On Completely Regular Spaces

by

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The concept of uniform space is due to Weil(3). Is is well known that a uniformity is generated by the family of all pseudometrics that are uniformly continuous(relative to the product uniformity)and that the topology of a uniformity is completely regular. Given a completely regular space similar results can be obtained but by using pseudometrics that are continuous. Thus the topology $\mathcal{T}$ of a completely regular space is the topology of the family G of all pseudometrics which are continuous (relative to the product topology); also G is the gage of the finest uniformity whose topology is $\mathcal{T}$. Every topology $\mathcal{T}$ has a finest completely regular topology S coarser than $\mathcal{T}$ and S is the topology of all the pseudometrics that are continuous(relative to the product topology $\mathcal{T}\times\mathcal{T}$).

Unless otherwise specified the terminology used in this paper is that of Kelley(1).

Let M be a set and $\mathcal{T}$ a topology for M. Denote by L the cartesian product of M with itself and by (L, S)the product of the topological spaces (M, $\mathcal{T}$), (M, $\mathcal{T}$). Let R denote the reals.

Definition 1. A non-negative function d from L to R is said to be a pseudometric for M iff for all x, y, z in M

(1) $d(x, x) = 0$

(2) $d(x, y) = d(y, x)$ and

(3) $d(x, y) \leq d(x, z) + d(z, x)$.

The ordered pair (M, d) is said to be a pseudometric space. d is said to be continuous(for M)iff d considered as a function from L to R is continuous with R having usual topology.

Theorem 1. Let $(M, \mathcal{T})$ be a topological space and d a pseudometric for M. Then d is continuous for M iff for each point x in M and each $r > 0$ the set $S(x, r) = \{y: d(x, y) < r\}$ is open in $(M, \mathcal{T})$.

Proof. Let x in M and $r > 0$ imply $S(x, r)$ is open in $(M, \mathcal{T})$. To prove d is continuous it is enough to prove that $(x, y)$ in L and $2r > 0$ imply there is a neighborhood A of $(x, y)$ such that $|d(x, y) - d(u, v)| < 2r$ for all $(u, v)$ in A. Take $A = S(x, r) \times S(y, r)$; then A is a neighborhood of $(x, y)$. Let $(u, v) \in A$. Now $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ and so $d(x, y) - d(u, v) \leq d(x, u) + d(v, y) < r + r = 2r$. Similarly $d(u, v) - d(x, y) < 2r$ and so $|d(x, y) - d(u,v)| < 2r$.

Next, to prove the converse, let d be continuous and suppose x in M and $r > 0$. Then d is continuous at $(x, x)$ and so there is a neighborhood A of $(x, x)$ such that $(u, v)$ in A implies $|d(x, x) - d(u,v)| < r$, i.e., $d(u, v) < r$. There are neighborhoods B, C of x such that $B \times C \subset A$. Then $T = B \cap C$ is a neighborhood of x in $(M, \mathcal{T})$ and $T \times T \subset A$. Hence u,v in T implies $(u, v)$ in A and so $d(u, v) < r$. Now x is in T and so v in T implies $d(x, v) < r$ and so $v \in S(x, r)$. Hence $x \in T \subset S(x, r)$. For $y \in S(x, r)$ there is $t > 0$ such that $S(y, t) \subset S(x, r)$ and so $S(x, r)$ will contain a $\mathcal{T}$-neighborhood of y. Hence $S(x, r)$ is $\mathcal{T}$-open.

In proving the converse it is enough to assume d is continuous at each $(x, x)$ for x in M. Hence the following corollary follows.

Corollary 1. d is continuous iff d is continuous on the diagonal $\Delta = \{(x, x): x \in M\}$.

Corollary 2. d is continuous for the pseudometric space $(M, d)$.

Lemma 1. Let $(M, \mathcal{T})$ be a topological space and f a function from M to the reals. Define d by $d(x, y) = |f(x) - f(y)|$. Then d is a pseudometric for M and d is continuous iff f is continuous.

Proof. That d is a pseudometric is obvious. Now f is continuous iff x in M and $r > 0$ imply $S(x, r) = \{y: d(x, y) < r\} = \{y: |f(x) - f(y)| < r\}$ is in $\mathcal{T}$.

Lemma 2. Let f be a continuous function from M to R, $\mathcal{R}$ the usual topology for R and $f^{-1}\mathcal{R}$ the inverse of $\mathcal{R}$ under f. Define d as in Lemma 1. Then $f^{-1}\mathcal{R}$ is the topology of d.

Definition 2. A topological space $(M, \mathcal{T})$ and its topology $\mathcal{T}$ are said to be completely regular iff A is a closed subset of M and x not in A imply there is a continuous function f from M to the closed unit interval $K = [0,1]$ such that $f(x) = 0$ and $fA = 1$.

Definition 3. A topology $\mathcal{T}$ for M is said to be finer than a topology S for M (or S is coarser than $\mathcal{T}$) iff S is a subfamily of $\mathcal{T}$.

Theorem 2. Let $(M, \mathcal{T})$ be a completely regular space. There is then a family F of pseudometrics for M such that $\mathcal{T}$ is the coarsest topology making each member of F continuous.

Proof. Let A be a closed subset of M and x not in A. There is then a continuous function f from M to K such that $f(x) = 0$ and $fA = 1$. For all u, v in M take $d(u, v) = |f(u) - f(v)|$. Then d is a continuous pseudometric for M and the pseudometric topology $\mathcal{T}_d$ of d is coarser than $\mathcal{T}$; also y in A implies $d(x, y) = 1$. For each closed subset A of M and each x not in A there is a pseudometric d with these properties; let F be the family of all such pseudometrics. For d in F, x in M and $r > 0$ take $S(d, x, r) = \{y: d(x, y) < r\}$ and let $\mathcal{B}$ be the family of all such $S(d, x, r)$. Obviously $\mathcal{B}$ is a subfamily of $\mathcal{T}$. Let $T \in \mathcal{T}$ and $x \in T$. Then the complement C of T is closed and so there is d in F such that $d(x, y) = 1$ for all $y \in C$. Then $x \in S(d, x, \frac{1}{2}) \subset T$. Hence $\mathcal{B}$

is a base for $\mathcal{T}$. It now follows easily that $\mathcal{T}$ is the coarsest topology such that each d in F is continuous.

Corollary. Let $\mathcal{U}$ be the uniformity for M generated by F. Then $\mathcal{T}$ is the topology of $\mathcal{U}$ and each d in F is uniformly continuous. Let $\delta$ be the proximity of $\mathcal{U}$ defined by $(A, B) \in \delta$ iff each U in $\mathcal{U}$ intersects $A \times B$ for $A, B \subset M$; then each d in F is a $\delta$-function if all the members of F are totally bounded with the exception of at most one of them.

The last part of the corollary follows from Thampuran (2).

Theorem 3. Let $(M, \mathcal{T})$ be a completely regular space and G the family of all continuous pseudometrics for M. Let $\mathcal{U}$ be the uniformity generated by G. Then $\mathcal{T}$ is the topology of $\mathcal{U}$, i.e., $\mathcal{T}$ is the coarsest topology making each member of G continuous. Also G is the gage of $\mathcal{U}$.

Proof. The topology of $\mathcal{U}$ is obviously coarser than $\mathcal{T}$, the F of Theorem 2 is a subfamily of G and so the first part of the theorem follows. For the last part it is only necessary to notice that each member of G is in the gage of $\mathcal{U}$ and if d is in the gage of $\mathcal{U}$ then d is continuous and so d is in G.

Corollary. Let $\mathcal{V}$ be a uniformity, for M, having $\mathcal{T}$ as its topology. Then the gage of $\mathcal{V}$ is a subfamily of G and hence $\mathcal{U}$ is the finest uniformity with $\mathcal{T}$ as its topology.

Definition 4. A family G of pseudometrics for a set M is said to be a gage iff there is a topology $\mathcal{T}$ for M such that G is the family of all continuous pseudometrics on $M \times M$ relative to $\mathcal{T} \times \mathcal{T}$. The family G is said to be the gage of $\mathcal{T}$.

Definition 5. Let F be a family of pseudometrics for a set M and let $\mathcal{U}$ be the uniformity, for M, generated by F. The topology $\mathcal{T}$ of $\mathcal{U}$ is called the topology of F or the topology generated by F. We will also say that F generates the gage of $\mathcal{T}$.

Let $(M, \mathcal{T})$ be a topological space and F the family of all continuous pseudometrics on M. Then the topology S of F is evidently coarser than $\mathcal{T}$. But $S = \mathcal{T}$ when the space $(M, \mathcal{T})$ is completely regular and conversely.

Theorem 4. A topological space $(M, \mathcal{T})$ is completely regular iff $\mathcal{T}$ is the topology of its gage.

Let $(M, \mathcal{T})$ be a topological space and G the gage of $\mathcal{T}$. Then the topology of G is the finest completely regular topology coarser than $\mathcal{T}$.

Theorem 5. Let $(M, \mathcal{T})$ be a topological space. There is then a finest completely regular topology S coarser than $\mathcal{T}$ and S is the topology of the gage of $\mathcal{T}$.

Let F be a family of pseudometrics for a set M and $\mathcal{T}$ the topology of F. Then the family of all sets of the form $S(d, x, r)$ for d in F, x in M and $r > 0$ is a sub-base for $\mathcal{T}$. Now a pseudometric e for M is continuous iff $S(e, x, r)$ is in $\mathcal{T}$ for each x in M and each $r > 0$ and so e is continuous iff for each x in M and each $r > 0$ there is $s > 0$ and there is a finite number $d_1, \ldots, d_n$ of members of F such that $S(d_1, x, s) \cap \ldots \cap S(d_n, x, s) \subset S(e, x, r)$.

Theorem 6. Let F be a family of pseudometrics for a set M and let G be the gage generated by F. Then a pseudometric d belongs to G iff x in M and $r > 0$ imply there is $s > 0$ and there is a finite subfamily $d_1, \ldots, d_n$ of F such that $\cap\{S(d_i, x, s) : i = 1, \ldots, n\} \subset S(d,x,r)$.

Let $(M, \mathcal{T})$ be a completely regular topological space and G the gage of $\mathcal{T}$. Then $\mathcal{T}$ is the coarsest topology for M such that the identity function from $(M, \mathcal{T})$ to $(M, d)$ is continuous for each d in G. Take P to be the product $\times\{M_d : d \in G\}$ where $M_d = M$ for each d in G and assign the product topology to P. Let $u_d$ denote the projection of u in P to $M_d$ and f the function from M to P defined by $f(x)_d = x$ for each d in G and each x in M. Then $\mathcal{T}$ is the coarsest topology for M such that f is continuous. But f is one to one and hence f is a homeomorphism.

Next assume $\mathcal{T}$ is Hausdorff and use the same notation as in the preceding paragraph. Now the pseudometric space (M, d) is isometric under a map $h_d$ to a metric space $(M_d*, d^*)$ and so $\mathcal{T}$ is the coarsest topology making each of the functions $h_d$ continuous. Let N be the Cartesian product of $M_d*$ for d in G and define the function h from M to N by $h(x)_d = h_d(x)$. Assigning the product topology to N we see that $\mathcal{T}$ is the coarsest topology for which h is continuous. Let x, y be two distinct points of M. If $h(x) = h(y)$ then $h_d(x) = h_d(y)$ for each d in G and so $d(x, y) = 0$ for each d in G. But then M is not Hausdorff. Hence h is one to one and in this case h is a homeomorphism. Hence we have:

Theorem 7. Let $(M, \mathcal{T})$ be a completely regular space and G the gage of $\mathcal{T}$. Then M is homeomorphic to a subspace (actually the diagonal) of the product of all the pseudometric spaces (M, d) for d in G. If $\mathcal{T}$ is Hausdorff then M is homeomorphic to a subspace of the product of all the metric spaces $(M_d*, d^*)$ for d in G.

We also have the following result. Let $(M, \mathcal{T})$ be a completely regular space, G the gage of $\mathcal{T}$ and $\mathcal{U}$ the uniformity generated by G. Then the proximity $\delta$ of $\mathcal{U}$ is the finest proximity whose topology is $\mathcal{T}$. This is because if $\delta'$ is a proximity whose topology is $\mathcal{T}$ then $\delta'$ is the proximity of some uniformity $\mathcal{U}'$ and $\mathcal{U}'$ is coarser than $\mathcal{U}$.

References


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