= \gamma x' + \beta \qquad where,\ \gamma = \frac{y-q}{x-p}\ and\ \beta = \frac{xq-yp}{x-p}. \quad (26)$$

According to Fig. 5, the amount of traffic from (p_1,q_1) to link (x,y), $T_{p_1,q_1}(x,y)$, along the line ($y' = \gamma_1 x' + \beta_1$, where $\gamma_1 = \frac{y-q_1}{x-p_1}$ and $\beta_1 = \frac{xq_1-yp_1}{x-p_1}$) consists of the traffic passing through (x,y) from node (p_1,q_1) to node (x_1,y_1) and the traffic passing through (x,y) from (p_1,q_1) to (x_2,y_2) and so on.

$$T_{p_1,q_1}(x,y) = P_{p_1,q_1}(x_1,y_1) + P_{p_1,q_1}(x_2,y_2) + P_{p_1,q_1}(x_3,y_3) + \cdots$$
$$\cdots + P_{p_1,q_1}(the\ end\ of\ line) \quad (27)$$

Here 'the end of line' means a crossing point with the network boundary and the line. All of the nodes along the line follow the line equation $y' = \gamma_1 x' + \beta_1$, so a point (p_1,q_1) can be written as $(p_1, \gamma_1 p_1 + \beta_1)$ and also (x_1,y_1) can be written as $(x_1, \gamma_1 x_1 + \beta_1)$. Then (27) can be written as

$$T_{p_1,q_1}(x,y) = P_{p_1,\gamma_1 p_1+\beta_1}(x_1, \gamma_1 x_1 + \beta_1) + P_{p_1,\gamma_1 p_1+\beta}(x_2, \gamma_1 x_2 + \beta_1) + \cdots$$
$$\cdots + P_{p_1,\gamma_1 p_1+\beta_1}(x_3, \gamma_1 x_3 + \beta_1) + P_{p_1,\gamma_1 p_1+\beta_1}(the\ end\ of\ line)$$
$$= \sum_{i=1}^{the\ end\ of\ line} P_{p_1,\gamma_1 p_1+\beta_1}(x_i, \gamma_1 x_i + \beta_1) \quad (28)$$

The traffic to link (x,y) from (p_2,q_2) which is on the same line will be,

$$T_{p_2,q_2}(x,y) = \sum_{i=1}^{the\ end\ of\ line} P_{p_2,\gamma_1 p_2+\beta_1}(x_i, \gamma_1 x_i + \beta_1) \quad (29)$$

Then, we have to sum each P_{p_i,q_i} which affects the link (x,y) along the line $y = \gamma_1 x + \beta_1$ to obtain the amount of traffic on link (x,y), $T(x,y)$,

$$T(x,y) = \sum_{j=1}^{(The\ other\ end\ of\ line)} \sum_{i=1}^{(The\ end\ of\ line)} P_{p_j,\gamma_1 p_j+\beta_1}(x_i, \gamma_1 x_i + \beta_1) \quad (30)$$

Finally, in order to calculate the total traffic amount, $T_{total}(x,y)$, rotate the line by α from 0 up to 2π while repeating the above procedure. Here the reason for using 2π instead of

π is to consider bi-directional traffic. The traffic intensity at link (x, y) can be determined as,

$$T_{total}(x,y) = \sum_{\alpha=0}^{2\pi} \sum_{j=1}^{(The\ other\ end\ of\ line)} \sum_{i=1}^{(The\ end\ of\ line)} P_{p_j,\gamma_\alpha p_j+\beta_\alpha}(x_i, \gamma_\alpha x_i + \beta_\alpha) \tag{31}$$

Here the first summation is approximate in an angular sense.

B.2 Continuous case

The continuous two-dimensional preference probability density function for node (p, q) is,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{pq}(x,y)dxdy = 1 \tag{32}$$

If the network is a circular network and the amount of traffic from a node (p, q) to the all other nodes $(x, y)'s$ is πR^2, then the preference function, F_{pq}, for node (p, q) has the following property:

$$\int_{-R}^{R}\int_{-R}^{R} F_{pq}(x,y)dxdy = L_{p,q} \tag{33}$$

If the preference is uniform, then $F_{pq}(x, y) = 1$, or $f_{pq}(x, y) = \frac{1}{\pi R^2}$ (only defined within $x^2 + y^2 \leq R^2$). Then:

$$\int_{-R}^{R}\int_{-R}^{R} F_{pq}(x,y)dxdy = \pi R^2 \tag{34}$$

The traffic intensity of an arbitrary node (u, v) of this circular network, $T_{total}(u, v)$, will be the integration of the preference function $F_{pq}(x, y)$ along a line passing through (u, v), while rotating the line and integrating again. The line equation is (also see (26)),

$$y = \gamma x + \beta \ \ , \ where \ \ \gamma = \frac{v-q}{u-p} \ and \ \beta = \frac{uq - vp}{u-p} \tag{35}$$

The circular network is divided into two parts, one part is east of the point (u, v) and the other part is west of (u, v).

First, in order to obtain the total traffic intensity, the crossing point between the line $y = \gamma x + \beta$ and the circle $x^2 + y^2 = R^2$ is needed,

$$x^2 + (\gamma x + \beta)^2 = R^2$$

$$then, \quad x = \frac{-\gamma\beta \pm \sqrt{(1+\gamma^2)R^2 - \beta^2}}{(1+\gamma^2)} \tag{36}$$

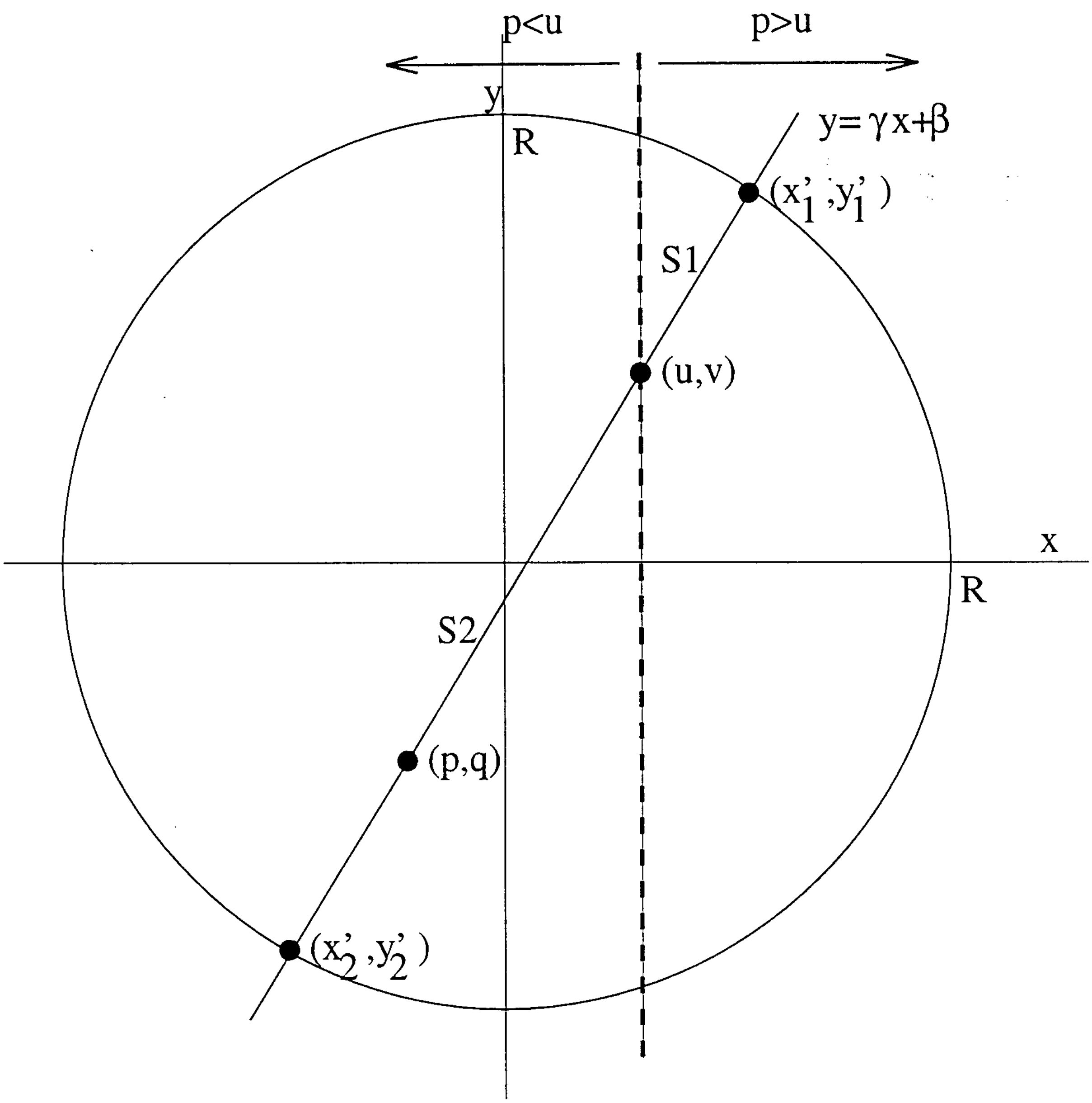

Fig. 6. Locality calculation for continuous 2-dimensional network

From Fig. 6, traffic from the node (p, q) which passes through point (u, v) along the line if (p, q) is located to the west of (u, v) (i.e. $p < u$), $T_{S1,p<u}$, is,

$$\begin{aligned} T_{S1,p<u} &= \int_v^{y'} \int_u^{x'} F_{pq}(x, y)dxdy \\ &= \int_u^{\frac{-\gamma\beta+\sqrt{(1+\gamma^2)R^2-\beta^2}}{(1+\gamma^2)}} F_{p,\gamma p+\beta}(x, \gamma x + \beta)dx \end{aligned} \tag{37}$$

Here the equation is in terms of the x coordinate (not the y coordinate) by replacing x

with $\gamma x + \beta$ and (x', y') is the crossing point between the line and the circle. The traffic from all other nodes $(p, q)'s$ located along the line which are also located to the west of (u, v) in Fig. 6 is,

$$T^{S2}_{S1,p<u} = \int_{\frac{-\gamma\beta-\sqrt{(1+\gamma^2)R^2-\beta^2}}{(1+\gamma^2)}}^{u} \int_{u}^{\frac{-\gamma\beta+\sqrt{(1+\gamma^2)R^2-\beta^2}}{(1+\gamma^2)}} F_{p,\gamma p+\beta}(x, \gamma x + \beta)dxdp \tag{38}$$

Up to now, only the linear network is considered and γ is fixed since only one line is considered (i.e. γ is independent of $(p, q)'s$ value on the line). In order to cover the circular network, the linear network has to be rotated starting γ from $-\infty$ to ∞.

$$T^{S2}_{S1,p<u}, rotation = \int_{-\infty}^{\infty} \int_{\frac{-\gamma\beta-\sqrt{(1+\gamma^2)R^2-\beta^2}}{(1+\gamma^2)}}^{u} \int_{u}^{\frac{-\gamma\beta+\sqrt{(1+\gamma^2)R^2-\beta^2}}{(1+\gamma^2)}} F_{p,\gamma p+\beta}(x, \gamma x + \beta)dxdpd\gamma \tag{39}$$

Here the first integration involves slope. Since $F_{p,q}$ is only defined in $x^2 + y^2 \leq R^2$, thus the equation (39) can be simplified as:

$$T^{S2}_{S1,p<u}, rotation = \int_{-\infty}^{\infty} \int_{-R}^{u} \int_{u}^{R} F_{p,\gamma p+\beta}(x, \gamma x + \beta)dxdpd\gamma \tag{40}$$

Here, it is assumed that the integral is zero if $x^2 + y^2 > R^2$.
If (p, q) is located on the east of (u, v), $p > u$, then:

$$T^{S2}_{S1,p>u}, rotation = \int_{-\infty}^{\infty} \int_{u}^{R} \int_{-R}^{u} F_{p,\gamma p+\beta}(x, \gamma x + \beta)dxdpd\gamma \tag{41}$$

Here, again, the first integration involves slope from ∞ to $-\infty$. Finally, the total traffic intensity at point (u, v) is

$$\begin{aligned} T_{total}(u, v) &= \underbrace{\int_{-\infty}^{\infty} \int_{-R}^{u} \int_{u}^{R} F_{p,\gamma p+\beta}(x, \gamma x + \beta)dxdpd\gamma}_{p<u} \\ &\quad + \underbrace{\int_{-\infty}^{\infty} \int_{u}^{R} \int_{-R}^{u} F_{p,\gamma p+\beta}(x, \gamma x + \beta)dxdpd\gamma}_{p>u} \end{aligned} \tag{42}$$

Again, the preference equation is general enough that it can be used not only for the traffic with locality but also for the traffic with preference if one knows the communication attractivity of the traffic between nodes. As a necessary but not sufficient use of this equation, let the preference function be uniform, that is $F_{p,q}(x, y) = 1$. Without loss of generality, let u=0, then one can obtain the traffic density along the y-axis.

Let's start from the equation (39).

$$\begin{aligned}
T_{S1,p<u}^{S2}, rotation &= \int_{-\infty}^{\infty}\int_{\frac{-\gamma\beta-\sqrt{(1+\gamma^2)R^2-\beta^2}}{(1+\gamma^2)}}^{0}\int_{0}^{\frac{-\gamma\beta+\sqrt{(1+\gamma^2)R^2-\beta^2}}{(1+\gamma^2)}} F_{p,\gamma p+\beta}(x,\gamma x+\beta)dxdpd\gamma \\
&= \int_{-\infty}^{\infty}\int_{\frac{-\gamma\beta-\sqrt{(1+\gamma^2)R^2-\beta^2}}{(1+\gamma^2)}}^{0}\int_{0}^{\frac{-\gamma\beta+\sqrt{(1+\gamma^2)R^2-\beta^2}}{(1+\gamma^2)}} 1\ \ dxdpd\gamma \\
&= \int_{-\infty}^{\infty}\int_{\frac{-\gamma\beta-\sqrt{(1+\gamma^2)R^2-\beta^2}}{(1+\gamma^2)}}^{0}\Big[\frac{-\gamma\beta+\sqrt{(1+\gamma^2)R^2-\beta^2}}{(1+\gamma^2)}\Big]dpd\gamma \\
&= \int_{-\infty}^{\infty}\Big[\frac{\gamma\beta+\sqrt{(1+\gamma^2)R^2-\beta^2}}{(1+\gamma^2)}\Big]\Big[\frac{-\gamma\beta+\sqrt{(1+\gamma^2)R^2-\beta^2}}{(1+\gamma^2)}\Big]d\gamma \\
&= \int_{-\infty}^{\infty}\frac{(1+\gamma^2)R^2-\beta^2-\gamma^2\beta^2}{(1+\gamma^2)^2}d\gamma \\
&= (R^2-\beta^2)\int_{-\infty}^{\infty}\frac{1}{1+\gamma^2}d\gamma \quad , let\ \ \gamma = tan\theta \\
&= (R^2-\beta^2)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \quad , where\ \ -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\
&= (R^2-\beta^2)\pi
\end{aligned} \tag{43}$$

And, also $T_{p>u}$ is,

$$\begin{aligned}
T_{p>u} &= \int_{-\infty}^{\infty}\int_{0}^{\frac{-\gamma\beta+\sqrt{(1+\gamma^2)R^2-\beta^2}}{(1+\gamma^2)}}\int_{\frac{-\gamma\beta-\sqrt{(1+\gamma^2)R^2-\beta^2}}{(1+\gamma^2)}}^{0} 1\ \ dxdpd\gamma \\
&= (R^2-\beta^2)\int_{-\infty}^{0}\frac{1}{1+\gamma^2}d\gamma \\
&= (R^2-\beta^2)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \\
&= (R^2-\beta^2)\pi
\end{aligned} \tag{44}$$

First of all, $\beta = \frac{uq-vp}{u-p}|_{u=0} = v$. Then total traffic intensity, $T_{total}(0,v)$, along the y-axis is,

$$\begin{aligned}
T_{total}(0,v) &= T_{p<u} + T_{p>u} \\
&= 2(R^2-v^2)\pi
\end{aligned} \tag{45}$$

This equation is the same as (9) since the preference is assumed to be uniform. Thus (45) represents an alternate derivation of (9) and confirmation of the utility of the preference equation (42).

V. Case Study : Locality Effect for Linear Discrete Network

In this section, we examine a linear discrete network exhibiting locality in terms of distance, a particular type of preference function. According to [3], a certain mobility model (in our case, the traffic moving can be assumed to be mobile objects) can be based on a gravity like function. Such models have been used in transportation research to model movement. The gravity model is examined below, along with a number of other possible locality functions.

Assumptions:

1) Distances between adjacent nodes are identical, d.

2) Each node generates N traffic units per unit time.

3) Pr {communication from node i to j } = $p_i(j) = p_{ij}$

4)

$$p_{ij} \propto \frac{1}{distance\ between\ node\ i\ and\ j} \tag{46}$$

i.e. $p_{ij} = \alpha_i \cdot K(|i-j| \cdot d) = \alpha_i \cdot K(md)$ where K($\cdot$) is a function of distance and m is the number of hops and α_i is a proportionality coefficient of probability for node i so that the sum of the probabilities equals to one.

If there are total N nodes along the linear network, then node 1 which is located at the leftmost end has the following associated probabilities,

$$\begin{aligned} p_{12} &= \alpha_1 \frac{1}{d} \\ p_{13} &= \alpha_1 \frac{1}{2d} \\ p_{14} &= \alpha_1 \frac{1}{3d} \\ \vdots &= \vdots \\ p_{1N} &= \alpha_1 \frac{1}{(N-1)d} \end{aligned} \tag{47}$$

Then:

$$\begin{aligned} 1 &= p_{12} + p_{13} + p_{14} + \cdots + p_{1N} \\ &= \alpha_1 (\frac{1}{d} + \frac{1}{2d} + \frac{1}{3d} + \cdots + \frac{1}{(N-1)d}) \end{aligned} \tag{48}$$

$$\Longleftrightarrow \alpha_1 = \frac{1}{\sum_{m=1}^{N-1} \frac{1}{md}} \tag{49}$$

where $d = \frac{length\ of\ the\ linear\ network}{number\ of\ nodes - 1}$

For node 2,

$$p_{21} = \alpha_2 \frac{1}{d},\ \ p_{23} = \alpha_2 \frac{1}{d},\ \ p_{24} = \alpha_2 \frac{1}{2d}, \cdots,\ \ p_{2N} = \alpha_1 \frac{1}{(N-2)d} \tag{50}$$

$$1 = \alpha_2 (\frac{1}{d} + \frac{1}{d} + \frac{1}{2d} + \cdots + \frac{1}{(N-2)d}) \tag{51}$$

$$\Longleftrightarrow \alpha_2 = \frac{1}{\frac{1}{d} + \sum_{m=1}^{N-2} \frac{1}{md}}$$

$$= \frac{1}{\sum_{k=1}^{1} \frac{1}{kd} + \sum_{m=1}^{N-2} \frac{1}{md}} \tag{52}$$

Then, for node i,

$$\alpha_i = \frac{1}{\sum_{k=1}^{i-1} \frac{1}{kd} + \sum_{m=1}^{N-i} \frac{1}{md}} \tag{53}$$

Thus the locality (preference) probability mass function (p.m.f) for the linear network will be,

$$\mathbf{p} = \begin{bmatrix} 0 & p_{12} & p_{13} & \cdots & p_{1N} \\ p_{21} & 0 & p_{23} & \cdots & p_{2N} \\ p_{31} & p_{32} & 0 & \cdots & p_{3N} \\ \vdots & & & \ddots & \vdots \\ p_{N1} & p_{N2} & \cdots & & 0 \end{bmatrix} \tag{54}$$

$$= \begin{bmatrix} 0 & \alpha_1 \frac{1}{d} & \alpha_1 \frac{1}{2d} & \alpha_1 \frac{1}{3d} & \cdots & \alpha_1 \frac{1}{(N-1)d} \\ \alpha_2 \frac{1}{d} & 0 & \alpha_2 \frac{1}{d} & \alpha_2 \frac{1}{2d} & \cdots & \\ \vdots & & & & \ddots & \vdots \\ \alpha_N \frac{1}{(N-1)d} & \cdots & & & & 0 \end{bmatrix} \tag{55}$$

The amount of traffic in the link 1, T_1, which is between node 1 and node 2 is

$$T_1 = N \cdot (p_{12} + p_{13} + \cdots + p_{1N}) + N \cdot p_{21} + N \cdot p_{31} + \cdots + N \cdot p_{N1}$$

$$= N \sum_{i=2}^{N} p_{1i} + N \sum_{j=2}^{N} p_{j1} \tag{56}$$

The amount of traffic in link 2 is,

$$
\begin{aligned}
T_2 &= N \cdot (p_{13} + p_{14} + \cdots + p_{1N}) + N \cdot (p_{23} + p_{24} + \cdots + p_{2N}) \\
&\quad + N \cdot (p_{31} + p_{32}) + N \cdot (p_{41} + p_{42}) + \cdots + N \cdot (p_{N1} + p_{N2}) \\
&= N \sum_{i=3}^{N} p_{1i} + N \sum_{j=3}^{N} p_{2j} + N \sum_{k=3}^{N} (p_{k1} + p_{k2}) \\
&= N (\sum_{m=1}^{2} \sum_{i=3}^{N} p_{mi}) + N \sum_{k=3}^{N} (\sum_{n=1}^{2} p_{kn}) \\
&= \sum_{m=1}^{2} \sum_{n=3}^{N} (P_{mn} + P_{nm}) \qquad (57)
\end{aligned}
$$

Here $P_{mn} = N p_{mn}$. The amount of traffic in link 3 is,

$$
\begin{aligned}
T_3 &= N \cdot (p_{14} + p_{15} + \cdots + p_{1N}) + N \cdot (p_{24} + p_{25} + \cdots + p_{2N}) \\
&\quad + N \cdot (p_{34} + p_{35} + \cdots + p_{3N}) + N \cdot (p_{41} + p_{42} + p_{43}) \\
&\quad + \cdots + N \cdot (p_{N1} + p_{N2} + p_{N3}) \\
&= \sum_{m=1}^{3} \sum_{n=4}^{N} (P_{mn} + P_{nm}) \qquad (58)
\end{aligned}
$$

Fig. 7 illustrates the spatial distribution with an uniform locality function and the distribution without locality which is given by (1). As we can see, these two graphs are same. The next graph, Fig. 8, shows the locality distributions for each node when the preference function is proportional to 1/distance.

Fig. 9 provides a comparison for the cases when the locality is proportional to $1/d$, $1/d^{1.3}$, and $1/d^2$. Here $1/d^2$ is the case of a gravity model. According to this graph, locality can have a significant effect on the spatial traffic distribution. As the power of distance becomes larger (i.e. becoming more localized), the traffic distribution becomes decentralized and the load in the center is reduced.

VI. Conclusion

We have found it to be possible to analytically model and solve for geographic traffic intensity with assumptions of either linear or circular network topology, shortest path routing and traffic preference. It is surprising that the spatial intensity for a network in a circular area with uniform loading and shortest path routing is a simple quadratic function of position. This work is of interest because it establishes a number of baseline analytical results for the spatial variation of traffic intensity. It is generic enough to be relevant for both circuit switched and packet switched networks in terms of questions of sizing and dimensioning networks .

REFERENCES

[1] B. A. Abernathy and T. G. Robertazzi, "Loading and spatial location in wire and radio communications," *IEEE Milcom 91*, vol. 2, 1991, pp. 391-395.

[2] Z. Sahinoglu and S. Tekinay, "On multimedia networks: self-similar traffic and network performance," *IEEE Communication Magazine*, vol. 37, no. 1, pp. 48-52.

[3] D. Lam, D. Cox, and J. Widom, "Teletraffic modeling for personal communications services," *IEEE Communication Magazine*, Feb. 1997, pp. 79-87.

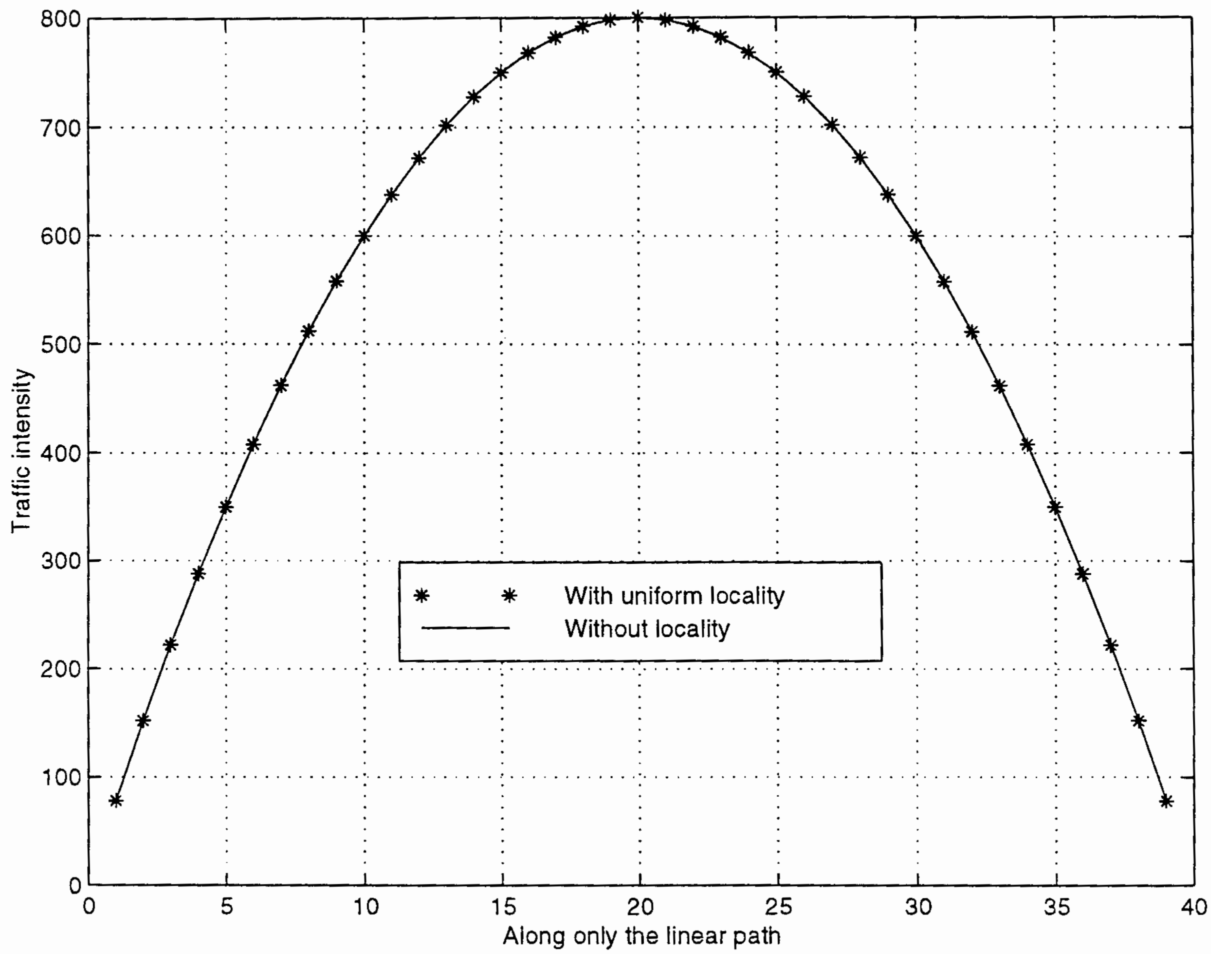

Fig. 7. Comparison between uniform-locality and no-locality

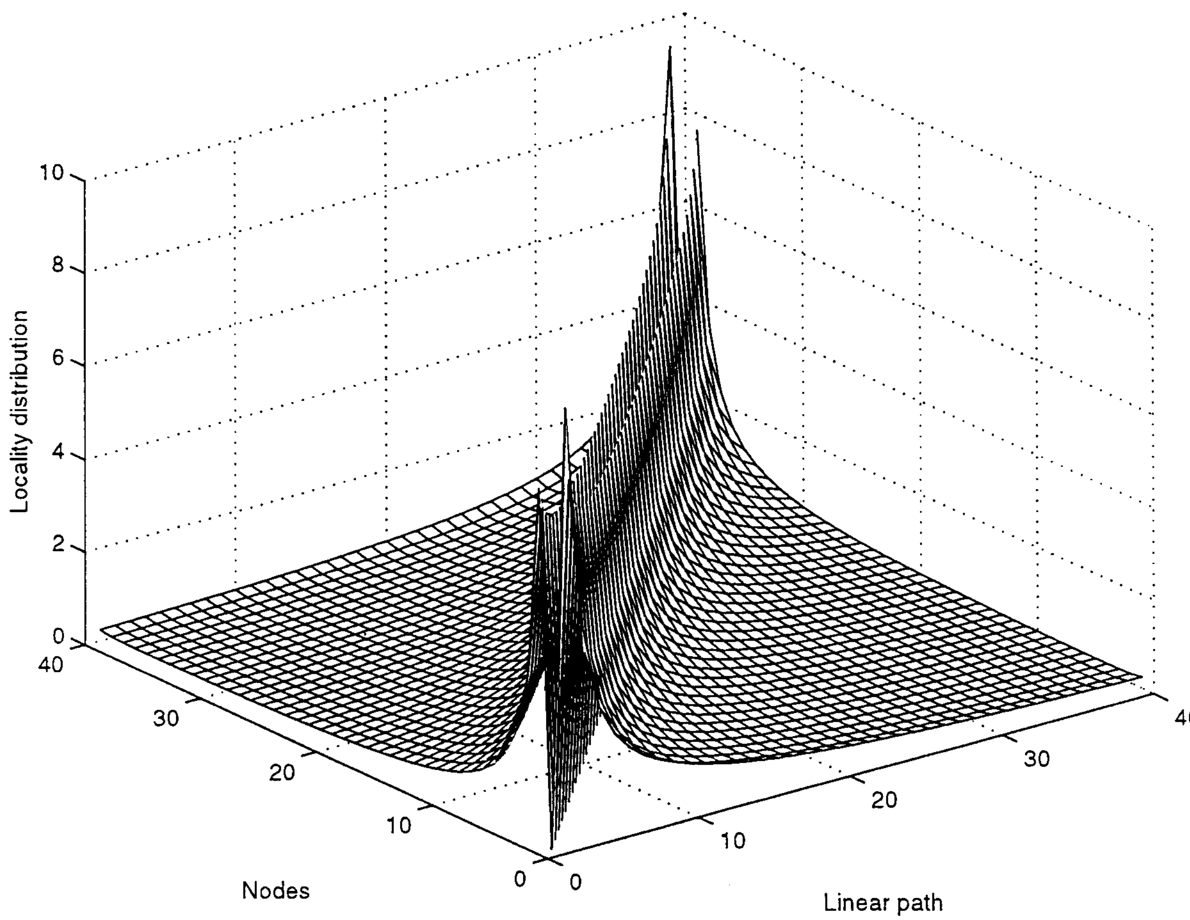

Fig. 8. Locality distribution for each node

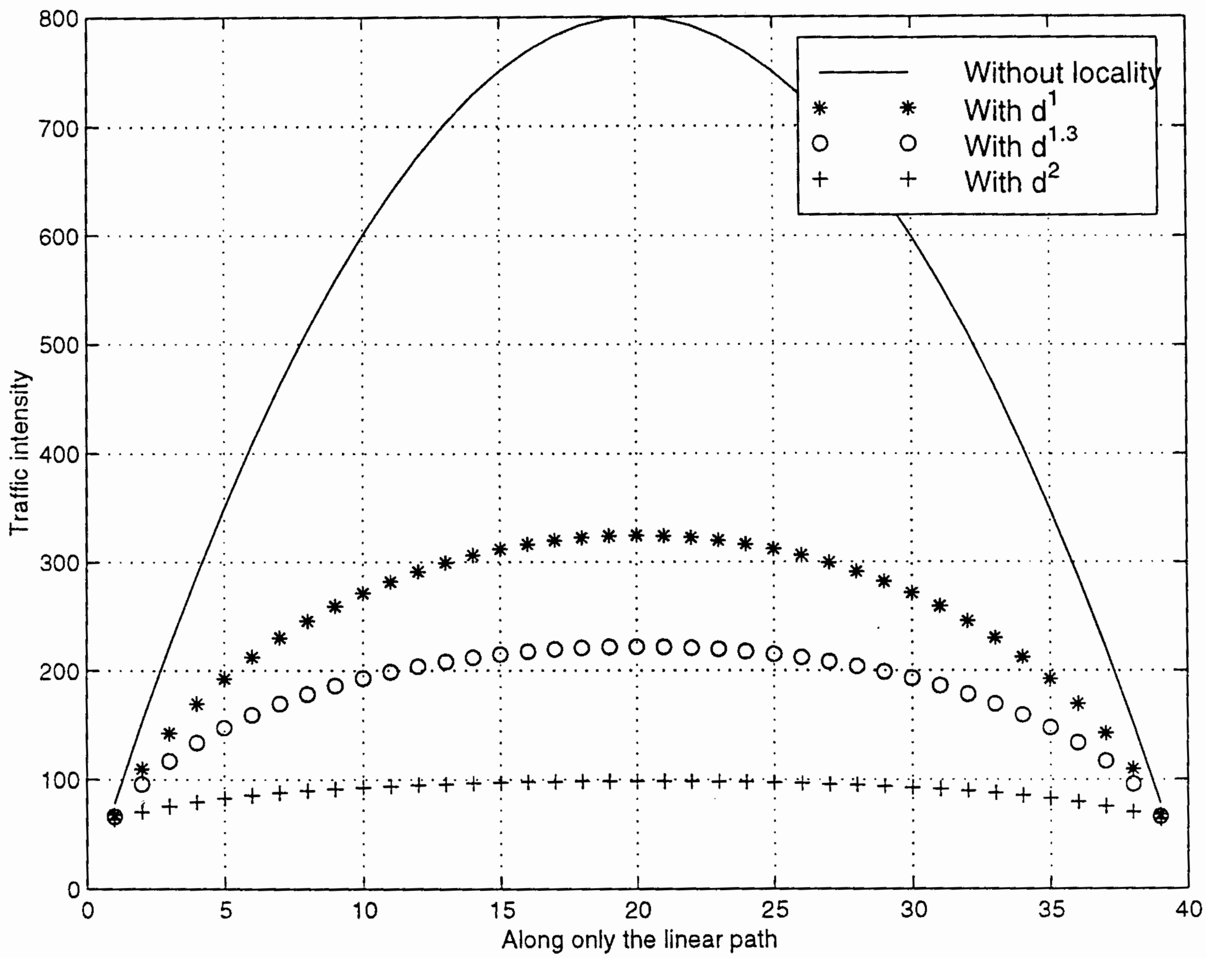

Fig. 9. Comparison of traffic distributions and locality