A SEVENTH ORDER NUMERICAL METHOD FOR SOLVING BOUNDARY VALUE NON-LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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SUMMARY

A numerical method for solving two-point boundary value problems associated with systems of first-order nonlinear ordinary differential equations is described. It needs three function evaluations for each sub-interval and is of order $O(h^7)$, where h is the space chop. Results of computational experiments comparing this method with other known methods are given.

INTRODUCTION

Accurate and fast numerical solution of two-point boundary value problems is necessary in many important application areas, e.g. boundary layer theory, the study of stellar interiors, control and optimization theory, and flow networks in biology.

In this paper a new method for the solution of two-point boundary value problems for nonlinear first-order differential equations is introduced. This method leads to a higher order accuracy than the presently known methods, without a corresponding increase in computation time. The method was tested numerically on the set of eight problems that were used in Reference 1, and in all cases led to significantly better results.

We consider the numerical solution of the system of differential equations

$$\underline{y}'(x)-\underline{f}(x,\underline{y}(x)) = 0, \tag{1}$$

with the two-point boundary conditions

$$\underline{g}(\underline{y}(0),\underline{y}(1)) = 0, \tag{2}$$

where $\underline{y}$, $\underline{f}$ and $\underline{g}$ are functions with values in R^m, and $0\leq x\leq 1$.

Let us subdivide the x range [0,1] into n equal parts, such that $h = 1/n$ and $x_i = ih$, $i = 0, 1,\ldots,n$. If we integrate the p^{th} equation of the system (1) in the interval $[x_{i-1},x_i]$ and denote the resulting quantity by $\phi_{pi}(\underline{y})$ then we have

$$\phi_{pi}(\underline{y}) = y_{pi}-y_{p,i-1}-\int_{x_{i-1}}^{x_i} f_p(x,\underline{y}(x))dx = 0, \tag{3}$$

where $y_{pi}=y_p(x_i)$ and $p = 1,2,\ldots,m$; $i= 1,2,\ldots,n$. The integral in (3) can be evaluated by the various numerical schemes. Let $f_{pi} = f_p(x_i ,\underline{y}(x_i))$ and $f_{p,i-\frac{1}{2}} = f_p(x_i-\frac{h}{2}, \underline{y}(x_i-\frac{h}{2}))$; $\underline{y}$ is now a vector with the components y_{pi} and the superscipts on f will denote its derivatives. In this paper we describe an $O(h^7)$ scheme that requires $f_{p,i-1}$, $f_{p,i-\frac{1}{2}}$, f_{pi} and only the first derivaties $f^1_{p,i-1}$ and f^1_{pi} to approximate the integral in (3). This is in contrast to the scheme described earlier by us in Reference 1, (viz. equation(1.4)), which needs not only $f_{p,i-1}$, f_{pi}, $f^1_{p,i-1}$ and $f^1_{p,i}$ but also the second derivatives $f^2_{p,i-1}$ and $f^2_{p,i}$ to achieve $O(h^7)$ accuracy.

THE $O(h^7)$ METHOD

Theorem: If

$$y_{p,i-\frac{1}{2}} = (y_{p,i-1} + y_{p,i})/2 + (5h/32)(f_{p,i-1} - f_{pi}) +$$

$$(h^2/64)(f^1_{p,i-1} + f^1_{pi}) + R \qquad (4)$$

and

$$\int_{x_{i-1}}^{x_i} f_p(x,\underline{y})\, dx = (h/30)[7(f_{p,i-1} + f_{p,i}) + 16\, f_{p,i-\frac{1}{2}} +$$

$$\frac{h}{2}(f^1_{p,i-1} - f^1_{p,i})] + E, \qquad (5)$$

where

$$f_{p,i-\frac{1}{2}} = f_p(x_i - \frac{h}{2}, \underline{y}(x_i - \frac{h}{2})) + S \qquad (6)$$

and y_p^6 , f_p^5 and all elements of the vector $\frac{\delta f_p}{\delta \underline{y}}$ are bounded then E is $O(h^7)$.

Proof: We make use of Taylor's expansion to express all the quantities on the right hand side of (4) about the point $i - \frac{1}{2}$, then after routine algebraic manipulations it follows that R is $O(h^6)$, since y_p^6 and f_p^5 are bounded. Since all the elements of the vector $\frac{\delta f_p}{\delta \underline{y}}$ are bounded, from (4) and (6) it follow that S is also $O(h^6)$. Consequently, the use of (6) in (5) will contribute at most an error of order $O(h^7)$.

We have to show now that integration formula (5) itself has an error $0(h^7)$. If we determine the quintic polynomial which uses the six points[2]

$(x_{i-1}, f^k_{i-1}), (x_i, f^k_i), (x_{i-\frac{1}{2}}, f^{(k)}_{i-\frac{1}{2}})$, $k = 0, 1$

and then integrate it between x_{i-1} and x_i, then we get (5). Since a symmetric fifth degree polynomial was integrated the error is $0(h^7)$. ∎

Now, in view of the above theorem, we write (3) as

$$\phi_{pi}(\underline{y}) = y_{pi} - y_{p,i-1} - (7h/30)(f_{p,i-1} + f_{pi}) - (8h/15)f_{p,i-\frac{1}{2}} - (h^2/60)(f^1_{p,i-1} - f^1_{pi}) + 0(h^7) \qquad (7)$$

where (4) and (6) with R and S replaced by zeros are used to compute $f_{p,i-\frac{1}{2}}$. The derivatives $f^1_{p,i-1}$ and f^1_{pi}, which are needed in (4) and (7), are computed as follows.

Since

$$f^1_p = \frac{\partial f_p}{\partial x} + \frac{\partial f_p}{\partial \underline{y}} \cdot \frac{\delta \underline{y}}{\delta x}$$

and using (1) we have

$$f^1_p = \frac{\partial f_p}{\partial x} + \frac{\partial f_p}{\partial \underline{y}} \cdot \underline{f}$$

From now on let us denote by ϕ and g the vectors with the components ϕ_{pi} and g_p respectively. Then neglecting the $0(h^7)$ term equation (7) can be written as

$$\phi(y) = 0 \qquad (8)$$

and (2) as

$$g(y) = 0. \qquad (9)$$

Clearly, $\phi \in R^{mn}$, $y \in R^{m(n+1)}$ and $g \in R^{m}$. It is worth noting that in (8), m components of y can be eliminated directly by expressing them as linear functions of the rest of the components by using (9), provided that g(y) is a linear function of y. We will assume that g(y) is linear and the necessary elimination of y components has been done so that we only have to solve (8).

COMPUTATIONAL RESULTS

Newton's method and a sparse linear equation solver were used to solve (8) for the set of problems used by us in Reference 1. All computations were done in double precision on UNIVAC 1100. The computational results are given in Table 1, where we have listed under each method the absolute value of the maximum error between the exact solution and the computed solution for each problem for the relevant values of n. Problem 2 leads to poor results because the higher derivatives of f_{2i} have large values. For example, f^{6}_{2i} have the factor $400\pi^{6} \approx 3.9 \times 10^{5}$. It is pointed out in [3] that for this problem an $O(h^{2})$ finite difference method required 2^{10} mesh points for 10^{-6} accuracy, and the deferred correction method needed 65 mesh points with 7 corrections. Problems 6 and 7 have discontinuities in f, and therefore, one sided derivatives are used. The starting solution used for all problems was a vector of all ones and convergence was obtained in at most seven iterations (the average number was 3.5).

It is evident from Table 1 that the present method yields excellent results.

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Table 1. Maximum errors in the solutions.

Problem	n	Method used	
		$0(h^7)$ in Ref.1.	Present Method
1	10	3.0(-6)	1.3(-8)
	20	3.0(-8)	2.1(-10)
	40	3.8(-10)	3.3(-12)
2	10	2.8(-1)	4.0(-3)
	20	2.0(-2)	7.6(-5)
	40	5.6(-4)	1.2(-6)
3	10	7.0(-8)	3.1(-11)
	20	1.2(-9)	5.7(-13)
	40	2.0(-11)	4.0(-14)
4	10	3.0(-6)	4.4(-9)
	20	3.2(-8)	7.0(-11)
	40	3.6(-10)	1.4(-12)
5	10	9.7(-6)	1.8(-7)
	20	8.3(-7)	3.7(-9)
	40	-----	5.6(-11)
6	10	----	8.8(-11)
	20	0	2.2(-11)
	40	-----	5.5(-12)
7	10	-----	9.5(-10)
	20	4.8(-8)	1.6(-11)
	40	7.9(-10)	3.3(-12)
8	10	1.7(-7)	3.2(-10)
	20	2.1(-9)	4.9(-12)
	40	3.0(-11)	7.7(-14)