CALCULATION OF THE ELASTIC MODULI OF BINARY COMPOSITES

Franklin F. Y. Wang

Department of Materials Science
State University of New York at Stony Brook
Stony Brook, New York   11790

ABSTRACT

A modification was introduced to Hashin and Shtrikamn's variational approach to calculate the elastic moduli of binary composites. The distribution effect of dispersed phase was taken into account. As a first approximation, a normal distribution was assumed. The calculations of moduli of nine systems were compared with experiments for concentrations up to 50%. The introduction of distribution effect showed distinctive improvement.

I. Introduction:

Recent technological advances have produced composite materials of
many kinds. Some composites are made from combination of materials, and
they can be either ceramics in plastics (such as fiberglass reinforced
plastics), or ceramics in metals (such as dispersed oxide particles in
metasl), or plastics in ceramics (such as plastic impregnated concrete),
or metal in ceramics (such as metal whiskers in ceramics). Some composites
are made from a single system in situ. These composites have been, in the
past, considered as single phase materials. They can be either pores in
single phase materials, or glass in crystalline materials (such as china-
ware or refractions), or crystals in glass (such as glass ceramics), or
eutectics in bulk matrix.

The physical properties of the composites are obviously related to the
corresponding properties of the individual phases in the composites. Only
a few properties, such as volume, and density, are additive, in that the lin-
ear combinations of properties of individual phases give the properties of
the composite.

All other physical properties such as elastic moduli, of the composites
are related to the corresponding properties of constituents when the following
factors are taken into account. The factors are:

1. volume concentrations of constituents

2. boundary conditions between phases

3. boundary shapes of phases

4. distribution of constituents

5. characteristics of interfaces

The last factor is a very significant factor. But it was not included in
any of the studies for the simple reason that the quantitative characteristics of

interfaces, such as the wettability, adhesion, the chemical compositions, volumes, density, and other physical properties of the interfaces, cannot be determined and are therefore totally unknown.

The first factor is naturally the most dominant factor. It is always accompanied by the third and fourth factors, which were not always considered. Most studies dealt with the third factor, i.e. the shape factor, by the assumption of a rigid spherical shape for the disperant.[1] Some attempts were made to consider the shape of the second phase as ellipsoidal[2]. Others treated the dispersed phase as fibers[3]. Few studies considered the fourth factor, i.e., the distribution of disperser at all.

The second factor took up the majority of studies. Up to now no formula has been derived which is theoretically rigorous and fits experimental results for the whole range of volume concentration. Reviews of these investigations may be found elsewhere in the literature[4]. In lieu of the exact formula, upper and lower bounds for the elastic moduli were obtained. In Paul's approach[5], a variational principle was used in which the same simple tension in matrix and spherical particles was taken as the admissible stress system and a simple tension deformation for an admissible displacement field. This approach produces bounds too far apart to be actually useful. A more satisfactory approach was made by Hashin and Shtrikman[6]. It employed the variational principle by considering the change in strain energy in a loaded homogeneous body due to the insertion of inhomogeneities. In an arbitrary reference cube in the composite material which is large compared to the size of the inhomogeneities, yet small compared to the whole body, the volume average of a quantity such as displacement, strains, stress or phase volume fraction is the same for the whole body and the reference cube.

For quasi-isotropic and quasi-homogeneous two phase materials, the lower and upper bounds for bulk modulus are given by

$$K_1^* = K_1 + \cfrac{v_2}{\cfrac{1}{K_2-K_1} + \cfrac{3v_1}{3K_1 + 4G_1}} \qquad (1)$$

$$K_2^* = K_2 + \cfrac{v_1}{\cfrac{1}{K_1-K_2} + \cfrac{3v_2}{3K_2+4G_2}} \qquad (2)$$

and $K_1^* < K^* < K_2^*$. K is the bulk modulus, G is the shear modulus, and v is the phase volume fraction. Subscripts 1 and 2 denote the phases, and asterick (*) denote the composite. K_2^* is the upper bound and K_1^* is the lower bound.

Most of the studies dealt with the fourth factor, i.e. the distribution of inhomogeneities, by assuming certain geometrical packings [2,3b,5] such as hexagonal arrays or a uniform random array. This paper extends the approach of Hashin and Shtrikman and applies a normal (gaussian) distribution of inhomogeneities with additional assumptions. The calculations agreed well with experimental results of several different composites.

II. General Method:

The general method in this study adopts, to a large extent, the approach of Hashin and Shtrikman[6]. They assumed that a reference cube exists in the body having the same volume average of any quantity, such as displacement, strain, stress, or phase volume fraction, as the whole body. This study assumes that the body can be subdivided into equal sized cubic cells. Within each cell i, there is a phase volume fraction of phase 1, $(v_1)_i$. Let $(v_1)_i$ be p. The distribution of the phase volume fraction of phase 1 is f(p). In the case of continuous distribution, the normalized condition hold such that

$$\int_{-\infty}^{\infty} f(p) \, dp = 1 \qquad (3)$$

Using Hashin and Shtrikman's upper bound expression for the modulus, eq.(2), the effective bulk modulus for the composite is therefore

$$\langle K \rangle = \int_{-\infty}^{\infty} K_2{}^* \ f(p) \ dp \tag{4}$$

Expanding the eq.(2) in a Taylor's series in terms of p, and substituting it into eq.(4), it becomes

$$\langle K \rangle = K_2 + \frac{1}{C+B} \int_{-\infty}^{\infty} pf(p)dp + \frac{B}{(C+B)^2} \int_{-\infty}^{\infty} p^2 f(p)dp + \cdots \tag{5}$$

where

$$C = \frac{1}{K_1 - K_2} \tag{6}$$

$$B = \frac{3}{3K_2 + 4G_2} \tag{7}$$

In principle, $f(p)$ can be measured by the various quantitative microscopic methods[7]. Then, the bulk modulus of the composite $\langle K \rangle$ can be evaluated directly with the use of eq.(5).

In cases where $f(p)$ was not experimentally known, a distribution form must be assumed. As a first approximation, a normal (gaussian) distribution is therefore assumed. It is

$$f(p) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left(- \frac{(p - v_1)^2}{2\sigma^2} \right) \tag{8}$$

Where σ is the standard deviation and v_1 is the mean. It is reasonable that the standard deviation σ should be dependent on the mean phase volume fraction v_1. In other words, the width of the distribution is a function of the mean value. For this relationship, we assume that the variance σ^2 is the sum of an arithmatic progression which starts at zero, increases $S/5000$ at each step, and has $100v_1$ terms. It is namely,

$$\sigma^2 = S\left(v_1^2 - v_1\right) \tag{9}$$

This relationship fulfills the boundary conditions that $\sigma = 0$ when $v_1 = 1$ or 0, which means no standard deviations for pure phase 1 or pure phase 2 body. Combining eqs.(8) and (9) with eq.(5), one obtains the following expression after

truncating the higher terms

$$<K> = K_2 + \frac{1}{C+B} -S \frac{B}{(C+B)^2} p + \frac{B}{(C+B)^2} (1+S) -S \frac{3B^2}{(C+B)^3} p^2 \qquad (10)$$

In order to compare results of eq.(10) with experiments, difficulties arise because most experiments measured Young's modulus $<E>$ rather than bulk modulus $<K>$ of the composite. In a composite, the relationship between the bulk and Young's moduli is difficult to ascertain. As a first approximation, we assume that

$$\frac{<E>}{<K>} = \frac{E_2}{K_2} \qquad (11)$$

and use eqs.(10) and (11) to obtain the calculated $<E>$ values.

III. Results;

: We have calculated the Young's moduli of nine types of binary composites and compared the calculations with known experiments. In some of the experiments, analytical expressions were derived from data. The simplest expression is the linear relationship[8], where

$$<E> = E_2 (1 - \alpha_E p) \qquad (12)$$

The other expression is of the exponential form[9], where

$$<E> = E_2 \exp (-Ap) \qquad (13)$$

It is obvious that eq.(12) is only the Taylor's series expansion of eq.(13) in the first leading terms. The elastic properties of the nine binary composite systems are listed in Table 1.

Substituting the elastic moduli values into eq.(10) and using the assumptions eq.(10) and eq.(11), and the Young's modulus of the composite $<E>$ can be calculated for any given S. value. It was found that the sample correlation coefficient, as calculated from the two variables linear regression formula[17], is not sensitive to the variations in S values. Instead, the standard deviation for the % differences

in Young's modulus, σ_E was found to be more indicative of the experimental fit, and σ_E is defined as

$$\sigma_E = \sqrt{\Sigma \left(\frac{<E>_{expt.} - <E>_{calc.}}{<E>_{expt.}} \right)^2} \tag{14}$$

Where $<E>_{expt.}$ is the Young's modulus from experimental data, either from table, figure, or analytical expression, and $<E>_{calc.}$ is the calculated Young's modulus.

In order to test the effect of normal distribution, one can substitute it with a constant. In that case, the combination of eqs.(10) and (11) gives

$$<E> \doteq \frac{E_2}{K_2} \left(K_2 + \frac{1}{C+B} p - \frac{B}{(C+B)^2} p^2 + \cdots \right) \tag{15}$$

As s first approximation, it can be made to have the form of eq.(13), namely,

$$<E> = E_2 \exp(-A'p)$$
$$A' = -\frac{E_2}{K_2(C+B)} \tag{16}$$

The degrees of experimental fit of the $<E>$ values, calculated from either eq.(16) or from eqs.(10) and (11), as indicated by their respective σ_E values, are listed in Table 2. The calculated $<E>$ values for concentions up to 50% in the nine composite systems are listed in Table 3.

IV. Discussion:

As shown in Tables 2 and 3, the calculated Young's moduli $<E>$ values from eq.(16) in all cases have larger deviations from experimental data than those calculated from eqs.(10) and (11). The former case assumes an uniform distribution of dispersed particles, and the latter case assumes a normal (gaussian) distribution of dispersed particles, with a specific assumption about the standard deviation (eq.(9)). The improvement of experimental fit, as shown in Tables 2 and 3, by the introduction of latter assumptions is very evident.

In view of the fact that these Young's moduli $<E>$ values were calculated by requiring only the knowledge of the elastic moduli values of each individual phase material and the volume fraction of the dispersed phase and a minimum of adjustable parameters, their agreements with experiments are all the more impressive, especially in the low concentration regions.

In three cases, namely, pores in Y_2O_3 (system no.1), alumina in glass (system no.6) and spherical tungsten in glass (system no.7), the σ_E values are small, i.e., less than 5%, whether they were calculated from eq. (16) or from eqs. (10) and (11). They represent the cases where the distribution of the dispersed phase is rather uniform, and the basic assumptions from the Hashin and Shtrikman's approach are essentially fulfilled. As shown in Table 3, the experimental fits in these cases were very good up to higher concentrations.

In system no.2, namely, pores in glass, the σ_E values are moderately low, i.e., 10.098% and 7.392%. The good agreements at low concentrations testify to the correctness of the elastic moduli values used for glass in these calculations. The increases in the deviations, from the eq.(16) calculations, at higher concentrations demonstrate that the uniform distribution of the dispersed phase, namely, pores, becomes a progressively unrealistic assumption with the increasing concentrations of pores. In other words, the effect of distribution is definitely significant in this case. This is confirmed by the lower σ_E value from eqs. (10) and (11). However, the assumptions used about the normal distribution are not completely correct. Hence, the σ_E value is not small. It is therefore shown that this type of calculation is more than an attempt to curve fit the data. Rather some additional information can be learnt about the composite even when the data fits are not good.

In system no.3, pores in MgO, the σ_E values are very large. The extremely poor agreements at the lowest concentration are probably due to the fact that the elastic moduli values used for MgO in the calculations are different than those of MgO in the experiments. The effect of distribution of pores is evident by the comparison of σ_E

values.

In the case of pores in alumina, experiments [9,13] showed two different fitting parameters, A' of either 4.36 or 3.95. This is a clear confirmation of the distribution effect of pores. Unfortunately, the elastic moduli values for alumina are probably also different than those in the experiments. But, the inproduction of distribution is definitely an improvement, as shown in systems no.4 and no.5.

In the case of C in ZrC, experiment [16] showed different moduli behaviors for the parallel or perpendicular cases of hot press directions. This is another demonstration of the effect of distributions of the dispersed phase. The improvements in σ_E values by the eqs. (10) and (11) calculations confirm this effect. However, the hot press operation may have also introduced an orientation effect. In that case, the distribution must be highly anisotropic. The assumption of normal distribution is obviously inadequate. This is also confirmed by the large σ_E values.

In the eqs. (10) and (11) calculations, the adjustable parameters S were selected at the minimum σ_E values. The dependence of σ_E on S values is shown in Table 4. This type of dependence was not found to be a sharp function of S. Therefore, the above discussion does not depend on the specific S value selected.

In summary, it can be concluded that;

1) The introduction of a distribution to Hashim and Shtrikman's approach is a valid improvement, i.e. eq.(5).

2) The introduction of a normal distribution is a valid first approximation, namely, e.q. (8).

3) In place of experimental σ vs. v_1 relationship, the assumed relationship, namely, eq.(9), is workable, but it requires improvement.

Acknowledgement

Reference

1. For example

    (a) A. Einstein, Investigations on the Theory of Brownian Motion, Dover Publications, New York 1956, pp. 36-62.

    (b) J. K. Mackenzie, "The Elastic Constants of a Solid Containing Spherical Holes", Proc. Phys. Soc. (London), B63, 2-11 (1950).

    (c) Z. Hashin, "The Elastic Moduli of Heterogeneous Materials", J. Appl. Mech. 29, Trans. ASME, 84, Series E, 143-50 (1962).

2. For example

    (a) R. C. Rossi, "Prediction of the Elastic Moduli of Composites". J. Am. Ceram. Soc. 51, (8), 433-9 (1968).

    (b) M. A. Sadowsky and E. Sternberg, "Stress Concentration Around a Triaxial Ellipsoidal Cavity", J. Appl. Mech. 19 (1), 19-30 (1951).

3. For example

    (a) B. W. Rosen, A. E. Ketler, and Z. Hashin, "Hollow Glass Fiber Reinforced Plastics", Contract No w61-0613-d, Final Report, Space Sciences Laboratory, General Electric Company, Philadelphia, Pa., November 1962.

    (b) Z. Hashin and B. W. Rosen, J. Appl. Mech. 31, (2), 223-32 (1964).

4. For example

    (a) H. L. Frisch and R. Simha, "The Viscosity of Colloidal Suspensions and Macro-Molecular Solutions", Chapter 14 in Rheology, F. R. Eirich, editor, vol. 1, Academic Press, New York, N. Y. 1956 .

    (b) M. Reiner, "Rheology" in Encyclopedia of PDysics, S. Fluggi, editor, vol 6, "Elasticity and Plasticity", Springer Verlag, Germany 1958, pp 522-534.

5. B. Paul, "Prediction of Elastic Constants of Multi-Phase Materials", Trans AIME, 218, (1) 36-41 (1960).

6. Z. Hashin and S. Shtrikman, "A Variational Approach to the Theory of the Elastic Behavior of Mulitphase Materials". J. Mech. Phys. Solids 11, 127-40 (1963).

7. For example, Quantitative Microscopy, edited by Robert T. DeHoff and Frederich N. Rhines, McGraw Hill, New York (1968).

8. D.P.H. Hasselman and R. M. Fulrath, "Elastic Properties of Multiphase Ceramic Systems", UCRL-16360, University of California, Lawrence Radiation Laboratory, September 22, 1965.

9. R. M. Spriggs, "Expression for Effect of Proosity on Elastic Modulus of Polycrystalline Refractory Materials, Particularly Aluminum Oxide", J. Am Ceram.Soc., 44, (12), 628-9 (1961).

10. W.R. Manning and Orville Hunter, Jr,, "Porosity Dependence of Young's and Shear Moduli of Polycrystalline Yttriam Oxide", J. Am. Ceram. Soc., 51. (9) 537-8 (1968).

11. D.P.H. Hasselman and R. M. Fulruth, "Effect of Small Fraction of Spherical Porosity on Elastic Moduli of Glass", J. Am. Ceramic Soc. <u>47</u>, (1), 52-53 (1964).

12. R. M. Spriggs, L.A. Brissette and T. Vasilos, "Effect of Porosity on Elastic and Shear Moduli of Polycrystalline Magnesium Oxide", J. Am. Ceram. Soc. <u>45</u>, (8) 400 (1962).

13. F.P. Knudson, "Effect of Porosity on Young's Modulus of Alumina", J. Am.Ceram. Soc., <u>45</u>, (9), 452-3 (1962).

14. D.P.H. Hasselman and R. M. Fulrath, "Effect of Alumina Dispersions on Young's Modulus of a Glass", J. Am,Ceram. Soc., <u>48</u>. (4) 218-9 (1965).

15. D.P.H. Hasselman and R. M. Fulrath, "Effect of Spherical Tungsten Dispersions on Young's Modulus of a Glass", J. Am. Ceram. Soc., <u>48</u>, (10) 548-9 (1965).

16. D.P.H. Hasselman, "Experimental and Calculated Young's Moduli of Zirconium Carbide Containing a Dispersed Phase of Graphite", J. Am. Ceram.Soc. <u>46</u>, (2), 103-4 (1963).

17. Edwin L. Crow, Francis A. Davis, and Margaret W. Maxfield, <u>Statistics Manual</u>. Dover New York p. 158, (1960).

Table 1
Elastic Properties of Binary Composites

Systems No.	Phase 1	Phase 2	Young's Modulus E_1	(Kilobar) E_2	Bulk Modulus K_1	(Kilobar) K_2	Shear Modulus G_1	(Kilobar) G_2	Ref. No.
1	pores	Y_2O_3	.0	1715.0	.0	1354.7	.0	665.0	$\langle E \rangle = 1715.0\ (1-2.02v_1)$ 10
2	pores	glass	.0	805.0	.0	442.8	.0	336.3	$\langle E \rangle = 805.0\ (1-2.0v_1)$ 11
3	pores	MgO	.0	3177.9	.0	1564.1	.0	1795.9	$\langle E \rangle = 3177.9 \exp(-4.74v_1)$ 12
4	pores	Al_2O_3	.0	4110.4	.0	2448.0	.0	1635.0	$\langle E \rangle = 4110.4 \exp(-4.36v_1)$ 9
5	pores	Al_2O_3	.0	4110.4	.0	2448.0	.0	1635.0	$\langle E \rangle = 4110.4 \exp(-3.95v_1)$ 13
6	$Al2O_3$	glass	4110.4	805.0	2448.0	442.8	1635.0	336.3	Table I of Ref. 8 and Ref. 14
7	W	glass	3550.0	805.0	2366.7	442.8	1481.0	336.3	Table II of Ref.8 and Ref. 15
8	C ($\perp$ Hot press) direction	ZrC	103.5	4750.3	69.0	2142.8	45.4	2030.1	Figure 8 of Ref.8 and Ref. 16
9	C ($\parallel$ Hot press) direction	ZrC	103.5	4750.3	69.0	2142.8	45.4	2030.1	Figure 8 of Ref.8 and Ref. 16

Table 2

Comparison of Experimental Fits

Systems No.	Phase 1	Phase 2	σ_E from \<E>calcu. from eq. (16)	S value used in eq. (10)	σ_E from \<E> calculated from eqs. (10) & (11)
1	pores	Y_2O_3	2.543%	0.0095 ± 0.0025	1.554%
2	pores	glass	10.098%	-1.015 ± 0.005	7.392%
3	pores	MgO	162.287%	0.997 ± 0.001	71.046%
4	pores	Al_2O_3	113.098%	0.227 ± 0.0002	37.009%
5	pores	Al_2O_3	83.821%	0.1845 ± 0.0005	27.366%
6	Al_2O_3	glass	2.795%	0.2005 ± 0.0005	2.261%
7	W	glass	1.903%	0.012 ± 0.003	1.480%
8	C ($\perp$ to Hot press direction)	ZrC	39.891%	0.873 ± 0.003	25.882%
9	C ($\parallel$ to Hot press direction)	ZrC	62.280%	-1.70 ± 0.005	29.044%

Table 3

Calculated Moduli from eq.(16) and from eqs.(10) and (11)

System No.1   Phase 1 = Pores   Phase 2 = Y_2O_3 S = 0.009

p	$\langle E \rangle_{expt}$ (Kilobar)	$\langle E \rangle_{calc.}$ from eq. (16)	$\dfrac{\langle E \rangle_{expt} - \langle E \rangle_{calc.\,eq.(16)}}{\langle E \rangle_{expt.}}$	$\langle E \rangle_{calc.}$ from eqs.(10) & (11)	$\dfrac{\langle E \rangle_{expt.} - \langle E \rangle_{calc.}}{\langle E \rangle_{expt.}}$ eqs. (10) & (11)
0.02	1647.0	1630.4	1.005%	1632.0	0.911%
0.04	1579.0	1550.1	1.833	1554.0	1.581
0.06	1511.0	1473.6	2.473	1481.1	1.976
0.08	1443.0	1401.0	2.913	1413.3	2.056
0.10	1375	1331.9	3.135	1350.6	1.775
0.12	1307.0	1266.2	3.120	1293.0	1.075
0.14	1239.0	1203.8	2.841	1240.4	-0.116
0.16	1171.0	1144.4	2.268	1193.0	-1.882

$\sigma_E = 2.543\%$ $\sigma_E = 1.554\%$

System No.2   Phase 1 = Pores   Phase 2 = Glass   S = -1.01

$-p$	$\langle E \rangle_{expt}$ Kilobar	$\langle E \rangle_{calc.}$ from eq.(16)	$\dfrac{\langle E \rangle_{expt} - \langle E \rangle_{calc.}\ eq.(16)}{\langle E \rangle_{expt.}}$	$\langle E \rangle_{calc.}$ from eqs.(10) & (11)	$\dfrac{\langle E \rangle_{expt.} - \langle E \rangle_{calc.}\ eqs.(10)\ \&\ (11)}{\langle E \rangle_{expt}}$
10^{-4}	804.8	804.8	-0.001%	804.7	0.019%
5×10^{-4}	804.2	804.2	-0.004%	803.4	0.095%
10^{-3}	803.3	803.4	-0.007%	801.8	0.190%
2.5×10^{-3}	800.9	801.0	-0.019%	797.1	0.473%
0.05	722.1	728.8	-0.936%	664.5	7.977%
0.10	639.2	659.9	-3.243%	559.9	12.397%
0.15	556.3	597.5	-7.409%	487.8	12.307%
0.20	473.3	540.9	-14.283%	444.5	6.088%
0.25	390.4	489.8	-25.446%	426.6	-9.258%

$$\sigma_E = 10.098\% \qquad\qquad\qquad\qquad \sigma_E = 7.392\%$$

System No.3   Phase 1 = Pores   Phase 2 = MgO   S = 0.997

p	$\langle E\rangle_{expt}$ Kilobar	$\langle E\rangle_{calc.}$ from eq.(16)	$\dfrac{\langle E\rangle_{expt} - \langle E\rangle_{calc.}\ eq.(16)}{\langle E\rangle_{expt.}}$	$\langle E\rangle_{calc.}$ from eqs.(10) & (11)	$\dfrac{\langle E\rangle_{expt.} - \langle E\rangle_{calc.}\ eqs.(10)\ \&\ (11)}{\langle E\rangle_{expt}}$
0.05	2507.4	2925.8	-16.688%	3045.2	-21.449%
0.10	1978.3	2693.7	-36.161%	2838.7	-43.495
0.15	1560.9	2479.9	-58.884%	2571.7	-64.761
0.20	1231.5	2283.2	-85.399%	2257.0	-83.271
0.25	971.6	2102.0	-116.338%	1907.7	-96.334
0.30	766.6	1935.3	-152.441%	1536.7	-100.454
0.35	604.9	1781.7	-194.569%	1157.2	-91.312
0.40	477.2	1640.4	-243.727%	782.0	-63.860
0.45	376.5	1510.2	-301.089%	424.2	-12.660

$\sigma_E = 162.287\%$ $\sigma_E = 71.046\%$

System No. 4 Phase 1 = Pores Phase 2 = Al_2O_3 S = 0.227

p	$\langle E\rangle_{espt}$ Kilobar	$\langle E\rangle_{calc.}$ from eq. (16)	$\dfrac{\langle E\rangle_{expt.} - \langle E\rangle_{calc.}\ eq.(16)}{\langle E\rangle_{expt}}$	$\langle E\rangle_{calc.}$ from eqs. (10) & (11)	$\dfrac{\langle E\rangle_{expt.} - \langle E\rangle_{calc.}\ eqs.(10)\ \&\ (11)}{\langle E\rangle_{expt.}}$
0.05	3305.3	3696.4	-11.833%	3720.0	-12.548%
0.10	2657.8	3324.1	-25.067	3316.3	-24.775
0.15	2137.2	2989.3	-39.867	2905.0	-35.921
0.20	1718.6	2688.2	-56.418	2491.5	-44.974
0.25	1382.0	2417.5	-74.927	2081.7	-50.631
0.30	1111.3	2174.0	-95.627	1681.0	-51.268
0.35	893.6	1955.0	-118.776	1295.2	-44.936
0.40	718.6	1758.1	-144.665	929.7	-29.387
0.45	577.8	1581.0	-173.617	590.4	- 2.173
0.50	464.6	1421.8	-205.995	282.7	39.158

$\sigma_E = 113.098\%$ $\qquad\qquad\qquad$ $\sigma_E = 37.009\%$

System No. 5   Phase 1 = Pores   Phase 2 = Al_2O_3 S = 0.184

p	$\langle E \rangle_{expt}$ Kilobar	$\langle E \rangle_{calc.}$ from eq. (16)	$\dfrac{\langle E \rangle_{expt.} - \langle E \rangle_{calc.}\ eq.(16)}{\langle E \rangle_{expt}}$	$\langle E \rangle_{calc.}$ from eqs. (10) & (11)	$\dfrac{\langle E \rangle_{expt.} - \langle E \rangle_{calc.}\ eqs.(10)\ \&\ (11)}{\langle E \rangle_{expt.}}$
0.05	3373.7	3696.4	-9.564%	3713.6	-10.073%
0.10	2769.1	3324.1	-20.043%	3310.5	-19.553
0.15	2272.8	2989.3	-31.524%	2905.8	-27.850
0.20	1865.5	2688.2	-44.103%	2504.0	-34.226
0.25	1531.1	2417.5	-57.885%	2109.5	-37.775
0.30	1256.7	2174.0	-72.986%	1727.1	-37.430
0.35	1031.5	1955.0	-89.530%	1361.3	-31.970
0.40	846.6	1758.1	-107.657%	1018.5	-20.305
0.45	694.9	1581.0	-127.518%	699.9	-0.725
0.50	570.4	1421.8	-149.278%	411.5	-27.847

σ_E = 83.821% σ_E = 27.366%

System No.6   Phase 1 = Al_2O_3 Phase 2 = Glass   S = 0.201

p	$\langle E \rangle_{expt}$ Kilobars	$\langle E \rangle_{calc}$ from eq.(16)	$\dfrac{\langle E \rangle_{expt.} - \langle E \rangle_{calc.}\ eq.(16)}{\langle E \rangle_{expt}}$	$\langle E \rangle_{calc.}$ from eqs.(10) & (11)	$\dfrac{\langle E \rangle_{expt} - \langle E \rangle_{calc.}}{\langle E \rangle_{expt.}}$ eqs.(10) & (11)
0.10	924.0	925.4	-0.147%	941.9	-1.937%
0.20	1071.0	1063.7	0.680%	1095.7	-2.310%
0.30	1228.0	1222.8	0.427%	1264.6	-2.980%
0.40	1447.0	1405.6	2.863%	1446.5	0.035%
0.45	1585.0	1507.0	4.922%	1541.7	2.730%
0.50	1678.0	1615.7	3.711%	1639.5	2.294%
			$\sigma_E = 2.795\%$		$\sigma_E = 2.261\%$

System No. 7   Phase 1 = W   Phase 2 = Glass $S = 0.012$

p	$\langle E \rangle_{expt}$ Kilobar	$\langle E \rangle_{calc}$ from eq.(16)	$\dfrac{\langle E \rangle_{expt.} - \langle E \rangle_{calc.}\ eq.(16)}{\langle E \rangle_{expt}}$	$\langle E \rangle_{calc.}$ from eqs.(10) & (11)	$\dfrac{\langle E \rangle_{expt} - \langle E \rangle_{calc}\ eqs.(10)\ \&\ (11)}{\langle E \rangle_{expt.}}$
0.10	909.0	923.7	-1.618%	924.3	-1.681%
0.20	1055.0	1059.9	-0.466%	1058.8	-0.358%
0.30	1180.0	1216.2	-3.068%	1208.4	-2.405%
0.40	1375.0	1395.5	-1.494%	1373.0	0.147%
			$\sigma_E = 1.903\%$		$\sigma_E = 1.480\%$

System No.8   Phase 1 = C   Phase 2 = ZrC $S = -0.875$

($\perp$ to Hot press direction)

p	$\langle E \rangle_{expt}$ Kilobar	$\langle E \rangle_{calc.}$ from eq.(16)	$\dfrac{\langle E \rangle_{expt} - \langle E \rangle_{calc.\ eq.(16)}}{\langle E \rangle_{expt}}$	$\langle E \rangle_{calc.}$ from eqs.(10) & (11)	$\dfrac{\langle E \rangle_{expt} - \langle E \rangle_{calc}\ eqs.(10)\ \&\ (11)}{\langle E \rangle_{expt}}$
0.20	2694.4	3387.3	-25.715%	2920.0	-8.372%
0.25	2277.8	3112.7	-36.655%	2683.9	-17.831%
0.35	1722.2	2628.5	-52.621%	2415.8	-40.270%
			$\sigma_E = 39.891\%$		$\sigma_E = 25.882\%$

System No. 9 Phase 1 = C Phase 2 = ZrC S = -1.70

(|| to Hot press direction)

p	$\langle E \rangle_{expt}$ Kilobar	$\langle E \rangle_{calc.}$ from eq. (16)	$\dfrac{\langle E \rangle_{expt} - \langle E \rangle_{calc.} \text{ eq.}(16)}{\langle E \rangle_{expt}}$	$\langle E \rangle_{calc.}$ from eqs. (10) & (11)	$\dfrac{\langle E \rangle_{expt} - \langle E \rangle_{calc}}{\langle E \rangle_{expt}}$ eqs. (10) & (11)
0.20	2277.8	3387.3	-48.711%	2482.6	-8.994%
0.25	2000.0	3112.7	-55.635%	2275.3	-13.765%
0.35	1472.2	2628.5	-78.538%	2172.2	-47.543%
			$\sigma_E = 62.280\%$		$\sigma_E = 29.044\%$

Table 4

Dependence of σ_E on S Values

Systems	Phase 1	Phase 2	S	σ_E from eqs. (10) & (11)
1	pores	Y_2O_3	0.001	3.100%
			0.01	3.096%
			0.1	3.516%
2	pores	glass	-0.60	15.440%
			-1.01	14.452%
			-1.25	14.775%
3	pores	MgO	0.93	71.714%
			0.997	70.691%
			1.07	71.899%
4	pores	Al_2O_3	0.19	38.304%
			0.227	37.009%
			0.25	37.495%
5	pores	Al_2O_3	0.15	28.400%
			0.184	27.364%
			0.19	27.389%
6	Al_2O_3	glass	0.080	3.336%
			0.200	2.261%
			0.290	2.901%
7	W	glass	-0.01	1.907%
			0.012	1.857%
			0.035	1.912%
8	C ($\perp$ to Hot press direction)	ZrC	-0.40	37.142%
			-0.872	36.182%
			-1.4	37.245%
9	C ($\parallel$ to Hot press direction)	ZrC	-1.10	31.622%
			-1.70	30.517%
			-2.30	31.600%