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ORTHONORMAL SERIES EXPANSIONS OF CERTAIN DISTRIBUTIONS AND DISTRIBUTIONAL TRANSFORM CALCULUS

by

A. H. Zemanian

Fifth Scientific Report

Contract No. AF 19(628)-2981

Project No. 5628

Task No. 562801

NOVEMBER 15, 1964

Prepared for

Air Force Cambridge Research Laboratories
Office of Aerospace Research
United States Air Force
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1. A. H. Zemanian, "A Time-Domain Characterization of Rational Positive-Real Matrices," First Scientific Report, AFCRL-63-390, College of Engineering Tech. Rep. 12, State University of New York at Stony Brook; August 5, 1963.

2. A. H. Zemanian, "A Time-Domain Characterization of Positive-Real Matrices," Second Scientific Report, AFCRL-63-391, College of Engineering Tech. Rep. 13, State University of New York at Stony Brook; August 16, 1963.

3. A. H. Zemanian, "The Time-Domain Synthesis of Distributions," Third Scientific Report, AFCRL-64-191, College of Engineering Tech. Rep. 19, State University of New York at Stony Brook; February 1, 1964.

4. A. H. Zemanian, "The Distributional Laplace and Mellin Transformations," Fourth Scientific Report, AFCRL-64-685, College of Engineering Tech. Rep. 26, State University of New York at Stony Brook; August 15, 1964.

Papers:

1. A. H. Zemanian, "The Time-Domain Synthesis of Distributions" Proceedings of the First Allerton Conference on Circuit Theory, University of Illinois; 1963.

2. A. H. Zemanian, "The Approximation of Distributions by the Impulse Responses of RLC Two-ports," Proceedings of the International Conference on Microwaves, Circuit Theory, and Information Theory; Tokyo; September, 1964.

3. A. H. Zemanian, "The Time-Domain Synthesis of Distributions," IEEE Transactions on Circuit Theory, accepted for publication in December, 1964 issue.

4. A. H. Zemanian, "A Characterization of the Inverse Laplace Transforms of Rational Positive-real Functions," Journal of SIAM, accepted for publication.

ABSTRACT

A technique for expanding certain Schwartz distributions into series of orthonormal functions is devised. The method works for all the classical orthogonal polynomials and many other sets of orthogonal functions. This result is then used to generalize various standard integral transforms, which are based on orthogonal series expansions, to distributions. As specific examples, the following distributional transforms are developed: the finite Fourier transform, the Laguerre transform, the Hermite transform, the Jacobi transform, the Legendre transform, the Chebyshev transform, the Gegenbauer transform, the finite Hankel transform of zero order. An application to the solution of differential equations is given.

1. Introduction. The objectives of this paper are to develop a method for the expansion of certain Schwartz distributions [1] into series of orthonormal functions and to generalize in a distributional way a variety of integral transforms. Any complete system of functions that are the eigenfunctions of a certain type of self-adjoint differentiation operator may be used in the expansions. The resulting integral transforms that are thereby generalized include the finite Fourier transform, the Laguerre transform, the Hermite transform, the Jacobi transform with its special cases: the Legendre, Chebyshev, and Gegenbauer transforms, and finally the finite Hankel transform of zero order. There is quite an extensive literature on these classical transforms. See for example [2] - [10] . Apparently, however, they have not as yet been extended to Schwartz distributions, except for the finite Fourier and Hermite transforms [1; Vol. II, pp. 80-87 and 116-119 .]

Other works that apply the technique of orthogonal series expansion to generalized functions are by O. Widlund [11] and M. Giertz [12] . Their procedures are developed for the generalized functions of Temple [13] and Lighthill [14] , whereas the present work is suitable for the distributions of Schwartz.

2. Some Definitions and Assumptions. Let I denote the open interval, $a < x < b$, on the real line. Here, $a = -\infty$

and $b = \infty$ are not excluded. As is customary, $L_2(I)$ shall denote the space of quadratically (Lebesgue) integrable functions on I (more precisely, the space of equivalence classes of such functions that equal each other almost everywhere.) $L_2(I)$ is a normed linear space with the norm,

$$\|f\| = \left[\int_a^b |f|^2 \, dx \right]^{1/2}$$

Thus, a sequence $\{f_\nu\}_{\nu=1}^{\infty}$ is said to converge in $L_2(I)$ if $\|f_\nu - f_\mu\| \longrightarrow 0$ as ν and μ tend to infinity independently. $L_2(I)$ is a sequentially complete space [15, pp. 216] ; that is, to each sequence $\{f_\nu\}$ that converges in $L_2(I)$, there exists a function $f \in L_2(I)$ such that $\|f - f_\nu\| \longrightarrow 0$ as $\nu \longrightarrow \infty$. The inner product in $L_2(I)$ is defined by

$$(f,g) = \int_a^b f(x)\overline{g(x)} \, dx$$

By a smooth function on I we shall mean a function that has ordinary derivatives of all orders at every point of I. θ_k $(k = 0, 1, 2, \ldots)$ shall always denote a real-valued smooth function on I such that $\theta_k(x) \neq 0$ on $a < x < b$. $\mathfrak{N}$ shall denote a linear differentiation operator of the form,

$$\mathfrak{N} = \theta_0 D^{n_1} \, \theta_1 D^{n_2} \, \ldots \, D^{n_\nu} \theta_\nu ,$$

where $D^p = d^p/dx^p$. We also impose the restrictions, $\theta_k = \theta_{\nu-k}$, $n_{k+1} = n_{\nu-k}$, and, if ν is odd, $n_{(\nu+1)/2}$ is even. Moreover, we shall assume that $\mathfrak{N}$ possesses eigenvalues λ_n

and normalized eigenfunctions ψ_n $(n = 0, 1, 2, \ldots)$ with the following properties. The ψ_n form a complete orthonormal system in $L_2(I)$. The λ_n are real and have no finite point of accumulation. We shall always number the λ_n such that $|\lambda_0| \leq |\lambda_1| \leq |\lambda_2| \leq \ldots$.

All the orthonormal systems arising from the classical orthogonal polynomials and many other standard sets of orthogonal functions appear as special cases of this general formulation. We list a number of these at the end of this paper.

$\mathcal{D}_I$ denotes the space of smooth functions whose supports are contained in I. We assign to it the topology that makes its dual $\mathcal{D}_I'$ the space of Schwartz distributions on I. This topology is rather complicated [1; Vol. I, pp. 65] but we can describe it simply by saying that a sequence $\{\varphi_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{D}_I$ if and only if the supports of the φ_ν are all contained within a fixed compact subset of I and, for each k, $\{D^k\varphi_\nu\}_{\nu=1}^{\infty}$ converges uniformly. This concept of convergence in $\mathcal{D}_I$ is all we shall need. Clearly, $\mathcal{D}_I$ is a sequentially complete space.

The number that $f \in \mathcal{D}_I'$ assigns to $\varphi \in \mathcal{D}_I$ is denoted by (f, φ). If f is an integrable function on I, we set

$$(f, \varphi) = \int_a^b f(x)\, \overline{\varphi(x)}\, dx$$

To $\mathcal{D}_I'$ we assign its weak topology, which is generated by the seminorms

$$\rho_\varphi(f) = |(f, \varphi)|$$

Thus, a sequence $\{f_\nu\}_{\nu=1}^{\infty}$ is said to converge in $\mathcal{D}_I'$ if, for each φ in $\mathcal{D}_I$, the numerical sequence $\{(f_\nu, \varphi)\}_{\nu=1}^{\infty}$ converges. It is a fact that $\mathcal{D}_I'$ is also sequentially complete [16; Sec. 2-2.]

Still another class of functions we shall make use of is the space $\mathcal{E}_I$ of all smooth functions on I. The topology of $\mathcal{E}_I$ is generated by the seminorms,

$$\rho_{\Omega,m}(\varphi) = \max_{0 \leq p \leq m} \sup_{x \in \Omega} |D^p \varphi(x)| \qquad (\varphi \in \mathcal{E}_I),$$

where Ω is an arbitrary compact subset of I. For every Ω and every m, we have another seminorm. Thus, a sequence $\{\varphi_\nu\}_{\nu=1}^{\infty}$ is said to converge in $\mathcal{E}_I$ if each φ_ν is in $\mathcal{E}_I$ and, for each p, $|D^p \varphi_\nu|$ converges uniformly on every Ω as $\nu \longrightarrow \infty$ [1; Vol. I, pp. 88]. The sequential completeness of $\mathcal{E}_I$ is clear. The dual of $\mathcal{E}_I$ is the space $\mathcal{E}_I'$ of all distributions whose supports are compact subsets of I [1; Vol. I, pp. 89].

3. The Testing Function Space $\mathcal{A}$. We shall now describe a certain space $\mathcal{A} = \mathcal{A}(\mathcal{R}, I)$ of testing functions. Its dual turns out to be a space of distributions which can be expanded into a series of the ψ_n. $\mathcal{A}$ consists of all complex-valued functions φ defined and smooth on I such that

$$\gamma_k(\varphi) = \left[\int_a^b |\mathcal{R}^k \varphi|^2 \, dx \right]^{1/2} < \infty \qquad (k = 0, 1, 2, \ldots)$$

and $(\mathfrak{N}^k\varphi, \psi_n) = (\varphi, \mathfrak{N}^k\psi_n)$ for each n and k. $\mathcal{Q}$ is a linear space over the field of complex numbers. The $\gamma_k(\varphi)$ are taken as the seminorms of $\mathcal{Q}$. They are a seperating set since $\gamma_0(\varphi)$ is the customary norm for $L_2(I)$. In accordance with the topology generated by these seminorms, we shall say that a sequence $\{\varphi_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{Q}$ if each φ_ν is in $\mathcal{Q}$ and, for each k, $\gamma_k(\varphi_\nu - \varphi_\mu) \longrightarrow 0$ as ν and μ tend to infinity independently.

Obviously, $\mathcal{D}_I$ is contained in $\mathcal{Q}$ and convergence in $\mathcal{D}_I$ implies convergence in $\mathcal{Q}$.

Moreover, each ψ_n is in $\mathcal{Q}$ because

$$\left[\gamma_k(\psi_n)\right]^2 = \int_a^b |\mathfrak{N}^k\psi_n|^2 \, dx = |\lambda_n|^{2k} \int_a^b |\psi_n|^2 \, dx < \infty$$

and, with δ_{nm} denoting the Kronecker delta,

$$(\mathfrak{N}^k\psi_n, \psi_m) = \lambda_n^k \, \delta_{nm}(\psi_n, \psi_m) = \lambda_m^k \, \delta_{nm}(\psi_n, \psi_m) = (\psi_n, \mathfrak{N}^k\psi_m).$$

We also note that the operator $\mathfrak{N}$ is a continuous linear mapping of $\mathcal{Q}$ into itself. This follows directly from the definition of $\mathcal{Q}$.

<u>Theorem 1:</u> $\mathcal{Q}$ <u>is</u> <u>sequentially complete</u>.

<u>Proof:</u> For each k, we have that $\mathfrak{N}^k\varphi_m$ converges in $L_2(I)$ as $m \longrightarrow \infty$. By the sequential completeness of $L_2(I)$, there exists a function $\chi_k \in L_2(I)$ such that $\mathfrak{N}^k\varphi_m$ converges to it in $L_2(I)$. We shall first show that $\mathfrak{N}\chi_k = \chi_{k+1}$ at each point of I and for every k.

Let x_1 be a fixed point of I and x a variable point in I. Also, let D^{-1} be the integration operator,

$$D^{-1} = \int_{x_1}^{x} \ldots dt.$$

Thus, for any continuous function ζ, $D^{-1} D\zeta = \zeta(x) - \zeta(x_1)$. Using Schwarz's inequality, we may write

$$\left|D^{-1}\theta_0^{-1} \; \mathfrak{N}^{k+1}(\varphi_m - \varphi_n)\right|^2 \leq \left|\int_{x_1}^{x} |\theta_0^{-1}|^2 \; dt\right| \int_a^b \left|\mathfrak{N}^{k+1}(\varphi_m - \varphi_n)\right|^2 dt.$$

The first factor on the right-hand side is a bounded function on every Ω (Ω denotes an arbitrary compact subset of I), whereas the second factor converges to zero as m and n tend to infinity independently. This shows that the left-hand side converges uniformly on every Ω.

We may remove the differentiations and multiplications by the θ_k in the operator $\mathfrak{N}$ step by step to obtain $\mathfrak{N}^{-1}\mathfrak{N}^{k+1}\varphi_m$ where

$$\mathfrak{N}^{-1} = \theta_\nu^{-1} \; D^{-n_\nu} \; \ldots \; \theta_1^{-1} \; D^{-n_1} \; \theta_0^{-1}$$

and $D^{-n} = (D^{-1})^n$. At each step the resulting quantity will converge uniformly on every Ω as $m \longrightarrow \infty$. Moreover,

$$\mathfrak{N}^k\varphi_m(x) = \mathfrak{N}^{-1}\mathfrak{N}^{k+1}\varphi_m(x) + \sum_j \varphi_m^{(j)}(x_1) \, g_j(x, x_1),$$

where

$$\varphi_m^{(j)}(x_1) = D\varphi_m \Big|_{x=x_1}.$$

The summation has only a finite number of terms, and the

$g_j(x,x_1)$ are linearly independent smooth functions on I that do not depend upon φ_m. As $m \to \infty$ the sum on j converges uniformly on every Ω if it converges at all. It must converge since $\mathfrak{N}^k\varphi_m$ converges in $L_2(I)$ and $\mathfrak{N}^{-1}\mathfrak{N}^{k+1}\varphi_m$ converges uniformly on every Ω. Since the $g_j(x,x_1)$ are linearly independent, it follows that, for each j, $\varphi_m^{(j)}(x_1)$ converges as $m \longrightarrow \infty$.

Since k is arbitrary, all this shows that we can interchange the limit on m with $\mathfrak{N}$ to get

$$\mathfrak{N}\chi_k = \mathfrak{N} \lim_{m\to\infty} \mathfrak{N}^k\varphi_m = \lim_{m\to\infty} \mathfrak{N}^{k+1}\varphi_m = \chi_{k+1}$$

To complete the proof, we have to show that $(\mathfrak{N}^k\chi_0, \psi_n) = (\chi_0, \mathfrak{N}^k\psi_n)$ for every n and k. Since the inner product is a continuous function under convergence in $L_2(I)$ [17; p. 75],

$$(\mathfrak{N}^k\chi_0, \psi_n) = \lim_{m\to\infty}(\mathfrak{N}^k\varphi_m, \psi_n) = \lim_{m\to\infty}(\varphi_m, \mathfrak{N}^k\psi_n)$$

$$= (\chi_0, \mathfrak{N}^k\psi_m)$$

Q. E. D.

We shall need the following lemmas.

<u>Lemma 1: If φ is in $\mathcal{a}$, then</u>

$$\varphi = \sum_{n=0}^{\infty} (\varphi, \psi_n)\psi_n,$$

<u>where the series converges in $\mathcal{a}$.</u>

<u>Proof:</u> By the definition of $\mathcal{a}$, $\mathfrak{N}^k\varphi$ is in $L_2(I)$ for each nonnegative integer k. Hence, we may expand $\mathfrak{N}^k\varphi$

into a series of the orthonormal functions ψ_n to obtain

$$\mathfrak{N}^k\varphi = \sum_{n=0}^{\infty} (\mathfrak{N}^k\varphi, \psi_n)\psi_n = \sum (\varphi, \mathfrak{N}^k\psi_n)\psi_n = \sum (\varphi, \lambda_n^k\psi_n)\psi_n$$

$$= \sum (\varphi, \psi_n)\lambda_n^k\psi_n = \sum (\varphi, \psi_n)\mathfrak{N}^k\psi_n \tag{3-1}$$

These series converge in $L_2(I)$. Consequently, for each k,

$$\gamma_k\left(\varphi - \sum_{n=0}^{N} (\varphi, \psi_n)\psi_n\right) = \left[\int_a^b \left|\mathfrak{N}^k\varphi - \sum_{n=0}^{N} (\varphi, \psi_n)\mathfrak{N}^k\psi_n\right|^2 dx\right]^{1/2}$$

$$\longrightarrow 0$$

as $N \to \infty$. This is what we had to show.

We can characterize orthonormal series that converge in $\mathfrak{a}$ in the following way.

Lemma 2: Let a_n denote complex numbers. Then, $\sum_{n=0}^{\infty} a_n\psi_n$ converges in $\mathfrak{a}$ if and only if $\sum_{n=0}^{\infty} |a_n|^2|\lambda_n|^{2k}$ converges for every k.

Proof: We employ the fact that the ψ_n form an orthonormal set to write

$$\int_a^b \left|\mathfrak{N}^k \sum_{n=q}^{p} a_n\psi_n\right|^2 dx = \int_a^b \left|\sum_{n=q}^{p} a_n\lambda_n^k\psi_n\right|^2 dx$$

$$= \int_a^b \sum_{n=q}^{p}\sum_{m=q}^{p} a_n\overline{a_m}\lambda_n^k\lambda_m^k\psi_n\overline{\psi_m}\, dx$$

$$= \sum_{n=q}^{p} |a_n|^2|\lambda_n|^{2k}$$

Our assertion follows directly from this equation.

$\mathfrak{N}$ is self-adjoint on $\mathfrak{a}$. That is, for every pair of elements, φ and χ, in $\mathfrak{a}$, we have

$$(\mathfrak{N}\varphi, \chi) = (\varphi, \mathfrak{N}\chi).$$

Indeed, using (3-1) and the fact that the inner product is a continuous function under convergence in $L_2(I)$, we obtain

$$(\mathfrak{N}\varphi, \chi) = \int_a^b \overline{\chi} \sum_n (\varphi, \psi_n)\, \mathfrak{N}\psi_n \, dx = \sum_n (\varphi, \psi_n) \int_a^b \overline{\chi}\, \mathfrak{N}\psi_n \, dx$$

$$= \sum_n (\varphi, \psi_n) \int_a^b \psi_n \mathfrak{N}\overline{\chi} \quad dx = \int_a^b \sum_n (\varphi, \psi_n)\, \psi_n \mathfrak{N}\overline{\chi} \, dx = (\varphi, \mathfrak{N}\chi).$$

$\mathcal{A}$ is obviously a subspace of $\mathcal{E}_I$. Moreover, convergence in $\mathcal{A}$ implies convergence in $\mathcal{E}_I$, as we shall now show. Again, Ω denotes an arbitrary compact subset of I. We have already demonstrated in the proof of theorem 1 that, if $\{\varphi_m\}_{m=1}^{\infty}$ converges in $\mathcal{A}$, then it converges uniformly on every Ω. We have also shown that $\{\mathfrak{N}_p \varphi_m\}_{m=1}^{\infty}$ converges uniformly on Ω at the same time, where

$$\mathfrak{N}_p = D^{n_p}\theta_p D^{n_{p+1}}\theta_{p+1} \quad \ldots D^{n_\nu}\theta_\nu$$

denotes a partial application of $\mathfrak{N}^k$ obtained by terminating the differentiations and multiplications by θ_k at some step. Repeatedly applying the rule for the differentiation of a product, we find that

$$\mathfrak{N}_p \varphi_m = \sum_{\mu=0}^{q-1} P_\mu D^\mu \varphi_m + \theta_p \theta_{p+1} \ldots \theta_\nu D^q \varphi_m ,$$

where

$$q = n_p + n_{p+1} + \ldots + n_\nu$$

and the P_μ denote polynomials in the variables $D^j\theta_k$.

Assuming that $\{D^\mu \varphi_m\}_{m=1}^\infty$ converges uniformly on every Ω for $\mu = 0, 1, \ldots, q-1$, we conclude that $\{D^q \varphi_m\}_m$ also converges uniformly on every Ω. By induction, therefore, $\{\varphi_m\}_m$ converges in $\mathcal{E}_I$.

4. <u>The Space of Distributions</u> $\mathcal{A}'$. $\mathcal{A}' = \mathcal{A}'(\mathfrak{N}, I)$ denotes the dual of $\mathcal{A}$. That is, f is in $\mathcal{A}'$ if and only if it is a continuous linear functional on $\mathcal{A}$. The number that $f \in \mathcal{A}'$ assigns to any $\varphi \in \mathcal{A}$ is again denoted by the inner product symbolism (f, φ). (We shall show later on that $L_2(I)$ is imbedded in $\mathcal{A}'$ by defining (f, φ) as the number that any $f \in L_2(I)$ assigns to any φ in $\mathcal{A}$.)

We assign to $\mathcal{A}'$ the (weak) topology, generated by the seminorms,

$$\eta_\varphi(f) = |(f, \varphi)|$$

where the φ are arbitrary elements of $\mathcal{A}$. Thus, a sequence $\{f_\nu\}_{\nu=1}^\infty$ is said to converge in $\mathcal{A}'$ if and only if every f_ν is in $\mathcal{A}'$ and for each $\varphi \in \mathcal{A}$, the numerical sequence $\{(f_\nu, \varphi)\}_{\nu=1}^\infty$ converges.

<u>Theorem 2</u>: $\mathcal{A}'$ <u>is sequentially complete</u>.

<u>Note</u>: In this case, sequential completeness means that for every sequence $\{f_\nu\}_{\nu=1}^\infty$ that converges in $\mathcal{A}'$ there exists a unique $f \in \mathcal{A}'$ such that $\lim (f_\nu, \varphi) = (f, \varphi)$ for each $\varphi \in \mathcal{A}$.

<u>Proof</u>: Assume that $\{f_\nu\}_{\nu=1}^\infty$ converges in $\mathcal{A}'$. Thus, $\lim (f_\nu, \varphi)$ exists for each $\varphi \in \mathcal{A}$ and defines, therefore, a unique functional f on $\mathcal{A}$. This functional is clearly linear.

To show that it is continuous, we modify M. S. Brodskii's proof of the sequential completeness of $\mathcal{D}'$, the space of all distributions [16; Sec. 2.2].

Assume that f is not continuous on $\mathcal{A}$. Then, there exists a sequence of testing functions which converges in $\mathcal{A}$ to the zero function and is such that the corresponding sequence of numbers assigned to it by f does not converge to zero. We can certainly choose a subsequence $\{\varphi_\nu\}_{\nu=1}^{\infty}$ such that

$$|(f, \varphi_\nu)| \geqslant c > 0 \qquad (\nu = 1, 2, 3, \ldots .)$$

and

$$\gamma_k(\varphi_\nu) = \left[\int_a^b |\eta^k \varphi_\nu|^2 dx \right]^{1/2} \leqslant 4^{-\nu} \quad (k = 0, 1, \ldots, \nu) \qquad (4\text{-}1)$$

Let $\Psi_\nu = 2^\nu \varphi_\nu$. By (4-1), the sequence $\{\Psi_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{A}$ to the zero function, whereas $\{|(f, \Psi_\nu)|\}_{\nu=1}^{\infty}$ tends to $+\infty$. We can choose a subsequence $\{\Psi_\nu'\}$ from $\{\Psi_\nu\}$ and a subsequence $\{f_\nu'\}$ from $\{f_\nu\}$ such that the following two conditions can be satisfied.

$$|(f_\nu', \Psi_\mu')| < \frac{1}{2^{\mu-\nu}} \qquad (\nu = 1, 2, \ldots, \mu - 1) \qquad (4\text{-}2)$$

$$|(f_\nu', \Psi_\nu')| > \sum_{\mu=1}^{\nu-1} |(f_\nu', \Psi_\mu')| + \nu \qquad (4\text{-}3)$$

Indeed, we first choose Ψ_1' and f_1' such that $|(f, \Psi_1')| > 1$ and $|(f_1', \Psi_1')| > 1$. Then, assuming that the first $\nu - 1$ elements of these subsequences have been chosen, we can select Ψ_ν' as an element from $\{\Psi_\nu\}_{\nu=1}^{\infty}$ such that (4-2) is satisfied as well as (4-3) with f'_ν replaced by f. This is because, for each fixed f'_ν, $(f'_\nu, \Psi_\mu) \longrightarrow 0$ as $\mu \longrightarrow \infty$ and because $|(f, \Psi_\nu)| \longrightarrow \infty$ as $\nu \longrightarrow \infty$. Then, f'_ν can be chosen to satisfy (4-3) since $(f_\nu, \varphi) \longrightarrow (f, \varphi)$ for each fixed $\varphi \in \mathfrak{a}$.

Next, consider the series, $\Psi = \sum_{\nu=1}^{\infty} \Psi'_\nu$. We know that

$$\gamma_k(\Psi_\nu) = 2^\nu \gamma_k(\varphi_\nu) \leqslant 2^{-\nu} \qquad (k = 0, 1, 2, \ldots, \nu)$$

Hence, using Minkowski's inequality, we can write for m > n

$$\gamma_k\Big(\sum_{\nu=n}^{m} \Psi'_\nu\Big) \leqslant \sum_{\nu=n}^{m} \gamma_k(\Psi'_\nu) \leqslant \sum_{\nu=n}^{\infty} \gamma_k(\Psi_\nu) \leqslant \sum_{\nu=n}^{\infty} 2^{-\nu}$$

The right-hand side converges to zero as $n \rightarrow \infty$, which proves that the series converges in $\mathfrak{a}$. By the sequential completeness of $\mathfrak{a}$, $\Psi \in \mathfrak{a}$.

Finally,

$$(f'_\nu, \Psi) = \sum_{\mu=1}^{\nu-1} (f'_\nu, \Psi'_\mu) + (f'_\nu, \Psi'_\nu) + \sum_{\mu=\nu+1}^{\infty} (f'_\nu, \Psi'_\mu) \qquad (4\text{-}4)$$

By (4-2)

$$\Big| \sum_{\mu=\nu+1}^{\infty} (f'_\nu, \Psi'_\mu) \Big| \leqslant \sum_{\mu=\nu+1}^{\infty} \frac{1}{2^{\mu-\nu}} = 1. \qquad (4\text{-}5)$$

Therefore, by (4-3), (4-4) and (4-5)

$$|(f'_\nu, \Psi)| \geqslant |(f'_\nu, \Psi'_\nu)| - \Big|\sum_{\mu=1}^{\nu-1} (f'_\nu, \Psi'_\mu)\Big| - \Big|\sum_{\mu=\nu+1}^{\infty} (f'_\nu, \Psi'_\mu)\Big| > \nu - 1$$

This shows that, as $\nu \rightarrow \infty$, $|(f'_\nu, \Psi)| \rightarrow \infty$, which contradicts the hypothesis that (f_ν, Ψ) converges as $\nu \longrightarrow \infty$. Thus, the proof is complete.

As is usual, $f = g (f, g \in \mathfrak{a}')$ means that $(f, \varphi) = (g, \varphi)$ for every $\varphi \in \mathfrak{a}$. We define the sum f + g by

$$(f + g, \varphi) = (f, \varphi) + (g, \varphi),$$

and multiplication by the complex number α by

$$(\alpha f, \varphi) = \alpha (f, \varphi).$$

It follows that $\mathfrak{a}'$ is a linear space.

The operator $\mathfrak{N}$ is defined on $\mathcal{A}'$ by $(\mathfrak{N} f, \varphi) = (f, \mathfrak{N}\varphi)$, where $f \in \mathcal{A}'$ and $\varphi \in \mathcal{A}$. In words, $\mathfrak{N} f$ is that functional on $\mathcal{A}$ which assigns to each $\varphi \in \mathcal{A}$ the same number that f assigns to $\mathfrak{N}\varphi$. Clearly, $\mathfrak{N} f$ is continuous and linear on $\mathcal{A}$ and, therefore, $\mathfrak{N} f \in \mathcal{A}'$. Since $\mathcal{A}'$ is a linear space, $\mathfrak{N}$ is a linear operator on $\mathcal{A}'$. Moreover, $\mathfrak{N}$ is a continuous operator because, if $\{f_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{A}'$ to zero, then

$$(\mathfrak{N} f_\nu, \varphi) = (f_\nu, \mathfrak{N}\varphi) \longrightarrow 0$$

In short, $\mathfrak{N}$ is a continuous linear mapping of $\mathcal{A}'$ into $\mathcal{A}'$.

Since $\mathcal{D}_I$ is a subspace of $\mathcal{A}$ and since convergence in $\mathcal{D}_I$ implies convergence in $\mathcal{A}$, every element of $\mathcal{A}'$ is a distribution in $\mathcal{D}_I'$. Moreover, the weak topology of $\mathcal{D}_I'$ is generated by the seminorms,

$$\eta_\varphi(f) = \left|(f, \varphi)\right|$$

where now φ is any element of $\mathcal{D}_I$. Consequently, convergence in $\mathcal{A}'$ implies convergence in $\mathcal{D}_I'$.

We imbed $L_2(I)$ (and, therefore, $\mathcal{A}$) into $\mathcal{A}'$ by defining the number that $f \in L_2(I)$ assigns to any $\varphi \in \mathcal{A}$ as

$$(f, \varphi) = \int_a^b f(x)\overline{\varphi(x)}\, dx$$

In this case, f can be shown to be a continuous linear functional on $\mathcal{A}$. More generally, if $f = \mathfrak{N}^k g$ for some k, where $g \in L_2(I)$, then $f \in \mathcal{A}'$. To see this let $\varphi \in \mathcal{A}$. Then

$$(f, \varphi) = (\mathfrak{N}^k g, \varphi) = (g, \mathfrak{N}^k \varphi),$$

so that (f,φ) exists. f is clearly linear on $\mathcal{A}$. Its continuity on $\mathcal{A}$ follows from the fact that, if $\{\varphi_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{A}$ to the zero function, then by the Schwarz's inequality

$$\left|(f,\varphi_\nu)\right| \leqslant \left[\int_a^b |g|^2\, dx\right]^{1/2} \gamma_k(\varphi_\nu) \longrightarrow 0.$$

Still another subspace of $\mathcal{A}'$ is $\mathcal{E}_I'$, the space of all distributions whose supports are compact subsets of I. This is a consequence of the facts that $\mathcal{A} \subset \mathcal{E}_I$ and convergence in $\mathcal{A}$ implies convergence in $\mathcal{E}_I$.

5. Orthonormal Series Expansions and Distributional Transforms. Any $f \in \mathcal{A}'$ can be expanded into a series of the orthonormal eigenfunctions ψ_n, as follows

$$f = \sum_{n=0}^{\infty} (f,\psi_n)\psi_n \tag{5-1}$$

Here, the series is understood to converge in $\mathcal{A}'$. That is, for every $\varphi \in \mathcal{A}$ we have

$$(f,\varphi) = \sum_{n=0}^{\infty} (f,\psi_n)(\psi_n,\varphi). \tag{5-2}$$

To show this, we need merely invoke lemma 1 and write

$$(f,\varphi) = (f,\sum(\varphi,\psi_n)\psi_n) = \sum (f,\psi_n)\overline{(\varphi,\psi_n)}$$

$$= \sum(f,\psi_n)(\psi_n,\varphi).$$

We can view the orthonormal series expansion (5-1) as the inversion formula for a certain distributional transform $\mathcal{T}$, defined by

$$\mathcal{T}f = F(n) = (f, \psi_n) \qquad (f \in \mathcal{a}', \ n = 0, 1, \ldots) \qquad (5\text{-}3)$$

Thus, $\mathcal{T}$ is a mapping on $\mathcal{a}'$ into the space of complex-valued functions defined on n. For the inverse mapping $\mathcal{T}^{-1}$ we shall use the notation

$$\mathcal{T}^{-1} F(n) = \sum_{n=0}^{\infty} F(n)\psi_n = f \qquad (5\text{-}4)$$

(Henceforth, the transform function (5-3) shall be denoted by the capital letter corresponding to the lower case letter used for the original distribution in $\mathcal{a}'$.) Clearly, $\mathcal{T}$ is a linear mapping and is continuous in the sense that, if $\{f_\nu\}_{\nu=1}^{\infty}$ converges in $\mathcal{a}'$ to f, then $\{F_\nu(n)\}_{\nu=1}^{\infty}$ converges to F(n) for each n.

<u>Theorem 3 (Uniqueness of $\mathcal{T}$): If f, g $\in \mathcal{a}'$ and if</u> F(n) = G(n), <u>then</u> f = g.

<u>Proof:</u>

$$f - g = \sum (f - g, \psi_n)\psi_n = \sum [(f, \psi_n) - (g, \psi_n)]\psi_n = 0$$

We now turn to the problem of precisely characterizing the functions F(n) that are generated by the transform $\mathcal{T}$. We do this by adapting some arguments due to O. Widlund [11] and M. Giertz [12].

<u>Theorem 4: Let b_n denote complex numbers. Then,</u>

$$\sum_{n=0}^{\infty} b_n \psi_n \qquad (5\text{-}5)$$

converges in $\mathcal{A}'$ if and only if there exists an integer q *such that*

$$\sum_{n=0}^{\infty} \frac{|b_n|^2}{(1 + |\lambda_n|^q)^2} \tag{5-6}$$

converges. Furthermore, if f *denotes the limit in $\mathcal{A}'$ of* (5-5), *then* $b_n = (f, \psi_n)$.

Proof: First, assume that (5-6) converges for some q. We wish to show that, for each $\varphi \in \mathcal{A}$,

$$\sum_{n=0}^{\infty} (b_n \psi_n, \varphi) \tag{5-7}$$

converges. Using the Schwarz inequality, we may write

$$\sum_{n=0}^{\infty} |(b_n \psi_n, \varphi)| = \sum |b_n| \, |(\varphi, \psi_n)|$$

$$= \sum \frac{|b_n|}{1 + |\lambda_n|^q} (1 + |\lambda_n|^q) \, |(\varphi, \psi_n)|$$

$$\leq \left[\sum \frac{|b_n|^2}{(1 + |\lambda_n|^q)^2} \sum (1 + |\lambda_n|^q)^2 \, |(\varphi, \psi_n)|^2 \right]^{1/2}$$

The first series in the right-hand side of this inequality converges by assumption, whereas the second series converges by lemmas 1 and 2. Thus, (5-7) truly does converge.

Next, assume that (5-5) converges in $\mathcal{A}'$. Its limit f must be in $\mathcal{A}'$ because of the sequential completeness of $\mathcal{A}'$. Since $\psi_m \in \mathcal{A}$, we may write

$$(f,\psi_m) = \sum_{n=0}^{\infty} b_n(\psi_n,\psi_m)$$

and by the orthonormality of the ψ_n we get $(f,\psi_m) = b_m$. This verifies the last statement of the theorem.

Still assuming that (5-5) converges in $\mathcal{a}'$, we finally wish to show that (5-6) converges for some q. Assume the opposite. Then, we can choose a sequence of integers, $m_0 = 0$, m_1, m_2, ... , which tend to infinity, such that

$$\sum_{n=m_{q-1}}^{m_q-1} \frac{|b_n|^2}{(1+|\lambda_n|^q)^2} > 1 \qquad (q = 1, 2, \ldots)$$

Setting $P_n = q$ for $m_{q-1} \leq n < m_q$, we obtain the divergent series

$$\sum_{n=0}^{\infty} \frac{|b_n|^2}{(1+|\lambda_n|^{P_n})^2} \tag{5-8}$$

Let ℓ_2 denote the normed linear space of infinite sequences of complex numbers, $\underset{\sim}{a} = \{a_n\}_{n=0}^{\infty}$, where

$$\|\underset{\sim}{a}\| = \left[\sum_{n=0}^{\infty}|a_n|^2\right]^{1/2} < \infty$$

It is a fact that ℓ_2 is its own dual; that is, j is a continuous linear functional on ℓ_2 if and only if

$$j(\underset{\sim}{c}) = \sum_{n=0}^{\infty} \overline{a_n}\, c_n$$

where $\underset{\sim}{a}$, $\underset{\sim}{c} \in \ell_2$ [17; pp. 108-109].

Now consider the subset of ℓ_2 consisting of the vectors

$$\underset{\sim}{c}_k = \left\{ \frac{b_0}{1 + |\lambda_0|^{p_0}}, \ \ldots \ , \frac{b_k}{1 + |\lambda_k|^{p_k}}, \ 0, \ 0, \ \ldots \right\}$$

If

$$\sup_{0 \le k < \infty} \left| j(\underset{\sim}{c}_k) \right| = \sup_k \left| \sum_{n=0}^{k} \frac{\overline{a}_n \, b_n}{1 + |\lambda_n|^{p_n}} \right| < \infty$$

for every continuous linear j on ℓ_2, then by the principle of uniform boundedness [18; pp. 202]

$$\sup_k \| \underset{\sim}{c}_k \| = \left[\sum_{n=0}^{\infty} \frac{|b_n|^2}{(1 + |\lambda_n|^{p_n})^2} \right]^{1/2} < \infty$$

Consequently, the divergence of (5-8) implies that

$$\sum_{n=0}^{\infty} \frac{\overline{a}_n \, b_n}{1 + |\lambda_n|^{p_n}}$$

diverges for at least one $\underset{\sim}{a} \in \ell_2$.

Finally, choosing such an $\underset{\sim}{a}$, set

$$\theta = \sum_{n=0}^{\infty} \frac{a_n}{1 + |\lambda_n|^{p_n}} \psi_n \, .$$

θ is in $\mathcal{a}$ because this series converges in $\mathcal{a}$ by lemma 2. The application of (5-5) to θ yields

$$\sum_{n=0}^{\infty} b_n (\psi_n , \theta) = \sum_{n=0}^{\infty} \frac{\overline{a}_n \, b_n}{1 + |\lambda_n|^{p_n}}$$

The left-hand side converges under our assumption that (5-5) converges in $\mathcal{A}'$, whereas the right-hand side has been shown to diverge. This contradiction completes the proof.

We can characterize the elements of $\mathcal{A}'$ in still another way.

Theorem 5: Necessary and sufficient conditions for f *to be in* $\mathcal{A}'$ *are that there be some nonnegative integer* q *and a* $g \in L_2(I)$ *such that*

(*i*) *if* $\lambda_n \neq 0$ *for every* n, *then* $f = \mathcal{N}^q g$, *and*

(*ii*) *if* $\lambda_n = 0$ *for* $0 \leq n < N$ *and* $\lambda_n \neq 0$ *for* $n \geq N$, *then*

$$f = \mathcal{N}^q g + \sum_{n=0}^{N-1} c_n \psi_n$$

where the c_n *denote complex numbers.*

Proof: *Sufficiency*: We have already shown in Sec. 4 that $\mathcal{N}^q g \in \mathcal{A}'$ and that $\mathcal{A} \subset \mathcal{A}'$. Since $\psi_n \in \mathcal{A}$ it follows that $f \in \mathcal{A}'$.

Necessity: Set $G(n) = F(n)/\lambda_n^q$ for $n \geq N$ and choose $G(n)$ arbitrarily for $0 \leq n < N$. In view of theorem 4 and the comparison test, $\sum_{n=0}^{\infty} |G(n)|^2$ converges. By the Riesz-Fischer theorem, there exists a $g \in L_2(I)$ such that $G(n) = (g, \psi_n)$. Hence,

$$f = \sum_{n=0}^{\infty} F(n)\psi_n = \sum_{n=N}^{\infty} \lambda_n^q G(n)\psi_n + \sum_{n=0}^{N-1} F(n)\psi_n$$

$$= \sum_{n=0}^{\infty} (g, \lambda_n^q \psi_n)\psi_n + \sum_{n=0}^{N-1} F(n)\psi_n$$

$$= \sum_{n=0}^{\infty} (\mathcal{N}^q g, \psi_n)\psi_n + \sum_{n=0}^{N-1} F(n)\psi_n$$

$$= \mathcal{N}^q g + \sum_{n=0}^{N-1} F(n)\psi_n$$

Q. E. D.

6. A Transform Calculus. We have already indicated that the differential operator $\mathcal{N}$ is a continuous linear mapping on $\mathcal{a}'$ into $\mathcal{a}'$. Therefore, we may write for every $f \in \mathcal{a}'$

$$\mathcal{N}^k f = \sum_{n=0}^{\infty} (f, \psi_n)\mathcal{N}^k\psi_n = \sum_{n=0}^{\infty} (f, \psi_n)\lambda_n^k\psi_n$$

We can use this fact to solve the differential equation, $P(\mathcal{N})u = g$, where P is a polynomial with real coefficients and the given g and unknown u are required to be in $\mathcal{a}'$. $P(\mathcal{N})$ is a self-adjoint operator whenever $\mathcal{N}$ is one. Through the transform $\mathcal{T}$, we obtain $P(\lambda_n)U(n) = G(n)$. If $P(\lambda_n) \neq 0$ for every n, we can divide by $P(\lambda_n)$ and then apply $\mathcal{T}^{-1}$ to get

$$u = \sum_{n=0}^{\infty} \frac{G(n)}{P(\lambda_n)} \psi_n \qquad (6\text{-}1)$$

By theorems 3 and 4, this solution exists and is unique in $\mathcal{a}'$. If $P(\lambda_n) = 0$ for some λ_n, say for λ_{n_k} $(k = 1, \ldots, m)$, and if $G(\lambda_{n_k})=0$, then the solution is no longer unique and we can add to (6-1) any complementary solution,

$$u_c = \sum_{k=1}^{m} a_k \psi_{n_k}$$

where the a_k are arbitrary numbers. The extension of this transform calculus to simultaneous differential equations of this type is straightforward. See [16; Secs. 11-7 and 11-8] for a discussion of such an extension in the case of ordinary Fourier series.

7. Special Cases. In this section we list a number of particular operators $\mathfrak{N}$, their eigenfunctions ψ_n and eigenvalues λ_n, and the corresponding intervals I on which our analysis can be made. In every case the assumptions made in Sec. 2 are found to be fulfilled so that our transform calculus can be applied simply by substituting the appropriate quantities. In most cases the name of the transform conforms with name of orthonormal functions that are generated as the eigenfunctions. A reference for the information listed here is [2; Vol.II]

A. The Finite Fourier Transform:

$$I = (-\pi, \pi)$$

$$\mathfrak{N} = D^2$$

$$\psi_n(x) = \begin{cases} \dfrac{1}{(2\pi)^{1/2}} & \text{for } n = 0 \\ \dfrac{\cos nx}{\sqrt{\pi}} & \text{for } n = 2k \\ \dfrac{\sin nx}{\sqrt{\pi}} & \text{for } n = 2k - 1 \end{cases}$$

$(k = 1, 2, 3, \ldots)$

$$\lambda_n = -n^2$$

B. The Laguerre Transform:

$$I = (0, \infty)$$

$$\mathfrak{N} = x^{-\alpha/2} e^{x/2} Dx^{\alpha+1} e^{-x} Dx^{-\alpha/2} e^{x/2}$$

$$= DxD - \frac{x}{4} - \frac{\alpha^2}{4x} + \frac{\alpha+1}{2}$$

Here, α is a real number greater than -1.

$$\psi_n(x) = \left[\frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)}\right]^{1/2} x^{\alpha/2} e^{-x/2} L_n^\alpha(x),$$

where the L_n^α are the generalized Laguerre polynomials,

$$L_n^\alpha(x) = \sum_{m=0}^{n} \binom{n+\alpha}{n-m} \frac{(-x)^m}{m!}$$

$$\lambda_n = -n \qquad (n = 0, 1, 2, \ldots)$$

The symbol $\binom{y}{v}$ denotes

$$\frac{\Gamma(y+1)}{\Gamma(v+1)\Gamma(y-v+1)}$$

C. The Hermite Transform:

$$I = (-\infty, \infty)$$

$$\mathfrak{N} = e^{x^2/2} De^{-x^2} \cdot De^{x^2/2}$$

$$= D^2 - x^2 + 1$$

$$\psi_n(x) = \frac{e^{-x^2/2} H_n(x)}{[2^n (n!) \sqrt{\pi}]^{1/2}}$$

where the $H_n(x)$ are the Hermite polynomials,

$$H_n(x) = n! \sum_{m=0}^{[n/2]} \frac{(-1)^m (2x)^{n-2m}}{m! (n-2m)!}$$

$[n/2]$ is $n/2$ or $(n-1)/2$ according to the evenness or oddness of n:

$$\lambda_n = -2n \qquad (n = 0, 1, 2, \ldots)$$

In this particular case, $\mathcal{A}$ happens to be the space $\mathcal{S}$ of testing functions of rapid descent and $\mathcal{A}'$ the space $\mathcal{S}'$ of distributions of slow growth. A proof of this is given in the appendix.

D. The Jacobi Transform:

$$I = (-1,1), \qquad w(x) = (1-x)^{\alpha}(1+x)^{\beta}$$

$$(\alpha, \beta \text{ are real and } \alpha > -1,\ \beta > -1)$$

$$\mathfrak{N} = \frac{1}{\sqrt{\omega(x)}} D\,(x^2 - 1)\,w(x)\,D \frac{1}{\sqrt{\omega(x)}}$$

$$\psi_n(x) = \sqrt{\frac{\omega(x)}{h_n}}\, P_n^{(\alpha,\beta)}(x) \qquad (n = 0, 1, 2, \ldots)$$

where

$$h_n = \frac{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{n!\,(2n+\alpha+\beta+1)\,\Gamma(n+\alpha+\beta+1)}$$

and the $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials,

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n} \sum_{m=0}^{n} \binom{n+\alpha}{m} \binom{n+\beta}{n-m} (x-1)^{n-m} (x+1)^m$$

$$\lambda_n = n\,(n+\alpha+\beta+1)$$

E. The Legendre Transform:

$$I = (-1,1)$$

$$\mathcal{N} = D\,(x^2 - 1)\,D$$

$$\psi_n(x) = (n + \tfrac{1}{2})^{1/2} P_n(x)$$

where the $P_n(x)$ are the Legendre polynomials,

$$P_n(x) = \frac{1}{2^n} \sum_{m=0}^{[n/2]} (-1)^m \binom{n}{m} \binom{2n - 2m}{n} x^{n-2m}$$

$[n/2]$ is $n/2$ or $(n-1)/2$ according to the evenness or oddness of n.

$$\lambda_n = n\,(n + 1) \qquad (n = 0, 1, 2, \ldots)$$

This is a special case of the Jacobi functions with $\alpha = \beta = 0$.

F. The Chebyshev Transform:

$$I = (-1,1)$$

$$\mathcal{N} = (1 - x^2)^{1/4} D\,(1 - x^2)^{1/2} D\,(1 - x^2)^{1/4}$$

$$\psi_0(x) = \frac{1}{\sqrt{\pi}\,(1 - x^2)^{1/4}}$$

$$\psi_n(x) = \sqrt{\frac{2}{\pi}}\,\frac{T_n(x)}{(1 - x^2)^{1/4}} \qquad (n = 1, 2, 3, \ldots)$$

where the T_n are the Chebyshev polynomials,

$$T_n(x) = \frac{n}{2} \sum_{m=0}^{[n/2]} \frac{(-1)^m\,(n - m - 1)!}{m!\,(n - 2m)!} (2x)^{n-2m}$$

$[n/2]$ is $n/2$ or $(n-1)/2$ according to the evenness or oddness of n.

$$\lambda_n = -n^2 \qquad (n = 0, 1, 2, \ldots)$$

This also is a special case of the Jacobi functions with $\alpha = \beta = -1/2$.

G. <u>The Gegenbauer Transform</u>:

$$I = (-1,1), \quad w(x) = (1 - x^2)^{\lambda - \frac{1}{2}} \qquad (\lambda \text{ real and } \lambda > -\tfrac{1}{2})$$

$$\eta = \frac{1}{\sqrt{w(x)}} D (1 - x^2)^{\lambda + \frac{1}{2}} D \frac{1}{\sqrt{w(x)}}$$

$$\psi_n(x) = \sqrt{\frac{w(x)}{h_n}} C_n^{\lambda}(x) \qquad (n = 0, 1, 2, \ldots),$$

where

$$h_n = \frac{\sqrt{\pi}\, \Gamma(\lambda + \frac{1}{2}) \Gamma(2\lambda + n)}{n!\, (n + \lambda) \Gamma(\lambda) \Gamma(2\lambda)}$$

and the $C_n^{\lambda}(x)$ are the Gegenbauer polynomials (also called ultraspherical polynomials),

$$C_n^{\lambda}(x) = \sum_{m=0}^{[n/2]} \frac{(-1)^m \Gamma(\lambda + n - m)}{m!\, (n - 2m)!\, \Gamma(\lambda)} (2x)^{n-2m}$$

$$\lambda_n = -n(n + 2\lambda)$$

This too is a special case of the Jacobi functions wherein we set $\alpha = \beta = \lambda - \frac{1}{2}$.

H. The Finite Hankel Transform of Zero Order:

a. First Form:

$I = (0, 1)$

$$\eta = \frac{1}{\sqrt{x}} D x D \frac{1}{\sqrt{x}}$$

$$\Psi_n(x) = \frac{\sqrt{2x}\; J_0(y_n x)}{J_1(y_n)}$$

where J_μ is the μth-order Bessel function of first kind and y_n denotes the successive roots of $J_0(x) = 0$.

$$\lambda_n = -y_n^2$$

b. Second Form:

$I = (0, 1)$

$$\eta = \frac{1}{\sqrt{x}} D x D \frac{1}{\sqrt{x}}$$

$$\Psi_n(x) = \sqrt{\frac{2x}{h_n}}\; J_0(z_n x)$$

where

$$h_n = \left[J_1(z_n)\right]^2 + \left[J_0(z_n)\right]^2$$

and z_n denotes the successive positive roots of $aJ_0(z) - zJ_1(z) = 0$. Here, a is real.

$$\lambda_n = -z_n^2$$

c. Third Form:

$$I = (a, b) \qquad \text{where} \qquad 0 < a < b < \infty$$

$$\eta = \frac{1}{\sqrt{x}} D x D \frac{1}{\sqrt{x}}$$

$$\psi_n(x) = \sqrt{\frac{\pi x}{2 h_n}} \left[J_0(\omega_n x) \; Y_0(\omega_n b) - Y_0(\omega_n x) J_0(\omega_n b) \right]$$

where

$$h_n = \frac{\left[J_0(\omega_n a) \right]^2 - \left[J_0(\omega_n b) \right]^2}{\left[\omega_n J_0(\omega_n a) \right]^2}$$

and ω_n denotes the succesive positive roots of

$$J_0(ax) \, Y_0(bx) - Y_0(ax) \, J_0(bx) = 0$$

$$\lambda_n = - \omega_n^2$$

Appendix

$\mathcal{S}$ denotes the space of all smooth functions φ on $-\infty < x < \infty$ such that for each pair of nonnegative integers, m and ν,

$$\left| x^{2m} D^{\nu} \varphi \right| < C_{m\nu} \qquad (-\infty < x < \infty)$$

where $C_{m\nu}$ denote constants. The topology of $\mathcal{S}$ is generated by the seminorms,

$$\gamma_{m\nu}(\varphi) = \sup_{x} \left| (1 + x^2)^m D^{\nu} \varphi(x) \right|$$

$\mathcal{S}'$ denotes the dual of $\mathcal{S}$.

O. Widlund [11] has proven that $a' = \mathcal{S}'$ using the Temple-Lighthill approach to generalized functions. In the present case where Schwartz distributions are used, a proof can be constructed as follows. We have the formulas,

$$D\psi_0 = -\frac{1}{\sqrt{2}} \psi_1$$

$$D\psi_n = \sqrt{\frac{n}{2}}\,\psi_{n-1} - \sqrt{\frac{n+1}{2}}\,\psi_{n+1} \qquad (n = 1, 2, \ldots)$$

$$x^2 \psi_n = D^2 \psi_n + (2n + 1)\psi_n .$$

The first two are easily shown from the standard properties of the Hermite polynomials and the last is explicitly given in [2; Vol. II, p. 193]. From these, it follows that $x^{2m} \; D^{\nu} \psi_n$ is a finite linear combination of the ψ_{μ} whose coefficients are of slow growth with respect to n. Moreover, from [2; Vol. II, p. 208, eq. (19)], we have that $|\psi_\mu(x)| \leq C$ where C is a constant that is independent of μ and x. Hence,

$$|x^{2m} D^\nu \psi_n| \leq K n^q$$

for some sufficiently large integer q and constant K.

<u>Lemma 3</u>: <u>If</u> $\varphi \in \mathcal{A}$, <u>then for all</u> k <u>and all sufficiently large</u> n,

$$|(\varphi, \psi_n)| \leq |\lambda_n|^{-k} \left[\int_a^b |\mathcal{R}^k \varphi|^2 dx \right]^{1/2}$$

<u>Proof</u>: Since $\mathcal{R} \psi_n = \lambda_n \psi_n$ and $\lambda_n \neq 0$ for all sufficiently large n,

$$(\varphi, \psi_n) = (\varphi, \lambda_n^{-1} \mathcal{R} \psi_n) = \lambda_n^{-1} (\mathcal{R} \varphi, \psi_n)$$

$$= \lambda_n^{-2} (\mathcal{R}^2 \varphi, \psi_n) = \ldots = \lambda_n^{-k} (\mathcal{R}^k \varphi, \psi_n)$$

Therefore, by Schwarz's inequality,

$$|(\varphi, \psi_n)| \leq |\lambda_n|^{-k} \int_a^b |\mathcal{R}^k \varphi| \, |\overline{\psi_n}| \, dx$$

$$\leq |\lambda_n|^{-k} \left[\int_a^b |\mathcal{R}^k \varphi|^2 dx \int_a^b |\psi_n|^2 dx \right]^{1/2}$$

Since the ψ_n are normalized, the last expression proves the lemma.

Next, let $\varphi \in \mathcal{A}$. Then,

$$\varphi = \sum_{n=0}^{\infty} (\varphi, \psi_n) \psi_n$$

where the series converges in $\mathcal{A}$ according to lemma 1 and the (φ, ψ_n) comprise a sequence of fast descent as $n \to \infty$ according to lemma 2. Hence, this series and every one of its derivatives converges uniformly and consequently, it may be differentiated term by term any number of times. Moreover, the series converges in $\mathcal{S}$ because by lemma 4

$$\left| x^{2m} D^{\nu} \sum_{n=0}^{\infty} (\varphi, \psi_n)\psi_n \right| \leq \sum_{n=0}^{\infty} \left|(\varphi, \psi_n)\right| \left|x^{2m} D^{\nu} \psi_n\right|$$

$$\leq K_1 \left[\int_a^b |\mathfrak{N}\varphi|^2 dx \right]^{1/2} \sum_{n=0}^{\infty} n^{-k} n^q \qquad \text{(A-1)}$$

where K_1 and q are fixed and k can be chosen arbitrarily. Since $\mathcal{S}$ is sequentially complete, φ is in $\mathcal{S}$. Stated another way, we have shown that $\mathcal{a} \subset \mathcal{S}$. (A-1) also shown that convergence in $\mathcal{a}$ implies convergence in $\mathcal{S}$. Consequently, we can conclude that $\mathcal{S}' \subset \mathcal{a}'$.

Conversely, if φ is in $\mathcal{S}$, then $\mathfrak{N}^k\varphi = (D^2 - x^2 + 1)^k\varphi$ is clearly in $L_2(-\infty, \infty)$ for every k. Thus, $\mathcal{S} \subset \mathcal{a}$. Moreover, convergence in $\mathcal{S}$ implies convergence in $\mathcal{a}$ because

$$\int_{-\infty}^{\infty} \left| (D^2 - x^2 + 1)^k \varphi_\nu \right|^2 dx$$

$$\leq \int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)} \sup_{-\infty < x < \infty} \left| (1 + x^2) \left[(D^2 - x^2 + 1)^k \varphi_\nu \right]^2 \right|$$

Hence, $\mathcal{a}' \subset \mathcal{S}'$. This completes the proof of the fact that $\mathcal{a}' = \mathcal{S}'$.

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A Technique for expanding certain Schwartz distributions into series of orthonormal functions is devised. The method works for all the classical orthogonal polynomials and many other sets of orthogonal functions. This result is then used to generalize various standard integral transforms, which are basedon orthogonal series expansions, to distributions. As specific examples, the following distributional transforms are developed: the finite Fourier transform, the Laguerre transform, the Hermite transform, the Jacobi transform, the Gegenbauer transform, the finite Hankel transform of zero order. An application to the solution of differential equations is given.

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