Regularity Structures

By

D. V. Thampuran

The study of uniform spaces was begun by Weil (3); many of the properties of pseudometric spaces have their analogs for uniform spaces. Also it is well known that a topological space is uniformizable iff it is completely regular. It is shown in this paper that regular spaces have also a similar property; a structure --- regularity --- similar to uniformity exists and a topological space is regular iff its topology is that of a regularity. Hence many of the properties of uniformities can be generalized to regularities.

Terms not defined in this paper have the same meaning as in Kelley (1).

Let M be a set. If A is a subset of M and U is a subset of $M \times M$ then write $AU = \{ y : (x,y) \in U$ for some x in A$\}$. If A contains only one point x then we will write xU for AU. By xUU is meant (xU)U. For a subset A of M take $cA = M - A$.

Let M be a set and $\mathcal{U}$ a family of subsets of $M \times M$ such that for all U in $\mathcal{U}$

1. (x,x) is in U for each x in M
2. the inverse of U is in $\mathfrak{U}$
3. x is in M and U is in $\mathfrak{U}$ imply there is V in $\mathfrak{U}$ such that $xVV \subset xU$
4. U, V in $\mathfrak{U}$ imply $U \cap V$ is in $\mathfrak{U}$
5. $U \subset V \subset M \times M$ and U in $\mathfrak{U}$ imply V is in $\mathfrak{U}$.

Definition 1. $\mathfrak{U}$ as defined above is said to be a regularity for M and $(M, \mathfrak{U})$ is said to be a regular space.

Let $(M, \mathfrak{U})$ be a regular space. Let $\mathfrak{T}$ be the family of all subsets T of M such that x in T implies $xU \subset T$ for some U in $\mathfrak{U}$. Then $\mathfrak{T}$ is a topology for M.

Definition 2. $\mathfrak{T}$ as defined above is said to be the topology and $(M, \mathfrak{U}, \mathfrak{T})$ the topological space of $\mathfrak{U}$ and $(M, \mathfrak{U}, \mathfrak{T})$ will also be called a regular space.

Let $\mathfrak{T}$ be the topology of a regularity $\mathfrak{U}$ for M. When a regular space $(M, \mathfrak{U})$ is considered as a topological space, we will write $(M, \mathfrak{U}, \mathfrak{T})$ or simply M if the context makes the meaning clear.

Theorem 1. Let $(M, \mathfrak{U}, \mathfrak{T})$ be a regular space. Then the interior iA of a subset A of M is given by
$iA = \{x : xU \subset A \text{ for some } U \text{ in } \mathfrak{U}\}$.

Proof. Let $B = \{x : xU \subset A \text{ for some } U \text{ in } \mathfrak{U}\}$.

Now $iA \subset B \subset A$ and so if B is open then $iA = B$. Let x be a point of B. Hence there is U in $\mathfrak{U}$ such that $xU \subset A$ and then there is V in $\mathfrak{U}$ such that $xVV \subset xU$. Let $y \in xV$. Then $yV \subset xVV \subset xU \subset A$ and so y is in B. Therefore $xV \subset B$ which implies B is open.

Corollary. xU is a neighborhood of x for each U in $\mathfrak{U}$.

Definition 3. Let $(M, \mathfrak{T})$ be a topological space. Then the space (and also $\mathfrak{T}$) is said to be regular iff A is a closed subset and x a point of M imply x and A have disjoint neighborhoods.

Theorem 2. Let $(M, \mathfrak{U})$ be a regular space. Then the topology of $\mathfrak{U}$ is regular.

Proof. Let A be a closed set of $(M, \mathfrak{U}, \mathfrak{T})$ and x a point of the complement cA of A. There is then U in $\mathfrak{U}$ such that $xU \subset cA$. Hence there is a symmetric V in $\mathfrak{U}$ such that $xVV \subset xU$. Then xV and AV are disjoint neighborhoods of x and A.

Definition 4. Let M be a set. Then a set-valued set-function m mapping the power set, of M, to itself is said to be a neighborhood function for M iff

1. $m\phi = \phi$
2. $A \subset mA$ for each $A \subset M$
3. $A \subset B \subset M$ imply $mA \subset mB$.

The ordered pair (M,m) is said to be a neighborhood space. A subset A is said to be a neighborhood, relative to (M,m), of a point x of M iff x ε cmcA.

Definition 5. Let (M,m), (L,p) be two neighborhood spaces, f a function from M to L, and x a point of M. Then f is said to be continuous at the point x iff B is a neighborhood of f(x) implies the inverse of B, under f, is a neighborhood of x. We will say f is continuous iff f is continuous at every point of M.

Let N denote the set consisting of 1,1/2, 1/3,...,0. Define a distance function e for N as follows. Let u,v represent points of N.

$$e(u,v) = \begin{cases} 0 \text{ if there is no member of N between } u,v \\ |u-v| \text{ otherwise.} \end{cases}$$

For r > 0 let V(r) be the set of all pairs (u,v) such that e(u,v) < r. Define a neighborhood function n for N as follows. For a subset A of N take nA to be the set of all points u such that uV(r) intersects A for each r > 0. Thampuran (2) has proved the following result: If (M, $\mathcal{T}$) is a regular space, A a closed subset and x a point not in A then there is a continuous function f from (M, $\mathcal{T}$) to (N,n) such that f(x) = 0 and f is 1 on A.

Theorem 3. Let (M, $\mathcal{T}$) be a topological space. Then $\mathcal{T}$ is regular iff it is the topology of a regularity for M.

Proof. First, let the space be regular. Then for each closed set A and each point x not in A, there is a continuous function f from M to (N,n) such that $f(x) = 0$ and f is 1 on A; for y,z in M write $d(y,z) = e(f(y), f(z))$. There is such a d for each closed subset A and each point x, of M, such that x is not in A; let D be the family of all such d. For d in D and $r > 0$ take $V(d,r) = \{(x,y): d(x,y) < r\}$. Consider a $U = V(d,r)$ and a point x in M. It is obvious xU is a neighborhood of x in $(M, \mathfrak{T})$; let cA be the interior of xU. Then A is a closed set and so there is a continuous function g from $(M, \mathfrak{T})$ to (N,n) such that $g(x) = 0$ and g is 1 on A. For y,z in M take $b(y,z) = e(g(y), g(z))$. Take $V = V(b, 1/8)$. Let s be in xV. Then $b(x,s) < 1/8$ and so $g(s) < 1/8$. If t is in sV then $g(t) < 1/4$ and so t is in cA. Hence $xVV \subset cA \subset xU$.

Let $\mathfrak{U}$ be the family of all subsets W of $M \times M$ such that W contains the intersection of a finite number of the sets V(d,r) for d in D and $r > 0$. It is clear that $\mathfrak{U}$ is a regularity for M.

Finally let $\mathfrak{T}'$ be the topology of $\mathfrak{U}$ and let T be a member of this topology. Then x in T implies there is U in $\mathfrak{U}$ such that $xU \subset T$. Now $U \supset$ the intersection of a finite number of V(d,r), d in D and $r > 0$; each xV(d,r) is a $\mathfrak{T}$-neighborhood of x and so xU is also a $\mathfrak{T}$-neighborhood of x. Hence $T \in \mathfrak{T}$ and $\mathfrak{T}' \subset \mathfrak{T}$. Next let $S \in \mathfrak{T}$ and $x \in S$. Then there is d in D such that

$xV(d, 1/4) \subset S$. Hence S is in $\mathcal{T}'$ and so $\mathcal{T} \subset \mathcal{T}'$.

That the topology of a regularity is regular has already been proved. This completes the proof.

References

1. J. L. Kelley, General Topology, Princeton (1968).

2. D. V. Thampuran, Regular spaces and functional separation (to appear).

3. A. Weil, Sur les espaces à structure uniforme et sur la topologie générale, Act. sci. et ind., 551 Paris (1937).