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REALIZABILITY OF REAL MATRICES AS TRANSFORMERLESS RESISTIVE n-PORTS

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Realizability of Real Matrices as Transformerless Resisitive n-ports *

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Abstract

The question of whether a matrix represents the conductance matrix of an n-port consisting exclusively of positive resistors is considered. A systematic method of testing the matrix is presented. The end result of this procedure either yields a particular realization (or realizations) of the matrix or determines that a realization does not exist. The question of whether the matrix is realizable as an impedance matrix can be treated by simply inverting the matrix and applying the same procedure.

1 Introduction

Let **F** be any real $n \times n$ matrix. Cederbaum [2] found necessary and sufficient conditions for the realizability of **F** as an n-port conductance or resistance matrix using only positive resistors and no transformers. However, the determination of realizability required a procedure for testing **F** as the principal submatrix of a matrix having a particular form. Weinberg [6] compiled and

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summarized many of the advances in this problem. In this paper we take a slightly different approach in where the matrix **F** is tested for all possible n-port assignations. The algorithm presented will either find a realization or exhaust all possibilities, proving **F** not realizable. For **F** to be considered as a possible conductance matrix of an n-port we will assume only the following necessary conditions:

Conditions 1

1. **F** *is symmetric*
2. **F** *is nonsingular*
3. *The main diagonal consists of positive values.*

2 The Indefinite Admittance Matrix

A network $\mathcal{N}$ containing m nodes and consisting only of (not necessarily positive) resistors can be described by its indefinite admittance matrix. If we assume a terminal going into each node and let $\mathbf{i}$ be the vector of currents flowing into these terminals and $\mathbf{v}$ be the vector of voltages from each node with respect to an arbitrary reference, we can write $\mathbf{i} = \mathbf{Gv}$. The matrix $\mathbf{G}$ is of the form:

$$\mathbf{G} = \begin{pmatrix} \sum_{k \neq 1} g_{1,k} & -g_{1,2} & -g_{1,3} & \cdots & -g_{1,m} \\ -g_{1,2} & \sum_{k \neq 2} g_{2,k} & -g_{2,3} & \cdots & -g_{2,m} \\ -g_{1,3} & -g_{2,3} & \sum_{k \neq 3} g_{3,k} & \cdots & -g_{3,m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -g_{1,m} & -g_{2,m} & -g_{3,m} & \cdots & \sum_{k \neq m} g_{m,k} \end{pmatrix}. \quad (1)$$

The element $g_{r,s}$ represent the value of the conductance connecting node r to node s. The matrix $\mathbf{G}$ is symmetric, singular, and each of its rows and columns add up to zero. We will call any matrix of this form an IA-matrix. Furthermore, if all $g_{r,s}$ values are nonnegative, then $\mathbf{G}$ will be called a nonnegative IA-matrix.

Note that $\mathbf{G}$ can be completely specified by either the values above or below the main diagonal. There are $m(m-1)/2$ free variables on an $m \times m$ IA-matrix. Let $\mathcal{D}$ be an operator that extracts a vector consisting of these values out of an IA-matrix. That is:

$$\mathcal{D}(\mathbf{G}) = (g_{1,2}, \cdots, g_{1,m}, g_{2,3}, \cdots, g_{2,m}, \cdots, g_{m-2,m-1}, g_{m-2,m}, g_{m-1,m})^T \quad (2)$$

It is obvious that any resistive network has an IA-matrix representation. Conversely, any IA-matrix can be associated to a resistive network. This network is made of the $m(m-1)/2$ conductances of $\mathcal{D}(\mathbf{G})$.

3 Multiport Networks

Let $\mathbf{F}$, an $n \times n$ matrix, be the conductance matrix of an n-port. Let $\mathbf{j}$ be the port currents vector and $\mathbf{u}$ be the port voltages vector. The network satisfies the equation $\mathbf{j} = \mathbf{F}\mathbf{u}$. The actual realization will consist of a network with at least $n+1$ nodes so as to make all port voltages u_k independent of each other. Counting internal nodes (i.e. nodes not associated with any port), the network may have as many of them as required. However, we will limit ourselves to realizations with up to $2n$ nodes. There is no loss in generality since the nodes that are not connected to the terminals of any port can be removed through a star-delta transformation. In general, we will assume that the realization consists of a resistive network $\mathcal{N}$ with m

nodes and thus has an associated $m \times m$ IA-matrix which we will call $\mathbf{G}$. Assuming that the nodes of the network $\mathcal{N}$ are indexed, we will use the term "n-port assignation" to a collection of n ordered pairs of nodes. A pair of nodes corresponds to the positive and negative terminal of a port. Figure (1) shows a resistive network in both its multiterminal and multiport representations.

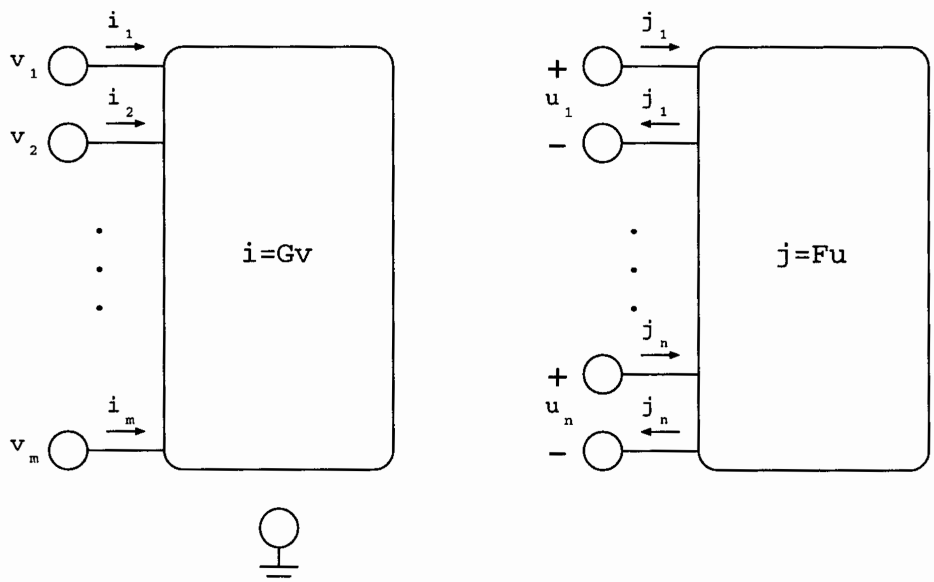

Figure 1: Multiterminal and multiport representation of a resistive network

Let $\mathbf{T}$ be the $n \times m$ matrix that maps $\mathbf{v}$ into $\mathbf{u}$ ($\mathbf{u} = \mathbf{Tv}$), and $\mathbf{H}$ be the $m \times n$ matrix that maps $\mathbf{j}$ into $\mathbf{i}$ ($\mathbf{i} = \mathbf{Hj}$). Each row of $\mathbf{T}$ has exactly one $+1$ and one -1, in the position corresponding to the node to which the positive and negative terminals are connected, respectively. As pointed in [2], if each port is represented by the branch connecting its positive and negative terminal, the n ports are distributed into one or more tree-like structures (since they can't form loops). In other words, the n ports form a forest that covers all m nodes. This condition limits the possible number of matrices $\mathbf{T}$ and $\mathbf{H}$ formed from all possible assignations.

Lemma 1 $H^T = T$.

Proof: Consider a network $\mathcal{N}$ both in its multiport and multiterminal representations. On the multiterminal case (indefinite admittance representation), let the last node be the reference node ($v_m = 0$). That makes it the return path of all currents. From each node k to node m connect a voltage source v_k. Then we have:

$$\mathbf{v} = (v_1, v_2, \cdots, v_{m-1}, 0)^T \tag{3}$$

$$\mathbf{i} = (i_1, i_2, \cdots, i_{m-1}, -\sum_{k=1}^{m-1} i_k)^T \tag{4}$$

The power delivered by $v_1, \ldots, v_{m-1}$ is $\sum_{k=1}^{m-1} v_k i_k$, which can also be written as $\mathbf{i}^T\mathbf{v}$. Now construct a replica of $\mathcal{N}$ but now let it be an n-port. On each port k we connect a source u_k according to the transformation $\mathbf{u} = \mathbf{T}\mathbf{v}$. The power delivered in this replica is $\mathbf{j}^T\mathbf{u}$ and must be the same as in the first one. Thus $\mathbf{j}^T\mathbf{u} = \mathbf{i}^T\mathbf{v} = (\mathbf{j}^T\mathbf{H}^T)\mathbf{v}$. That means $\mathbf{j}^T\mathbf{u} - \mathbf{j}^T\mathbf{H}^T\mathbf{v} = \mathbf{j}^T(\mathbf{u} - \mathbf{H}^T\mathbf{v}) = 0$. Therefore $\mathbf{j} = \mathbf{F}\mathbf{u} \perp (\mathbf{u} - \mathbf{H}^T\mathbf{v})$. Since $\mathbf{F}$ is nonsingular it spans all of the n-dimensional Euclidean space R^n. This implies that $(\mathbf{u} - \mathbf{H}^T\mathbf{v})$ is the zero vector, i.e. $\mathbf{u} = \mathbf{H}^T\mathbf{v}$. The choice of reference voltage in the multiterminal case is arbitrary so the previous equation holds for all possible vectors $\mathbf{v}$. Since $\mathbf{u} = \mathbf{T}\mathbf{v}$ we can finally conclude that $\mathbf{H}^T = \mathbf{T}$.

Note that $\mathbf{H}$ has the form of an incidence matrix. In fact, it is the incidence matrix of the forest of oriented trees described above. Therefore, this matrix alone is enough to characterize one n-port assignation.

Lemma 2 *Let $\mathcal{N}$ be a network with m indexed nodes. Let $\mathbf{H}$ be an $m \times n$ matrix that gives $\mathcal{N}$ an n-port assignation, with $n < m$. Then any matrix $n \times$*

n $\mathbf{F}$ *that satisfies conditions 1 can be realized by the resistors that constitute the IA-matrix* $\mathbf{G} = \mathbf{HFH}^T$.

Proof: For any input $\mathbf{v}$, $\mathbf{Gv} = \mathbf{i} = \mathbf{Hj} = \mathbf{H}(\mathbf{Fu}) = \mathbf{HFH}^T\mathbf{v}$. That is, any nonsingular symmetric $n \times n$ matrix with positive main diagonal elements has an n-port conductance realization with only linear conductances (which are not necessarily positive).

Now we will need to prove an additional property of the incidence matrix $\mathbf{H}$. First, we present a preliminary result.

Lemma 3 *A forest of oriented trees described by an* $m \times n$ *incidence matrix* H *has* $m - n$ *trees.*

Proof: First we note that, on a network with $q_k + 1$ nodes, a tree must have q_k branches. Now, let p be the number of parts of a forest. If the forest is described by an $m \times n$ incidence matrix (i.e. it has m nodes and n branches, the following must be satisfied:

$$
\begin{aligned}
m &= \sum_{k=1}^{p}(q_i + 1) \\
n &= \sum_{k=1}^{p} q_i.
\end{aligned}
$$

Substracting the second equation from the first, we conclude that $p = m - n$.

Now we can establish the following:

Theorem 1 *The* $m \times n$ *incidence matrix of a forest has* n *independent columns.*

Proof: A known result of network theory is that the incidence matrix of any graph consisting of p parts and $r + p$ nodes has rank r [1]. Identifying $r + p = m$ and $p = m - n$ we have rank($\mathbf{H}$)=m-(m-n)=n.

From here on, we will refere to the matrix $\mathbf{H}$ as an assignation matrix, that is, an incidence matrix with all columns linearly independent. Also, we will say that $\mathbf{F}$ is produced by $\mathbf{G}$ if for the resistive network described by $\mathbf{G}$ there exists an assignation matrix $\mathbf{H}$ such that $\mathbf{F}$ is the conductance matrix of the n-port.

Lemma 4 *Given an assignation matrix* $\mathbf{H}$ *and an IA-matrix* $\mathbf{G}$ *it is always possible to find the conductance matrix* $\mathbf{F}$ *produced by* $\mathbf{G}$*.*

Proof: It is a known result from linear algebra that if $\mathbf{H}$ is composed of linearly independent columns, then $\mathbf{H}^T\mathbf{H}$ is square, symmetric and invertible [5]. Then, we may write

$$\begin{aligned} \mathbf{G} &= \mathbf{HFH}^T \\ \mathbf{H}^T\mathbf{GH} &= \mathbf{H}^T\mathbf{HFH}^T\mathbf{H} \\ (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{GH}(\mathbf{H}^T\mathbf{H})^{-1} &= \mathbf{F} \end{aligned} \tag{5}$$

We can thus obtain $\mathbf{F}$ from equation (5).

4 Multiple realizations of an n-port

Consider the conductance matrix $\mathbf{F}$ and an associated assignation matrix $\mathbf{H}$:

$$\mathbf{F} = \begin{pmatrix} 8 & 6 \\ 6 & 9 \end{pmatrix} \quad \mathbf{H}^T = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}. \tag{6}$$

and two nonnegative IA-matrices:

$$\mathbf{G}_1 = \begin{pmatrix} 14 & -12 & -2 & 0 \\ -12 & 15 & 0 & -3 \\ -2 & 0 & 14 & -12 \\ 0 & -3 & -12 & 15 \end{pmatrix} \quad \mathbf{G}_2 = \begin{pmatrix} 15 & -13 & -1 & -1 \\ -13 & 16 & -1 & -2 \\ -1 & -1 & 15 & -13 \\ -1 & -2 & -13 & 16 \end{pmatrix} \tag{7}$$

$\mathbf{F}$ is produced by both $\mathbf{G}_1$ and $\mathbf{G}_2$ using the same assignation matrix $\mathbf{H}$.

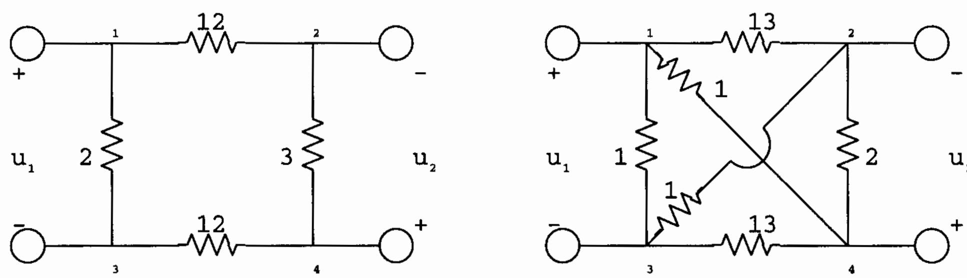

Figure 2: Two realizations of the same conductance matrix

This indicates that there might be a whole class of realizations of the matrix $\mathbf{F}$ for a particular assignation matrix $\mathbf{H}$. Given an $n \times n$ matrix $\mathbf{F}$ and an $m \times n$ assignation matrix $\mathbf{H}$ $(m > n)$ we inmediately get one realization of the n-port, namely $\mathbf{G}_1 = \mathbf{HFH}^T$. This may be the only realization (for the selected $\mathbf{H}$) or might be one representative of the whole class. The problem is to find all $\mathbf{G}$ such that:

$$\mathbf{F} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{G}\mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{G}_1\mathbf{H}(\mathbf{H}^T\mathbf{H})^{-1} \tag{8}$$

The equivalence class of all $\mathbf{G}$ that produce $\mathbf{F}$ can be defined by:

$$\mathbf{H}^T\mathbf{G}\mathbf{H} = \mathbf{H}^T\mathbf{G}_1\mathbf{H}. \tag{9}$$

Since $\mathbf{G}$ is an IA-matrix, it is uniquely determined by $\mathcal{D}(\mathbf{G})$. Let $\mathbf{g} = \mathcal{D}(\mathbf{G})$. For a given $\mathbf{G}_1$, equation (9) can be rewritten as a linear matrix equation on $\mathbf{g}$:

$$\mathbf{Kg} = \mathcal{B}(\mathbf{H}^T\mathbf{G}_1\mathbf{H}) = \mathbf{b} \tag{10}$$

where $\mathcal{B}$ is an operator that rewrites the matrix $\mathbf{H}^T\mathbf{G}_1\mathbf{H}$ as a vector $\mathbf{b}$ taking elements in row-wise order, and $\mathbf{K}$ is some matrix acting on $\mathbf{g}$. Equation (10) has at least one solution $\mathbf{g}_1 = \mathcal{D}(\mathbf{G}_1)$. Depending on the dimension of the null space of $\mathbf{K}$ it may have more. The general solution is:

$$\mathbf{g} = \mathbf{g}_h + \mathbf{g}_1 \tag{11}$$

where $\mathbf{g}_h$ is an element of the null space of $\mathbf{K}$.

Example. Given matrices $\mathbf{F}$ and $\mathbf{H}$ as before, we first get a particular solution $\mathbf{G}_1$:

$$\begin{aligned}\mathbf{G}_1 &= \mathbf{HFH}^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 8 & 6 \\ 6 & 9 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 8 & -6 & -8 & 6 \\ -6 & 9 & 6 & -9 \\ -8 & 6 & 8 & -6 \\ 6 & -9 & -6 & 9 \end{pmatrix} \end{aligned} \tag{12}$$

The corresponding vector solution $\mathbf{g}_1$ is:

$$\mathbf{g}_1 = \mathcal{D}(\mathbf{G}_1) = (6, 8, -6, -6, 9, 6)^T. \tag{13}$$

It contains two negative elements so $\mathbf{G}_1$ is not a nonnegative IA-matrix. We now search the space of solutions to find a realization with only positive resistors. We have:

$$\mathbf{G} = \begin{pmatrix} g_{12}+g_{13}+g_{14} & -g_{12} & -g_{13} & -g_{14} \\ -g_{12} & g_{12}+g_{23}+g_{24} & -g_{23} & -g_{24} \\ -g_{13} & -g_{23} & g_{13}+g_{23}+g_{34} & -g_{34} \\ -g_{14} & -g_{24} & -g_{34} & g_{14}+g_{24}+g_{34} \end{pmatrix}. \tag{14}$$

The equation $\mathbf{H}^T\mathbf{G}\mathbf{H} = \mathbf{H}^T\mathbf{G}_1\mathbf{H}$ reads:

$$\begin{pmatrix} g_{12}+4g_{13}+g_{14}+g_{23}+g_{34} & g_{12}-g_{14}-g_{23}+g_{34} \\ g_{12}-g_{14}-g_{23}+g_{34} & g_{12}+g_{14}+g_{23}+4g_{24}+g_{34} \end{pmatrix} = \begin{pmatrix} 32 & 24 \\ 24 & 36 \end{pmatrix}. \tag{15}$$

Noticing that one equation is redundant, we rewrite the above in vector form:

$$\mathbf{K}\mathbf{g} = \begin{pmatrix} 1 & 4 & 1 & 1 & 0 & 1 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} g_{12} \\ g_{13} \\ g_{14} \\ g_{23} \\ g_{24} \\ g_{34} \end{pmatrix} = \begin{pmatrix} 32 \\ 24 \\ 36 \end{pmatrix} = \mathbf{b}. \tag{16}$$

We use MATLAB to find that the null space of $\mathbf{K}$ is of dimension 3 and get a basis for it. The general form of a solution $\mathbf{g}$ is:

$$\mathbf{g} = c_1 \begin{pmatrix} -0.8165 \\ 0.2041 \\ -0.2041 \\ -0.2041 \\ 0.2041 \\ 0.4082 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0.1610 \\ 0.4685 \\ -0.7905 \\ 0.1610 \\ -0.3221 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ -0.3148 \\ 0.6368 \\ -0.0073 \\ -0.3148 \\ 0.6295 \end{pmatrix} + \begin{pmatrix} 6 \\ 8 \\ -6 \\ -6 \\ 9 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} -0.8165 & 0 & 0 \\ 0.2041 & 0.1610 & -0.3148 \\ -0.2041 & 0.4685 & 0.6368 \\ -0.2041 & -0.7905 & -0.0073 \\ 0.2041 & 0.1610 & -0.3148 \\ 0.4082 & -0.3221 & 0.6295 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} 6 \\ 8 \\ -6 \\ -6 \\ 9 \\ 6 \end{pmatrix}. \quad (17)$$

Let $c_1 = -7.3485, c_2 = -5.7970, c_3 = 11.3311$; we get $\mathbf{g} = (12, 2, 0, 0, 3, 12)^T$.

Let $c_1 = -8.5732, c_2 = -6.7632, c_3 = 13.2197$; we get $\mathbf{g} = (13, 1, 1, 1, 2, 13)^T$.

These are the vectors that generate the IA-matrices of equation (7). In fact, any nonnegative solution can be obtained by choosing an arbitrary cost function $\mathbf{f}$ that is a linear combination of the conductances (i.e $\mathbf{f}(\mathbf{g}) = \mathbf{cg}$ for some nonzero row vector $\mathbf{c}$ with $m(m-1)/2$ components) and then solving the linear programming problem: Maximize $\mathbf{f} = \mathbf{cg}$ subject to $\mathbf{Kg} = \mathbf{b}$ and $\mathbf{g} \geq \mathbf{0}$. [4]

5 An algorithm

Now we can establish a systematic procedure to test a matrix for realizability as an n-port with positive resistors only.

1. Is $\mathbf{F}$ symmetric, nonsingular and with positive main diagonal?
   NO: $\mathbf{F}$ is not realizable. Stop.
   YES: Continue

2. Let $m = 2n$.

3. Choose one of finitely many assignation matrices $\mathbf{H}$ of dimension $m \times n$ not already tested.

4. Let $\mathbf{G}_1 = \mathbf{HFH}^T$.

5. Is $\mathcal{D}(\mathbf{G}_1) \geq 0$?

   YES: $\mathbf{F}$ is produced by $\mathbf{G}_1$, a nonnegative IA-matrix. Stop.

   NO: Continue.

6. Solve for $\mathbf{G}$ in $\mathbf{H}^T\mathbf{GH} = \mathbf{H}^T\mathbf{G}_1\mathbf{H}$. That is, choose a random row vector $\mathbf{c}$ and solve the linear programming problem: maximize $\mathbf{f} = \mathbf{cg}$ subject to $\mathbf{Kg} = \mathbf{b}$, $\mathbf{g} \geq \mathbf{0}$.

7. Are there any feasible solutions to the linear program?

   YES: $\mathbf{F}$ is produced by the matrix $\mathcal{D}^{-1}(\mathbf{g})$ (it may be one of many). Stop.

   NO: Continue.

8. Are there any more $m \times n$ assignation matrices left?

   YES: Go to step 3.

   NO: Continue.

9. Decrease the value of m by 1.

10. Is $m < n + 1$?

    YES: No realization is possible (all possibilities exhausted). Stop.

    NO: Go to step 3.

The number of assignation matrices $\mathbf{H}$ to be tested is finite and thus the algorithm must end in finite time. It will either provide one nonnegative IA-matrix that corresponds to a network that realizes the matrix $\mathbf{F}$ as a conductance or it will prove by exhaustion that no realization with positive resistors is possible. To test if the matrix is realizable as a resistance matrix, we simply apply the algorithm to $\mathbf{F}^{-1}$.

6 Assignation Matrices

The generation of assignation matrices to be tested is an important part of the procedure. Graph theoretical methods exist to count the number of oriented trees with a specified number of nodes [3]. Each of these trees give rise to an incidence matrix. All of the forests corresponding to assignation matrices with n columns consist of different groupings of these incidence matrices.

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