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TWO MENGER-TYPE THEOREMS FOR TRANSFINITE GRAPHS AND SOME OPEN QUESTIONS INSPIRED BY THEM

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Abstract — Two Menger-type theorems are proven herein for transfinite graphs. They are established under some strong restrictions on the transfinite graphs. This leads to some open questions concerning more general Menger-type theorems for transfinite graphs.

Let us refer to a conventional graph as a "0-graph" and to their vertices as "0-vertices." Transfinite graphs of rank 1 (synonymously, 1-graphs) are constructed by connecting together infinite 0-graphs at their infinite extremities through 1-vertices. This can also be done for higher ranks of transfiniteness by connecting 1-graphs at their infinite extremities through 2-vertices to obtain a 2-graph, and so on to get ν-graphs where ν can be any natural-number rank or even a transfinite-ordinal rank. The theory of these structures was originally presented in [4] and developed further in [5]. All the terminology we use herein is explicated in those references except that we now speak of "ν-vertices" instead of "ν-nodes" and "edges" instead of "branches." Moreover, we restrict our attention herein to "pristine" transfinite graphs, in which no vertex of any rank contains a vertex of lower rank.

There is a large literature on Menger's theorem [1, page 121], which includes versions appropriate for conventionally infinite graphs [2], [3]. However, there have been no such results so far for transfinite graphs. The two Menger-type theorems stated herein are based on the assumption that the transfinite graph $\mathcal{G}^\nu$ at hand is "stout" in addition to being pristine.

Let us start by recalling some definitions. We confine the rank ν of the transfinite graph $\mathcal{G}^\nu$ to $0 \leq \nu \leq \omega$, $\nu \neq \vec{\omega}$, where ω is the first transfinite ordinal and $\vec{\omega}$ is the arrow rank preceding ω. The same arguments work for ranks higher than ω. A ν-vertex is defined in

[5, pages 30 and 42], and a ν-graph is defined in [5, pages 31 and 43]. A $(\nu-1)$-section of $\mathcal{G}^\nu$ is a maximal subgraph, every two edges of which are connected through a two-ended ρ-path for some rank $\rho = 0, \ldots, \nu - 1$ [5, page 49]. ρ-paths are defined in [5, pages 33 and 44]. Two ν-vertices are said to be ν-adjacent if they are incident to the same $(\nu-1)$-section [5, page 87]. Two endless paths (of any ranks) are called disjoint if no single vertex (of any rank) is in both paths. [5, page 33].

As for new definitions, we need the following. Two two-ended ν-paths are called independent if no single vertex (of any rank) is in both paths except possibly for one or two terminal vertices of both paths. In other words, the independent paths may meet at their terminal vertices but no place else. A set of such pairwise independent two-ended ν-paths is also called independent.

The ν-graph $\mathcal{G}^\nu$ is called stout if there exists a set $\mathcal{P}^{\nu-1}$ of pairwise disjoint endless $(\nu-1)$-paths satisfying the following two conditions:

- There is a bijection between $\mathcal{P}^{\nu-1}$ and the set of all pairs of ν-adjacent nonsingleton ν-vertices.
- Each path in $\mathcal{P}^{\nu-1}$ reaches the two ν-vertices in the corresponding pair under that bijection.

For example, the 1-graph of Fig 1(a) is stout whereas the 1-graph of Fig. 1(b) is not stout.

A two-ended ν-path in $\mathcal{G}^\nu$ that terminates at two ν-vertices n_a^ν and n_b^ν is called an $n_a^\nu n_b^\nu$-path. A set of ν-vertices is said to separate two given ν-vertices n_a^ν and n_b^ν if that set contains neither n_a^ν nor n_b^ν and if every $n_a^\nu n_b^\nu$-path meets at least one ν-vertex of that set.

The stoutness of $\mathcal{G}^\nu$ enables the lifting of Menger's conventional theorem into the context of transfinite graphs as follows:

Theorem 1. Assume that the ν-graph $\mathcal{G}^\nu$ has the following properties:

(a) $\mathcal{G}^\nu$ is ν-connected.

(b) $\mathcal{G}^\nu$ has only finitely many nonsingleton ν-vertices.

(c) $\mathcal{G}^\nu$ is stout.

Let n_a^ν and n_b^ν be two nonsingleton ν-vertices in $\mathcal{G}^\nu$ that are not ν-adjacent. Then, the maximum number of independent $n_a^\nu n_b^\nu$-paths in $\mathcal{G}^\nu$ is equal to the minimum number of ν-vertices that separate n_a^ν and n_b^ν.

Proof. Replace each $n_a^\nu n_b^\nu$-path P^ν in $\mathcal{G}^\nu$ by an $n_a^\nu n_b^\nu$-path Q^ν such that each endless $(\nu-1)$-path in P^ν reaching two consecutive ν-vertices of P^ν is replaced by the unique endless $(\nu-1)$-path in $\mathcal{P}^{\nu-1}$ reaching those two ν-vertices. Let $\mathcal{Q}^\nu$ be the set of those $n_a^\nu n_b^\nu$-paths Q^ν. $\mathcal{Q}^\nu$ is a subset of all the $n_a^\nu n_b^\nu$-paths in $\mathcal{G}^\nu$. By the stoutness of $\mathcal{G}^\nu$, the maximum number of independent $n_a^\nu n_b^\nu$-paths in $\mathcal{G}^\nu$ cannot be greater than the maximum number of independent $n_a^\nu n_b^\nu$-paths in $\mathcal{Q}^\nu$, nor can it be less obviously. Let K be that number.

Now, create a "surrogate" 0-graph $\mathcal{G}^{\nu\to 0}$ from $\mathcal{G}^\nu$ as follows. Replace bijectively each nonsingleton ν-vertex n^ν in $\mathcal{G}^\nu$ by a 0-vertex m^0 in $\mathcal{G}^{\nu\to 0}$; thus, n_a^ν and n_b^ν are replaced by m_a^ν and m_b^ν respectively. Also, replace bijectively each endless $(\nu-1)$-path $P^{\nu-1}$ in $\mathcal{P}^{\nu-1}$ by an edge that is incident to the two 0-vertices in $\mathcal{G}^{\nu\to 0}$ corresponding to the two ν-vertices that $P^{\nu-1}$ reaches. This determines a bijection between the paths in $\mathcal{Q}^\nu$ and the $m_a^0 m_b^0$-paths in $\mathcal{G}^{\nu\to 0}$. By Menger's theorem, K equals the minimum number of 0-vertices in $\mathcal{G}^{\nu\to 0}$ that separate m_a^0 and m_b^0.

Clearly, a set of ν-vertices in $\mathcal{G}^\nu$ separates n_a^ν and n_b^ν if and only if the corresponding set of 0-vertices in $\mathcal{G}^{\nu\to 0}$ separates m_a^0 and m_b^0. Therefore, K equals the minimum number of ν-vertices in $\mathcal{G}^\nu$ that separate n_a^ν and n_b^ν. □

In the event that $\mathcal{G}^\nu$ has infinitely many ν-vertices, we can state the following.

Theorem 2. Assume that the ν-graph $\mathcal{G}^\nu$ has the following properties.

(a) $\mathcal{G}^\nu$ is ν-connected.

(b) $\mathcal{G}^\nu$ has infinitely many nonsingleton ν-vertices.

(c) $\mathcal{G}^\nu$ is stout.

Let n_a^ν and n_b^ν be two nonsingleton ν-vertices in $\mathcal{G}^\nu$ that are not ν-adjacent. Then, there are infinitely many independent $n_a^\nu n_b^\nu$-paths in $\mathcal{G}^\nu$ if and only if no finite set of ν-vertices in $\mathcal{G}^\nu$ separates n_a^ν and n_b^ν.

Proof. The "only if" part is clear, for no finite set of ν-vertices other than n_a^ν and n_b^ν can meet infinitely many independent $n_a^\nu n_b^\nu$-paths.

So consider the "if" part. Let $\mathcal{Q}^\nu$ be the set of $n_a^\nu n_b^\nu$-paths defined in the proof of Theorem 1. choose any $n_a^\nu n_b^\nu$-path Q_1^ν in $\mathcal{Q}^\nu$. Q_1^ν has only finitely many ν-vertices. Since no finite set of ν-vertices separates n_a^ν and n_b^ν, there exists an $n_a^\nu n_b^\nu$-path P_2^ν that does not meet any of the ν-vertices of Q_1^ν. Since $\mathcal{G}^\nu$ is stout, we can replace P_2^ν by an $n_a^\nu n_b^\nu$-path Q_2^ν in $\mathcal{Q}^\nu$ that does not meet any vertex (of any rank) of Q_1^ν. Q_2^ν also has only finitely many ν-vertices. Therefore, there exists an $n_a^\nu n_b^\nu$-path P_3^ν that does not meet any of the ν-vertices of Q_1^ν and Q_2^ν. By the stoutness again, we can replace P_3^ν by an $n_a^\nu n_b^\nu$-path Q_3^ν in $\mathcal{Q}^\nu$ that does not meet any of the vertices (of any ranks) of Q_1^ν and Q_2^ν. Continuing in this way, we generate an infinite set of independent $n_a^\nu n_b^\nu$-paths $\{Q_1^\nu, Q_2^\nu, Q_3^\nu, \ldots\}$ in $\mathcal{G}^\nu$. $\square$

All this leads to the following open questions: Can a Menger-type theorem be stated for two vertices of different ranks? What can be said if the assumption of stoutness is dropped? Also, what if $\mathcal{G}^\nu$ is not pristine? These questions appear to be much harder to answer.

References

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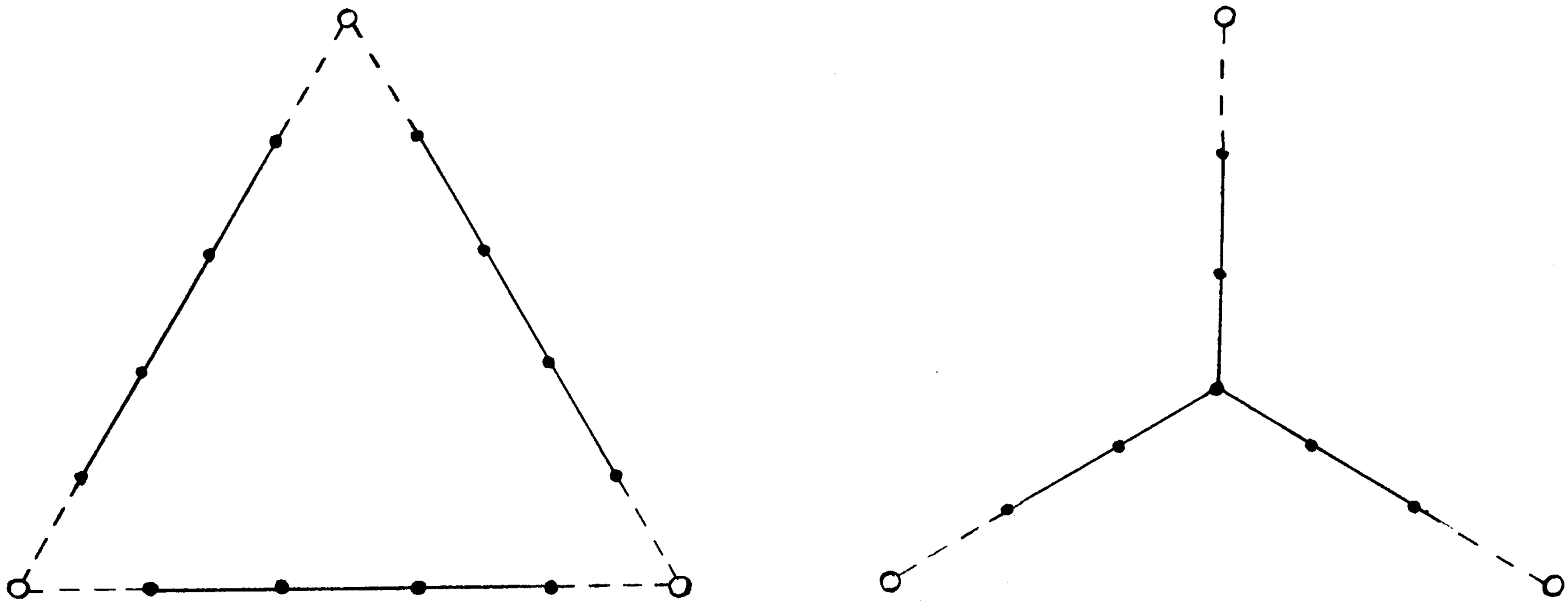
Caption for Fig. 1.

Fig. 1. Two 1-graphs. The first one shown in part (a) is stout, and the second one shown in part (b) is not stout. The dots represent 0-vertices, the small circles represent 1-vertices and the lines represent edges. Each pair of 1-nodes is connected through an endless 0-path in both cases.



(a) (b)