A NETWORK THEORY FOR THE INCREMENTAL DYNAMIC BEHAVIOR OF INTERCONNECTED SPATIALLY DISTRIBUTED MARKETS

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I. Introduction

Our purpose herein is to construct a network theory for the variations in prices and flows of commodities in a collection of markets interconnected by transportation facilities. We do this under the assumption that the deviations of these prices and commodity flows from their static equilibrium values are so small that the various supply, demand, and transportation characteristics can be taken to be linear.

The model we set up for such a system of markets is similar to a grounded electrical network; prices correspond to node voltages, and commodity flows correspond to electronic flows (not to conventional current). We exploit this analogy by borrowing a number of results from electrical network theory. In fact, one of the contributions we claim for this work is that our network models make the large body of literature concerning the dynamic behavior of electrical networks available to the analysis of spatially distributed markets interconnected by transportation facilities. Actually, analogies between electrical networks and market behavior have previously been pointed out and exploited, especially with regard to analog computation. For example, Morehouse, Stotz, and Horwitz [12] have developed an electrical analog for the dynamic behavior of a single one-commodity market. Enke [8] has proposed a piecewise-linear electrical analog for the determination of static equilibrium in a one-commodity multimarket system. Our work is distinguished from these in that it is addressed to the dynamic behavior of a multimarket system; moreover, our analysis can be extended to a multicommodity system. (It may be worth mentioning that electronic analogs have also been used for various feedback-control models in macroeconomics; see, for example, [1; Chapter 9], [15], and the references therein.)

We first assume that a single homogeneous commodity is traded in a finite number of purely competitive markets, which are situated in different locations. Some or perhaps all pairs of markets are connected by transportation facilities. In addition to the producers and consumers at the various markets, there is another (not necessarily disjoint) class of traders, called the "shippers" who transport goods between markets because of price differences. Under static conditions, the behavior of the producers, consumers, and shippers are specified respectively by supply, demand, and transportation curves. The prices in the various markets are then determined by the market-clearance conditions, which equate the goods supplied per unit time in every market to the goods demanded per unit time, including those supplied or demanded by the shippers. A result from the theory of nonlinear resistive electrical networks is invoked to establish the existence of a unique set of equilibrium prices in the markets, given certain assumed shapes for the supply, demand, and transportation curves.

Next, we turn to the incremental dynamic behavior of our system of markets. The first step is to postulate the behavior of the producers, consumers, and shippers by means of various mathematical expressions or equivalently by network configurations, which will be taken to be linear. This assumes in effect that the variations in each price around its equilibrium value are small and that the behavior of the producers, consumers, or shippers at each market varies in a smooth fashion as the equilibrium price changes. The equivalent networks are then connected together in accordance with the interconnections between the markets. The result is a "market network", which can be analyzed by means of Kirchhoff's node and loop laws to obtain the price variations in each market, and thereby the variations in commodity flows between producers and consumers and along the transportation facilities.

Our market networks correspond to certain kinds of electrical networks, about which much is known. We cite some results of this nature. For example, the concept of passivity, which has been such a fruitful idea in electrical network theory, can also be interpreted in terms of our system of markets. We discuss how the "customary" behavior of producers, consumers, and shippers is passive, whereas speculative behavior is not passive. Apparently, the passivity concept has not been taken into consideration in the economics literature.

In addition, we establish restrictions on transient variations in price on the following sort. Sudden shifts in demand or supply can be represented by suddenly applied commodity-flow sources at certain points. The resulting transients propagate throughout the market network, but the further away a market is from the point of disturbance the more slowly will these effects be felt. Our results state quantitative restrictions on the way these propagated disturbances build up.

In the last section of this work we consider the case of many commodities traded in a number of locations interconnected by transportation facilities. For each different commodity we have a separate market network. Under the assumptions that at each locality the total static-equilibrium consumer expenditure on all commodities remains fixed with regard to different static-equilibrium points and that the variations in prices and commodity flows remain small as compared to their static equilibrium values, it is shown that the separate market networks are coupled together by ideal transformers. Thus, electrical network theory can also be used to analyze the dynamic behavior of this multimarket multicommodity system.

II. Static Nonlinear Market Networks

In this section we discuss the static equilibrium of a spatially distributed but interconnected system of markets for a single commodity. First, consider one of those markets. Typical supply and demand curves are shown in Figure 1(a); for a given price P, $Q^s(P)$ is the commodity units per unit of time supplied to the market by the producers and $Q^d(P)$ is the commodity units per unit of time taken from the market by the domestic consumers. The difference $Q = H(P) = Q^s(P) - Q^d(P)$ is shown in Figure 1(b) for two different forms for $Q^s(P)$; if the market remains cleared, Q represents the commodity units per unit of time shipped out through the various transportation facilities servicing that market. In order to establish the existence of a unique equilibrium in our market network, we shall extend H(P) for negative values of P (as is indicated by the dotted lines in Figure 1(b)) and shall assume that for every market in our network H(P) satisfies the following conditions.

Condition A. H is a continuous, strictly increasing function of P such that $H(P) \rightarrow +\infty$ $(-\infty)$ as $P \rightarrow +\infty$ (respectively, $-\infty$). Moreover, $H^{-1}(0) \geq 0$ (i.e., the curve H(P) cuts the P-axis at a nonnegative point E).

We shall subsequently show that the manner in which H(P) is extended for negative P is immaterial because the equilibrium price (which, because of the transportation connections, is in general different from E) is always nonnegative.

Set $r(Q) = H^{-1}(Q) - E$. In words, r is the function of Q obtained by shifting the function H downward to make it pass through the origin. Since $P = H^{-1}(Q)$, it follows that P and Q can be symbolically related by the electrical circuit of Figure 2 consisting of a fixed voltage source E in series with a nonlinear resistor $r(\cdot)$. (In Figure 2 as well as in subsequent network diagrams, we use conventional electrical symbolism.) Here, the market is represented by a single node whose price P is taken to be the node voltage measured with respect to a hypothetical ground node. The commodity flow Q is analagous to the flow of electrons toward the market node and produces a voltage rise in the direction of its flow. Thus, voltages such as E have units of price, and flows are measured in commodity units per unit time.

Next, we consider the collection of all transportation facilities connecting two given markets, say, market i and market k. Its aggregate behavior is a function r_{ik} relating the price difference $P_k - P_i$ to the flow Q_{ik} of commodities from market i to market k through those facilities. This is illustrated in Figure 3(a). We shall assume that the following is satisfied.

Condition B. For every i and k, r_{ik} is a continuous, strictly increasing function $Q_{ik} \mapsto P_k - P_i$ such that $r_{ik}(0) = 0$ and $r_{ik}(Q_{ik}) \to +\infty\ (-\infty)$ as $Q_{ik} \to +\infty$ (respectively, $-\infty$). Moreover, $r_{ki}(Q_{ki}) = - r_{ik}(Q_{ik})$, where by definition $Q_{ik} = - Q_{ki}$.

r_{ik} can be represented as a nonlinear resistor connecting the market nodes i and k, as shown in Figure 3(b).

Upon connecting together the equivalent circuits of Figures 2 and 3(b) in accordance with the interconnections between the various markets, we obtain a network such as that shown in Figure 4. We shall refer to such a network as a static market network if the following conditions are satisfied: The network is finite. There is a node, called the ground, which is adjacent to every other node. (Those other nodes will be called market nodes.) Between every market node and ground, there is exactly one branch, and that branch has the form of Figure 2. Not every pair of market nodes need be adjacent, but those that are are connected by a single branch having the form of Figure 3(b). There are no other elements in the network.

Any arbitrary assignment of the market-node prices P_i will yield price differences across the branches that satisfy Kirchhoff's loop law, namely, the sum of the branch price rises minus the sum of the branch price drops found by tracing once around any loop is equal to zero. Moreover, these price differences determine commodity flows in the branches if conditions A and B are fulfilled. But, the commodity flows will not in general satisfy Kirchhoff's node law, which states that the sum of the commodity flows going towards a node minus the sum of the commodity flows going away from that node is equal to zero. However, clearance at every market is equivalent to the fulfillment of the latter law. So, the question at hand is whether there exists a set of market prices for which Kirchhoff's laws are satisfied and, if so, whether that set is unique. Some results of Duffin [7; Theorems 1 and 3] immediately yield the following answer.

Theorem 1. Let there be given a static market network whose ground node is at zero price and whose branches satisfy Conditions A and B. Then, there exists ond and only one set of market-node prices for which Kirchhoff's laws are satisfied (i.e., for which every market is cleared).

A theorem of Desoer and Katzenelson [6; Theorem I] implies that the monotonicity requirements in Conditions A and B need not be strict. H and r_{ik} may have zero slopes on parts of their domains.

The signs of the equilibrium prices have yet to be settled.

Theorem 2. Under the hypothesis of Theorem 1, every market price is nonnegative.

Proof. Assume that the ith market node has a negative price: $P_i < 0$. We first show that there is at least one market node, say, node j adjacent to node i for which $P_j < P_i$. Indeed, if all adjacent nodes had prices no more negative than P_i, then, by Kirchhoff's node law and Condition B, there would have to be a nonnegative commodity flow from the ground to the ith node through the branch that connects ground to the ith node. But, in view of Condition A, this implies that $P_i \geqslant 0$, a contradiction to our first assumption.

We now apply the same argument to node j to conclude that there exists a node k adjacent to node j such that $P_k < P_j$. We must have that node k is distinct from node i since $P_k < P_j < P_i$. Continuing this argument, we see that there must be an infinite sequence of distinct nodes in the market network. But, by definition a market network is finite. Hence, $P_i \geqslant 0$.

Given a static market network, the equilibrium market prices can be computed by using any one of a number of suitable algorithms. See, for example, [4], [5], [9], [10], [11], and the references therein.

III. Dynamic Linear Behavior

We now examine possible types of aggregate dynamic behavior for the producers, consumers, and shippers when commodity flows and prices are allowed to vary with time t by only small amounts. By virtue of this restriction, we take that incremental behavior to be linear, and we do so by postulating such behavior by means of either linear operators or their equivalent electrical networks.

Producers' one-ports. Consider the collection of all producers supplying a particular market. We shall represent the collection by a two-terminal network whose behavior is the same as the aggregate behavior of those producers, and, in conformity with customary electrical-network terminology, will refer to that network as a producers' one-port.

First of all, we assume that the equilibrium price $\tilde{P}$ and the equilibrium commodity flow $\tilde{Q}^s$ for our producers' one-port are linearly related by

$$\tilde{Q}^s = -a + b\tilde{P}, \quad b > 0. \tag{3.1}$$

This could be taken to be a tangent or a chord of the static supply curve in the region of variation for the dynamic behavior.

As one possible mode of dynamic behavior in the vicinity of the equilibrium point [2; pp. 435-436], we then assume that the commodity units per unit time $Q^s(t)$ supplied by the producers is related to the price $P(t)$ at the market by

$$Q^s(t) = -a - h(t) + bP(t) + cP'(t), \tag{3.2}$$

where

$$Q^s(t) = \tilde{Q}^s + q^s(t) \tag{3.3}$$

and

$$P(t) = \tilde{P} + p(t). \tag{3.4}$$

In (3.2) the prime denotes the first derivative, and a, b, and c are constants. The term $h(t)$ is inserted to allow for the shifting of the supply function with time; in effect, we are allowing the static equilibrium line (3.1) to shift its location but not its slope. Insert (3.1) into (3.2) and note that $\dot{P}$ is constant with respect to time. This yields the following expression for the incremental dynamic behavior of our producers' one-port.

$$q^s(t) \;=\; - h(t) + bp(t) + cp'(t). \tag{3.5}$$

An equivalent representation is provided by the parallel network of Figure 5. Going from left to right, we have the following elements: A commodityflow source that propels h(t) commodity units per unit of time in the direction shown by the arrow. A linear "conductance" of b commodity units per unit of time and per unit of price. A linear "capacitance" of c commodity units per unit of price.

A positive value for h(t) corresponds to a shift in the static equilibrium line in the direction of decreasing supply. Also, if the static supply curve is an increasing function of P in the region of variation, then it is natural to assume that $b > 0$. (If the static supply curve is backward sloping in the region of variation for P, then b should be taken to be negative. The presence of negative conductances - or negative cpapcitances - leads to less stable dynamic behavior.) Furhtermore, c can be either positive, negative, or zero, depending on the kind of behavior we wish to assume for the producers' one-port. A positive value of c implies that producers respond with more goods per unit of time not only to higher prices but also to increasing prices. This seems to be reasonable behavior for producers of consumption goods. On the other hand, a negative value of c implies that rising prices tend to decrease the supply rate of commodities. This may be characteristic of a speculative market, where, when prices are rising, producers will tend to hold onto their stocks in hopes of higher prices, and will tend to sell when prices are falling. In line with this reasoning, we may view the "charge" appearing on the capacitance c as the increment in the inventories of the producers.

Furthermore, inertia in altering production or marketing goods can be taken into account by introducing inductances. Morehouse, Strotz, and Horwitz [12] have proposed a piecewise-linear equivalent network with just this provision. The corresponding incremental network is shown in Figure 6. Here, $q^p(t)$ represents the increment in goods produced per unit of time, whereas $q^i(t)$ is the goods coming out of inventory per unit of time. l_1 and l_2 are inductances whose units are price units times the square of time units divided by commodity units. They reflect respectively the inertia in altering $q^p(t)$ and $q^s(t)$. (Actually, Morehouse, Strotz, and Horwitz include a nonlinear element in series with the capacitance to account for an inventory policy.)

Actually, far greater generality can be encompassed by our models if use is made of distributional convolution operators [18; Chapter 5]. Beckman and Wallace [3] have used this approach with ordinary convolution operators to determine the stability of market equilibria. Let $\mathcal{D}'$ be the space of real (Schwartz) distributions on the real line R, and $\mathcal{E}'$ the subspace of distributions of compact support. Let $f \in \mathcal{E}'$ and Q^s, a, h, $P \in \mathcal{D}'$, with a being a constant distribution. Then, we may hypothesize the following behavior for a producers' one-port.

$$Q^s = -a - h + f * P \qquad (3.6)$$

With the notation of (3.3) and (3.4), where $\tilde{Q}^s$ and $\tilde{P}$ are constant distributions, and with the condition

$$\tilde{Q}^s = -a + f * \tilde{P}, \qquad (3.7)$$

(3.6) is equivalent to

$$q^s = -h + f * p. \qquad (3.8)$$

By regularization [18; pp. 132 and 135], we have $f * \tilde{P} = b\tilde{P}$, where b is the value f assigns to the function that equals one everywhere. Thus, (3.7) is the same as (3.1). Equations (3.2) and (3.5) are special cases of (3.6) and (3.8) respectively obtained by setting $f = b\delta + c\delta'$, where δ is the delta functional. Higher-order derivatives and discrete time lags can be encompassed by allowing in f the shifted delta functional and its derivatives. Finally, by restricting P to appropriate subspaces of $\mathcal{D}'$ that contain the constant distributions, we can allow f to have an unbounded support. Thus, for example, f can be the ordinary function that equals e^{-t} for $t \geq 0$ and 0 for $t < 0$ if P is any distribution of slow growth.

We note in passing that time-varying networks also arise naturally in our market networks. For example, not only may the supply line shift with time parallel to itself, but its slope b may also change. In this case, the conductance b in Figure 5 becomes a time-varying one.

<u>Consumers' one-ports</u>. The aggregate behavior of all consumers trading in a particular market shall be represented by a two-terminal network, which will be called a <u>consumers' one-port</u>. We assume that the equilibrium price $\tilde{P}$ and equilibrium commodity flow $\tilde{Q}^d$ to the consumers are related by the straight line

$$\tilde{Q}^d = \alpha - \beta P , \qquad \beta > 0. \qquad (3.9)$$

An appropriate expression for (3.9) can be obtained by taking a tangent or a chord of the static demand curve in the vicinity of incremental operations. For small variations we postulate that the dynamic behavior of the consumers' one-port is given by

$$Q^d = \alpha + \eta - j * P \qquad (3.10)$$

where $j \in \mathcal{E}'$ and Q^d, α, η, $P \in \mathcal{D}'$, α being a constant distribution. Upon setting $Q^d = \tilde{Q}^d + q^d$, $P = \tilde{P} + p$, and b equal to the value that j assigns to the function that equals one everywhere, we obtain as before

$$q^d = \eta - j * p. \qquad (3.11)$$

As a special case, we set $j = \beta\delta + \gamma\delta'$ and obtain

$$q^d = \eta - \beta p - \gamma p', \qquad (3.12)$$

which can be represented by the equivalent network of Figure 7. Here too, we can relax the restriction on the support of j if we restrict the growths of j and p appropriately.

We have taken β to be a positive number to give the static demand line (3.9) its usual negative slope. On the other hand, γ can be either positive, negative, or zero. A positive value means that consumers tend to buy fewer goods if price is increasing [13] A negative value means that increasing price encourages consumers to acquire more goods, as in a speculative market. Consumer inertia, such as that arising from the reluctance of consumers to change their buying habits, can be accommodated in Figure 7 by inserting an inductance into the line carrying $q^d(t)$.

<u>Shippers' one-ports</u>. A two-terminal network representing the aggregate behavior of all transportation activities between two given markets will be called a <u>shippers' one-port</u>. By taking a tangent of chord of the curve of Figure 3(a), we get the following relationship between the equilibrium commodity flow

$\tilde{Q}_{ik}$ and the price rise $\tilde{P}_k - \tilde{P}_i$ from market i to market k along the shippers' one-port:

$$\tilde{Q}_{ik} = d + g(\tilde{P}_k - \tilde{P}_i), \quad g > 0. \tag{3.13}$$

Here, d and g are constants.

As for dynamic behavior it does not seem reasonable to allow a shifting term because shippers do not create or consume goods. Hence, we postulate the relationship:

$$Q_{ik} = d + s * (P_k - P_i), \tag{3.14}$$

where $s \in \mathcal{E}'$ and Q_{ik}, d, P_k, $P_i \in \mathcal{D}'$. By letting g be the value that s assigns to the function that equals one everywhere, we obtain the following relationship for the incremental variations.

$$q_{ik} = s * (p_k - p_i) \tag{3.15}$$

IV. Passive Behavior

A concept that has proved to be useful in the examination of energy transference in physical systems is that of passivity [19; Section 8.2]. This section examines the corresponding idea in the present context. As we shall see, passivity is a behavioral attribute of producers, consumers, and shippers.

Consider a producers' one-port whose static supply curve has the form of one of the increasing curves of Figure 1(a). At the static equilibrium point $(\tilde{P}, \tilde{Q}^s)$, the quantity $\tilde{P}\tilde{Q}^s$ represents the expenditures on the goods flowing from the producers to the market. However, for the incremental dynamic variables variables $p(t)$ and $q^s(t)$, the quantity $p(t)q^s(t)$ does not represent the increment in those expenditures; for, the latter quantity is $\tilde{P}q^s(t) + p(t)\tilde{Q}^s + p(t)q^s(t)$. Nevertheless, it is useful to examine $p(t)q^s(t)$ because it allows one to classify the behavior of the one-port as being either "customary" or "unusual".

Producers may increase quantities produced when prices rise and lower them when prices fall. Such behavior is reflected in a nonnegative value for $p(t)q^s(t)$ whatever be $p(t)$. This need not be the case however when producers react in a more complicated way to variations in price, as for example when producers exhibit a continuous lag in their adjustments to $p(t)$. Indeed, if producers react in accordance with (3.5) wherein $h(t) \equiv 0$, b is small, and c is large and if $p(t)$ equals t for $0 < t \leq 1$, $2 - t$ for $1 \leq t < 2$, and 0 everywhere else, then $p(t)q^s(t)$ can be negative during part of the interval $0 < t < 2$. Nonetheless, the integral

$$\int_{-\infty}^{t} p(x)q^s(x)\, dx \tag{4.1}$$

will remain nonnegative for all t so long as b and c are positive, that is, so long as the static supply curve is not backward sloping and the producers do not exhibit speculative behavior. It appears justifiable to view this as the "customary" behavior of producers and contrary behavior as "unusual". Indeed, it is a fact that if the equivalent circuit for a producers' one-port is a finite network containing only positive conductances, capacitances, and inductances and no sources, then (4.1) will remain nonnegative for every value of t and every p in broad classes of functions [14].

Similarly, if a consumers' one-port has a static equilibrium curve (like that of the decreasing curve of Figure 1(a)), then under static-equilibrium conditions $\tilde{P}\tilde{Q}^d$ represents the expenditures on the flow of goods from the market to the consumers. Moreover, if consumers raise (lower) their consumption when prices fall (respectively, rise), then $p(t)q^d(t)$ will be nonpositive. For the more complicated behavior of (3.12), this need not be so; however, the integral

$$- \int_{-\infty}^{t} p(x)q^{d}(x)\, dx \tag{4.2}$$

will remain nonnegative for all t and every p in certain broad classes of functions so long as β and γ are positive. that is, so long as the demand curve does not have a positive slope and the consumers do not speculate. If we attach the minus sign in (4.2) to $q^d(t)$, we obtain the same situation as that for producers. Namely, when the incremental commodity flow is measured as a flow toward the market node, the nonnegativity of (4.1) and (4.2) is a characteristic of the customary behavior of producers and consumers.

The same arguments can be advanced to support the contention that the "customary" behavior of shippers is such that

$$\int_{-\infty}^{t} [p_k(x) - p_i(x)]\; q_{ik}(x)\, dx \;\geqslant\; 0$$

for all t and all suitably restricted p_k and p_i. Needless to say, the nonnegativity of these integrals may not hold for a particular system of markets, in which case the consequences of passivity cannot be invoked.

To make these ideas precise, we introduce some terminology common in electrical-network theory. As before, a one-port is a two-terminal network representing the relationship between a commodity flow q and the price rise through which it flows. This is shown schematically in Figure 8, where the plus (minus) sign designates the terminal of higher (respectively, lower) price. In general, p and q are real Schwartz distributions on the real time axis. The one-port may define a single-valued mapping $\mathfrak{N}: p \mapsto q$ on some domain in $\mathcal{D}'$ into $\mathcal{D}'$; in this case, $\mathfrak{N}$ will be called the admittance operator for the one-port. For example, (3.8) with h = 0 defines an admittance operator on all of $\mathcal{D}'$.

Similarly, $\mathcal{Z}: q \mapsto p$ is called the <u>impedance operator</u> for the one-port whenever it exists.

Now, let $\mathcal{D}$ be the customary space of real-valued functions on R whose supports are bounded and whose derivatives of all orders are continuous. An operator $\mathcal{H}: p \mapsto q$ mapping $\mathcal{D}$ into $\mathcal{D}'$ will be called <u>passive</u> if, for every $p \in \mathcal{D}$, q is locally Lebesgue integrable and

$$\int_{-\infty}^{t} p(x)q(x)\, dx \geqslant 0$$

for every $t \in R$.

The convolution operators appearing in (3.8), (3.11), and (3.15) have a variety of properties that make them analytically attractive [18], [19]. We mention just a few results: Let $f \in \mathcal{E}'$. The mapping $p \mapsto f * p$ is passive if and only if the Laplace transform F of f exists and is a positive-real function [18; p. 186]. (A positive-real function $F(\zeta)$ is a function of the complex variable ζ which satisfies the following three conditions on the right-half plane $\{\zeta : \operatorname{Re} \zeta > 0\}$: F is analytic. $\operatorname{Re} F(\zeta) \geqslant 0$. $F(\zeta)$ is real whenever ζ is real and positive.) Another consequence of passivity is that the support of f is contained in the nonpositive real axis [19; Theorems 5.11-1 and 8.2-1].

Now, assume in addition that the positive-real function $F(\zeta)$ is analytic at infinity and tends to zero as ζ approaches infinity. The positive-reality of $F(\zeta)$ now implies that $F(\zeta) \sim K\zeta^{-1}$ as $|\zeta| \to \infty$, where $K > 0$. It also follows that f is a continuous function for $0 < t < \infty$ and has no distributional singularities in the neighborhood of the origin, that $f(0+) = K$, and that $|f(t)| \leq K$ for all $t > 0$. (The proof of the last bound, although given for rational $F(\zeta)$ in [17], extends readily to the $F(\zeta)$ considered here.) Actually, every one-port consisting of a finite network of positive conductances, capacitances, and inductances, having no internal sources , and having at least one purely capacitive path connecting its terminals satisfies the conditions of this paragraph. In this case, K is the reciprocal of the total capacitance between the one-port's terminals remaining after every conductance and inductance is replaced by an open circuit.

V. Market Networks

A network suitable for the incremental dynamic analysis of our system of markets can be obtained by interconnecting appropriate producers', consumers', and shippers' one-ports in accordance with the given transportation connections. We shall refer to such a configuration as an incremental market network. ¶An example of such a network is shown an Figure 9. In that diagram the producers' and consumers' one-ports at each market have been combined into a single one-port consisting of a conductance and capacitance in parallel. Also, it is assumed that there is a shift in supply and demand only at the first market; thus, only the first market's one-port has a commodity-flow source.

Although an incremental market network can be much more complicated than its corresponding static market network, the former does have the following properties in common with the latter: The network is finite. There exists a hypothetical node, called the ground node, with respect to which every price in the market system is measured. Each market is represented by a unique node, called a market node, and the price at that market appears as the voltage at the market node. (Since we are now dealing with incremental quantities, market prices may be negative.) Between every market node and ground there exists a one-port consisting of producers' and consumers' one-ports and possibly other parameters such as tax-induced price sources. Not every pair of market nodes need be adjacent, but those that are are connected by shippers' one-ports.

However, in contrast to static market networks, our incremental market networks are linear, and therefore we may apply results from the vast literature on linear electrical networks. Indeed, for a spatially distributed and interconnected system of markets, the incremental deviations of the market prices from their equilibrium values due to shifts in the various supply and demand functions can be determined by setting up the corresponding incremental market network, inserting appropriate commodity-flow sources to represent these shifts, and then analyzing the network under Kirchhoff's laws. For, market clearance in every market at every instant of time is equivalent to the satisfaction of Kirchhoff's node law at every market node. At the other nodes (such as those inside the producers', consumers', and shippers' one-ports), Kirchhoff's node law is satisfied simply by the definition of these one-ports as electrical networks. Also, the condition that each node has one and only one price associated with it is equivalent to the satisfaction of Kirchhoff's loop law. Finally, the one-ports characterizing the behavior of the producers, consumers, and shippers impose the remaining restrictions required by electrical network theory, namely, relationships between commodity flows and price differences along the network's branches.

If the number of markets is large, then the analysis of the incremental market network can become onerous. However, a variety of conclusions may be drawn from the general behavior of electrical networks. For example, if the market network consists only of conductances, capacitances, and inductances, all of which are positive and if every capacitance (inductance) is in parallel (respectively, in series) with a conductance, then the market system is stable in the following sense. The transient price or commodity-flow variations due to a sudden shift in supply or demand in one of the markets will die away exponentially, and all prices and commodity flows will approach a new stable equilibrium. More particularly, if all the producers' and consumers' one-ports have the forms shown in Figures 5 and 7 and every shippers' one-port is a single conductance, if all conductances and capacitances are positive, and if all commodity-flow sources are zero except for the one at the first market which is instead the delta functional, then the incremental variation in the price at that first market is given by

$$p_1(t) = \sum_{k=1}^{n} \lambda_k e^{-\rho_k t}, \qquad t > 0,$$

where the λ_k and ρ_k are all positive numbers [16, pp. 312-313]. In other words, the market network as seen from the first market and ground terminals is a relaxive one-port [20].

VI. Bounds on Transient Responses

As was mentioned above, the determination of transient responses in large incremental market networks may require extended computations. However, the special form that market networks have allow us in many cases to write down bounds on those transient responses merely by inspecting the networks. These bounds indicate how slowly the prices and commodity flows respond to variations in supply and demand.

As an example, consider again the incremental market network shown in Figure 9. Assume there occurs a shift in either supply or demand in the first market resulting in the commodity-flow source h(t). If all capacitances and conductances are positive, the price variation at the fourth market lags h(t). In fact, when h(t) is the delta functional, the price $p_4(t)$ at the fourth market satisfies

$$0 \leq p_4(t) \leq \frac{g_a g_c}{c_1 c_2 c_4} \cdot \frac{t^3}{3!} + \frac{g_b g_d}{c_1 c_3 c_4} \cdot \frac{t^3}{3!} + \frac{g_a g_e g_d}{c_1 c_2 c_3 c_4} \cdot \frac{t^4}{4!} + \frac{g_b g_e g_c}{c_1 c_3 c_2 c_4} \cdot \frac{t^4}{4!} . \tag{7.1}$$

When h(t) has a more general form with h(t) = 0 for t < 0, the resulting price, say, $v_4(t)$ at the fourth market can then be bounded by estimating the convolution integral

$$v_4(t) = \int_0^t h(t-x) p_4(x)\, dx, \quad t > 0.$$

(7.1) will be justified by establishing a more general result. Consider an arbitrary incremental market network where every market one-port (i.e., the parallel combination of the producers' and consumers' one-ports for the market) satisfies the following condition, wherein Z_k is the driving-point impedance of the kth-market one-port. (A driving-point impedance of a one-port is the Laplace transform of p in Figure 8 when q is the delta functional.)

Condition C. For every k, Z_k is a positive-real function analytic at infinity, and $Z_k(\zeta) \sim (c_k \zeta)^{-1}$ as $\zeta \to \infty$.

Let us introduce some additional terminology: A path in a network is a finite alternating sequence of nodes and branches such that the first and last terms are nodes, each branch is incident to the nodes immediately preceding and succeeding it in the sequence, and no node appears more than once in the sequence. Now, consider a market network where every shippers' one-port, no matter how complicated, is treated as a single branch. By an <u>admissible 1 to m path</u> we shall mean a path in a market network which terminates at nodes 1 and m and has only shippers' one-ports as branches. Next, assume in addition that every shippers' one-port is a single conductance and every market one-port satisfies Condition C. The <u>conductance-capacitance ratio</u> of an admissible 1 to m path is the product of all the conductances in the branches of the path divided by the product of the constants c_k for the market one-ports connecting the path's nodes to ground.

<u>Theorem 3</u>. Assume that, in a given incremental market network, the impedance of every market one-port satisfies Condition C, and every shippers' one-port is a single positive conductance. Let $p_m(t)$ be the incremental price variation at the mth market node ($m \neq 1$) resulting from the application of a delta-functional commodity-flow source at $t = 0$ directed from the first market node to ground. Then, for $t \geqslant 0$,

$$|p_m(t)| \leqslant \sum_j \frac{A_j t^{n_j}}{n_j!} . \qquad (7.2)$$

Here, A_j is the conductance-capacitance ratio for the jth admissible 1 to m path, n_j is the number of branches in that path, and the summation is over all the admissible 1 to m paths.

<u>Proof</u>. Assume there are r shippers' one-ports of conduntances $g_a, g_b, \ldots, g_r$ incident to the first market node. Let $p_1(t)$ be the price variation at that node. We replace our market network by another one obtained by deleting the first market one-port and the delta-functional source and connecting r price sources from ground to those shippers' one-ports that are incident at

the first market node. The procedure is illustrated by the transition from Figure 10(a) to 10(b). If each price source generates $p_1(t)$, then this transition does not alter the price and commodity-flow variations throughout the rest of the market network. By superposition, $p_m(t)$ is the sum of r terms, each of which is obtained by shorting out all but one of the price sources indicated in Figure 10(b). Consider the resulting network for one such term, say, the one $p_m^a(t)$ for which all price sources other than that connected to g_a are shorted out. Make a change of source by replacing the series combination of $p_1(t)$ and g_a by the parallel combination of a commodity-flow source $g_a p_1(t)$ and the conductance g_a. This is indicated in Figure 10(c).

Assume that g_a connects node 1 to node 2 and that the price variation at node 2 is $v_2(t)$. We now repeat the above procedure by deleting the parallel combination (including the market one-port) that connects node 2 to ground, separating all leads to node 2, and connecting a separate price source of value $v_2(t)$ from ground to each and every one of those leads. This yields a set of connections similar to those of Figure 10(b). It follows by superposition again that $p_m^a(t)$ is a sum of terms, each of which results from the shorting out of all but one of the price sources just introduced.

Continuing this procedure, we see that $p_m(t)$ is a sum of terms, each of which corresponds to a distinct admissible 1 to m path in the original market network. Indeed, our procedure results in a contribution to the terms comprising $p_m(t)$ whenever we trace out an admissible 1 to m path. On the other hand, if we apply our procedure along a sequence of shippers' one-ports in such a fashion that we meet a node for a second time, that node will be shorted to ground. Therefore, its price in the resulting network is identically zero and cannot contribute anything to $p_m(t)$.

We can now estimate each term in the aforementioned sum for $p_m(t)$ as follows. Let W be the driving-point impedance between the first market node and ground for the entire network, but with the commodity-flow source $\delta(t)$ removed, and let X be the corresponding driving-point impedance with the first market one-port also removed. It is a fact that the driving-point impedance of a one-port consisting only of an interconnection of one-ports having positive-real driving-point impedances is also positive-real. Hence, W is positive-real. Moreover, X is a rational function of the impedances in the market network, all of which are analytic at infinity by Condition C. Consequently, X is either analytic or has a pole at infinity. X cannot have a zero at infinity because every shippers' one-port incident to the first market node is a finite conductance. Since W is the parallel combination of X and the impedance Z_1 of the first market one-port, it follows that $W(\zeta)$ is asymptotic to $Z_1(\zeta)$ as $\zeta \to \infty$. That is, $W(\zeta) \sim (c_1 \zeta)^{-1}$ as $\zeta \to \infty$. Since p_1 is the inverse Laplace transform of W, we may now invoke [17; Theorem 1] to obtain $|p_1(t)| \leq c_1^{-1}$.

In Figure 10(c), let Z_2 be the impedance of the market one-port connected between ground and node 2. Then, the same argument shows that the driving-point impedance W_2 between those two nodes (with $g_a p_1(t) = 0$) is asymptotic to $Z_2(\zeta)$ and hence to $(c_2 \zeta)^{-1}$ as $\zeta \to \infty$. Let w_2 be that right-sided distribution whose Laplace transform is W_2. Then, we see as before that

$|w_2(t)| \leq c_2^{-1}$. But, for $t \geqslant 0$,

$$v_2(t) = \int_0^t g_a p_1(x) w_2(t - x)\, dx$$

and therefore

$$\left| v_2(t) \right| \leq \frac{g_a}{c_1 c_2} t .$$

Continuing this sequence of estimations along an admissible 1 to m path, we obtain one of the terms in the sum (7.2). This completes the proof of Theorem 3.

The significance of Theorem 3 is a natural one: The further away one market is from another market having a shifting supply or demand function, the greater must be the lag in the response of the former market to that shift.

(7.2) justifies the right-hand inequality in (7.1). The left-hand inequality in (7.1) can be established by using the following two facts. The convolution of two nonnegative functions is again nonnegative. Every function appearing in the sequence of convolution integrals in the preceding proof is nonnegative due to the fact that the altered networks arising from Figure 9 are all relaxive one-ports.

The <u>driving-point admittance</u> of a one-port is the Laplace transform of q in Figure 8 when p is the delta functional. A result similar to Theorem 3 can be stated if every shippers' one-port satisfies the following condition, wherein Y_k denotes the driving-point admittance of the kth shippers' one-port.

<u>Condition D</u>. Y_k is a positive-real function analytic at infinity, and $Y_k(\zeta) \sim (\underline{l}_k \zeta)^{-1}$ as $\zeta \to \infty$.

With the notation introduced in Conditions C and D, we define the <u>inductance-capacitance factor</u> of an admissible 1 to m path as the reciprocal of the product of all the constants c_k for the market one-ports connecting the path's nodes to ground and all the constants $\underline{l}_k$ for the shippers' one-ports through which the path proceeds.

Theorem 4. Assume that, in a given incremental market network, the driving-point impedance of every market one-port satisfies Condition C and the driving-point admittance of every shippers' one-port satisfies Condition D. Let $p_m(t)$ be as in Theorem 3. Then, for $t \geqslant 0$, $|p_m(t)|$ is bounded as in (7.2) except that now A_j is the inductance-capacitance factor for the jth admissible 1 to m path and n_j is the number of $\underline{l}_k$ and c_k in A_j.

This theorem is proven as is Theorem 3 except that, instead of making the changes of sources indicated in the transition from Figures 10(b) to 10(c), we proceed as follows. To obtain the appropriate figures for the present case, replace the conductances $g_a, \ldots, g_r$ by shippers' one-ports having the admittances $Y_a, \ldots, Y_r$. In Figure 10(b), the commodity flow q_a passing to the left through Y_a is equal to the convolution of p_1 with the function that q_a would be were p_1 the delta functional. This function is zero for $t < 0$, and it can be shown as before to be bounded in magnitude by $\underline{l}_a^{-1}$, where $Y_a(\zeta) \sim (\underline{l}_a \zeta)^{-1}$ as $\zeta \to \infty$. Then, in Figure 10(c), a commodity-flow source of value $q_a(t)$ is applied between node 2 and ground (but now $g_a p_1(t)$ and g_a do not appear.) The proof continues in this fashion, and the desired bounds are obtained by estimating a sequence of convolution integrals.

Actually, we can combine the proofs of Theorems 3 and 4 to write down a bound on $p_m(t)$ even when some of the shippers' one-ports are conductances g_i and the remaining shippers' one-ports are admittances Y_k satisfying Conditions D. In this case the factor A_j contains those g_i in its numerator and $\underline{1}_k$ in its denominator corresponding to the shippers' one-ports occurring in the jth admissible 1 to m path; n_j is again the number of $\underline{1}_k$ and c_k in the denominator of A_j.

It is worth mentioning that with no further assumptions on our market network, the bound (7.2) cannot be improved. For, networks can be constructed for which the difference between the two sides of (7.2) can be made arbitrarily small for any given t.

Upon replacing n_j by $n_j + 1$ in (7.2), we obtain a bound on how quickly a market price can start responding to a unit jump in supply or demand. Another consideration of interest is the delay inherent in the transition of that market price toward its new equilibrium value. There are a variety of results in the electrical-network literature on this question; we describe just one of them.

Assume there is a unit jump in supply of demand at the first market. This is represented by a commodity-flow source $h(t) = 1_+(t)$ connected between the first market node and ground. ($1_+(t)$ equals 1 for $t \geq 0$ and 0 for $t < 0$.) Assume further that the driving-point impedance Z of the entire market network as seen from those two nodes is positive-real, analytic at infinity and on the closed right-half plane $\{\zeta : \text{Re}\, \zeta \geq 0\}$, and such that $Z(\zeta) = (c\zeta)^{-1}$ as $\zeta \to \infty$ and $Z(0) = r > 0$. In many cases, c and r can be determined simply by inspecting the market network.

Under these assumptions, the incremental price $p_1(t)$ at the first market node is a continuous function, zero for $t < 0$, which rises toward and exponentially approaches (possibly in an oscillatory fashion) the constant value r as $t \to \infty$. With

$0 < \varepsilon < 1$, we define the <u>settling time</u> τ_ε as the least time beyond which $|p(t) - r| < \varepsilon r$. There is a lower bound on τ_ε depending only on c, r, and ε [20; Corollary 2]. It is

$$\tau_\varepsilon > rc \max_{0 \le y < 1} \left\{ \frac{\pi y}{\sin \pi y} - 2y^3 \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j(j^2 - y^2)} - \varepsilon \left[\frac{\pi y}{\sin \pi y} + 2y^3 \sum_{j=1}^{\infty} \frac{1}{j(j^2 - y^2)} \right] \right\} .$$

The right-hand side is plotted in [20; Figure 5]. Thus, here again, we can readily ascertain an informative restriction on the transient behavior of our market network.

Similar restrictions exist on the settling times of the prices $p_k(t)$ at the other markets $(k \neq 1)$; see [21] and [22]. But now, some extended computations may be needed to see if the market network satisfies the hypotheses under which the restrictions hold.

VII. Several Commodities

We now relax the assumption that we are dealing with only one commodity. We assume instead that, at each of m different market locations, n different commodities are traded. By the (k,i) market we shall mean the market at the kth location that trades in the ith commodity. Thus, $k = 1, \ldots, m$ and $i = 1, \ldots, n$. The prices and commodity flows in each market will be denoted by a similar double subscript notation.

It follows that we have n different market networks, one for each commodity. In order to relate them, we restrict the behavior of consumers as follows.

<u>Postulate E.</u> At each market location, a shift in demand for any commodity causes simultaneous shifts in the demands for the other commodities in such a fashion that the total

static-equilibrium expenditure of the consumers on all the commodities traded at that location remains the same before and after the shift.

This implies that the consumers at each market location have a budget constraint. If their demand for one commodity increases, then their demand for another decreases, and their total expenditure before the transients have started equals their total expenditure after the transients have died away.

We could have used another postulate; namely, the supply curves for the commodities at a given market location shift together in such a way that the total static-equilibrium value of all goods delivered by the producers at that location remains the same before and after the transient period. However, the analysis in this case is virtually the same as what we present below.

Let $(\tilde{P}_{ki}, \tilde{Q}^d_{ki})$ and $(\hat{P}_{ki}, \hat{Q}^d_{ki})$ be two static-equilibrium points as seen by the consumers at the (k, i) market. Postulate E is equivalent to the equations

$$\sum_{i=1}^{n} \tilde{P}_{ki}\tilde{Q}^d_{ki} = \sum_{i=1}^{n} \hat{P}_{ki}\hat{Q}^d_{ki}, \quad k = 1, \ldots, m. \tag{8.1}$$

Set $\hat{P}_{ki} = \tilde{P}_{ki} + v_{ki}$ and $\hat{Q}^d_{ki} = \tilde{Q}^d_{ki} + u_{ki}$. Then, (8.1) is the same as

$$\sum_{i=1}^{n} \left[v_{ki}(\tilde{Q}^d_{ki} + \tfrac{1}{2}u_{ki}) + (\tilde{P}_{ki} + \tfrac{1}{2}v_{ki})u_{ki}\right] = 0.$$

We now assume in addition that u_{ki} is small as compared to $\tilde{Q}^d_{ki}$ and that v_{ki} is small as compared to $\tilde{P}_{ki}$. Then, upon discarding the product $u_{ki}v_{ki}$ in the last equation, we get an approximation to zero, which we will take as an equality:

$$\sum_{i=1}^{n} (v_{ki}\tilde{Q}^d_{ki} + \tilde{P}_{ki}u_{ki}) = 0, \quad k = 1, \ldots, m \tag{8.2}$$

We base the rest of our analysis on (8.2). In other words, instead of using Postulate E precisely, we use an approximation to it given by (8.2). The smaller the increments in the static-equilibrium points, the better will be this approximation.

We take the increments v_{ki} and u_{ki} to be the results of shifts in the demand curves with no slope changes as before. The latter shifts appear as commodity-flow sources f_{ki} across the market one-ports. To compute the v_{ki} for a given i, we alter the incremental market network for the ith commodity by replacing all capacitances by short circuits and all inductances by open circuits. This assumes that all transients due to shifts in supply and demand die out with time, which will certainly be the case for jump-type shifts whenever the network is passive with only positive parameters and every loop in the network contains a nonzero conductance The result will be a linear grounded network having the form of Figure 4, except that each branch incident to ground consists of a conductance in parallel with a commodity-flow source f_{ki}. In this network the v_{ki} ($k = 1, \ldots, m$) are the prices at the market nodes. A nodal analysis yields a system of linear algebraic equations in the unknown v_{ki}, with the known terms being linearly homogeneous in the f_{ki}. If every conductance in the network is positive, then the network's structure insures that the determinant of coefficients for our system of equations is dominated by its diagonal elements and is therefore nonzero. Thus, the equations can be solved for the v_{ki}, which in turn immediately determine the u_{ki} as the commodity flows through the consumers' one-ports. All this shows that the solutions for the v_{ki} and u_{ki} are linearly homogeneous in the f_{ki}. Consequently, upon substituting these solutions into (8.2), we obtain an expression of the form

$$\sum_{i=1}^{n} T_{ki} f_{ki} = 0 , \quad k = 1, \ldots , m, \tag{8.3}$$

where the T_{ki} are constants depending upon the various equilibrium values $\tilde{P}_{ki}$ and $\tilde{Q}^{d}_{ki}$ and the conductance values in the incremental market networks.

Equations (8.3) are precisely the restrictions that would be imposed by m ideal transformers with n coils each, whose coils are connected in series with the commodity-flow sources f_{ki} [14; pp. 9-11]. These connections are illustrated in the incremental market network of Figure 11 for the case of 2 market locations and 3 commodities.

Now, a shift in demand at one or more markets produces nonzero values f_{ki} for the commodity-flow sources, which must satisfy the transformer conditions (8.3) according to our approximation to Postulate E. The resulting transients and final equilibrium values can be determined by making a standard electrical-network analysis of the transformer-coupled market network.

VIII. Conclusions

Electrical network theory can be used for the incremental dynamic analysis of spatially distributed markets. In addition to providing a means of calculating the price and commodity-flow variations in a particular market system for one or more commodities, it provides a variety of results that can be adapted to the general theory of market networks. Thus, for example, a theorem concerning nonlinear resistive networks provides an existence and uniqueness theorem for a set of market prices under static equilibrium. Also, the powerful concept of passivity is related to the behavior of producers, consumers, and shippers. This in turn leads to bounds on the transient responses of market prices to variations in demand or supply. Above all, this paper introduces a method of analysis which should be fruitful in extending the theory of interconnected spatially distributed markets.

Finally, we mention that value-added taxes, sales taxes, tariffs, and various kinds of subsidies can be represented by appropriate price sources inserted at various points of our market networks. The burdens and benefits they place upon the different groups of producers, consumers, and shippers can then be determined by analyzing the resulting network. This will be the subject of a subsequent paper.

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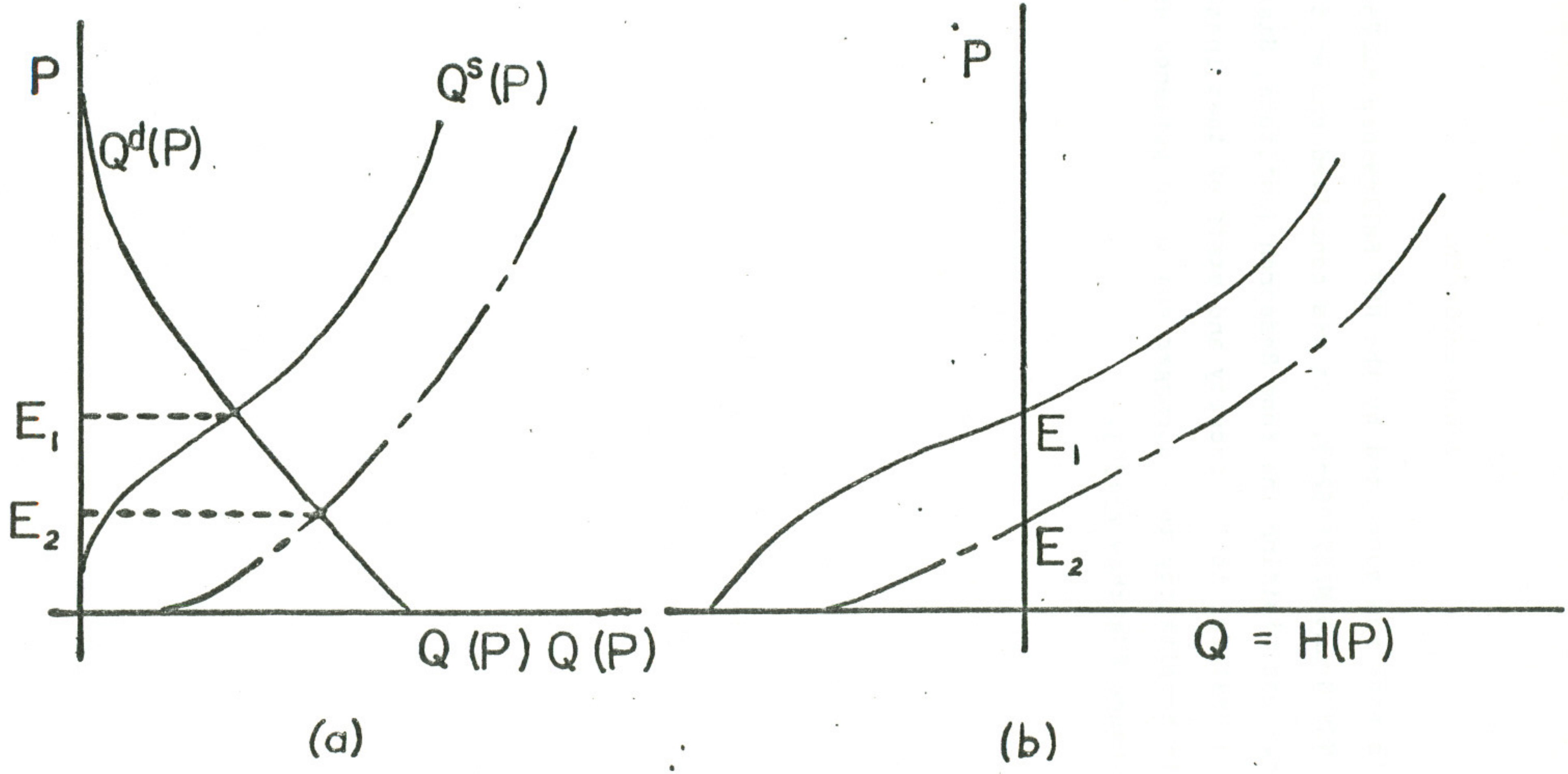
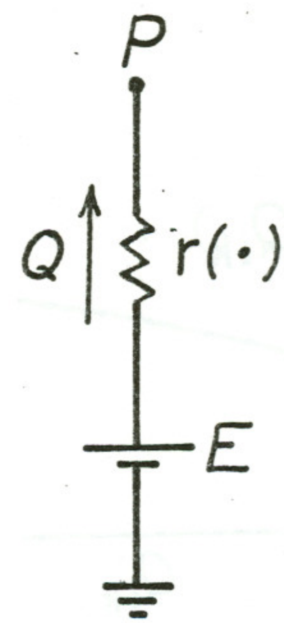


Figure 1



$$P = E + r(Q)$$

Figure 2

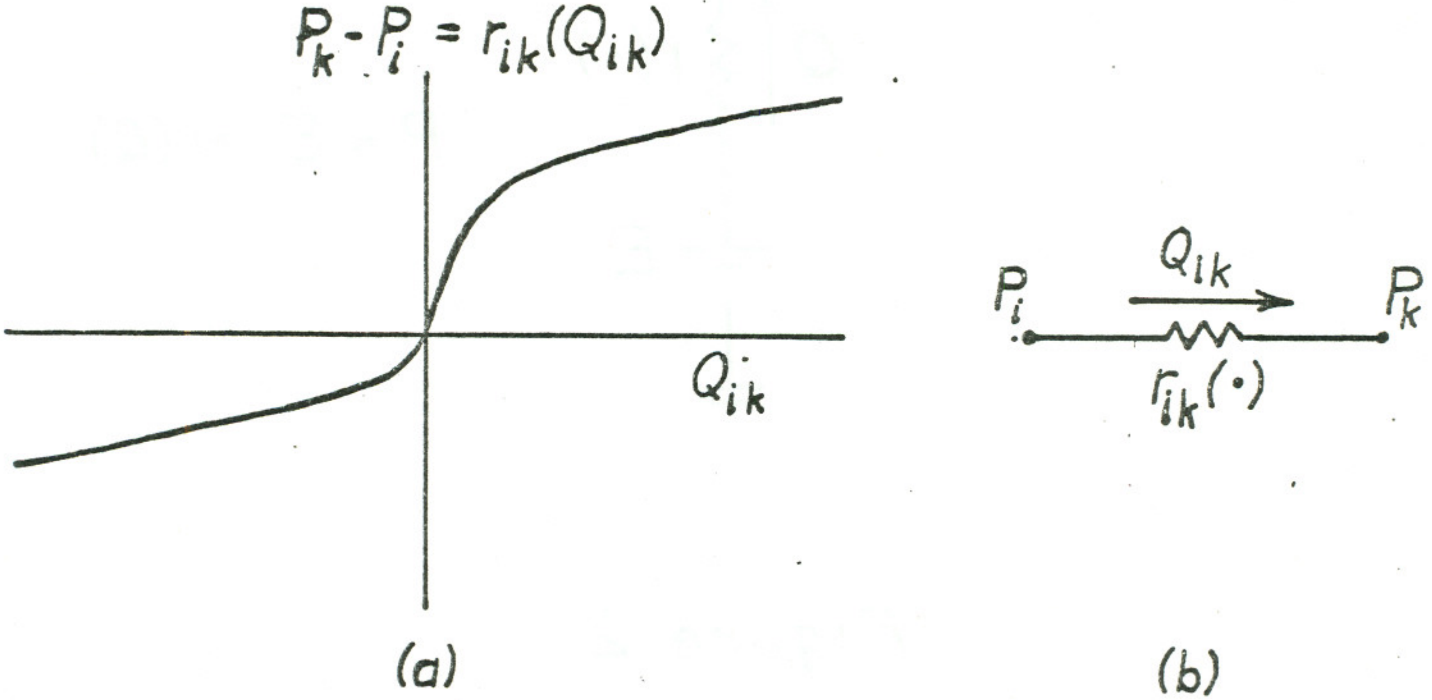


Figure 3

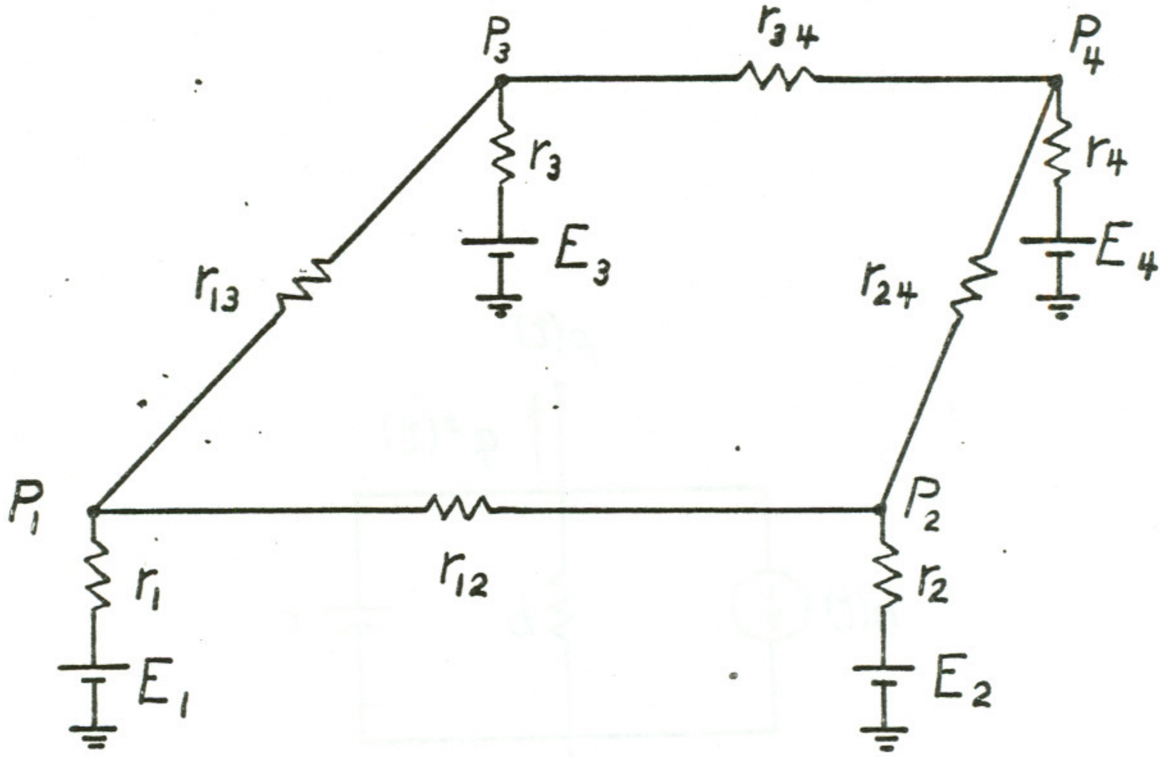


Figure 4

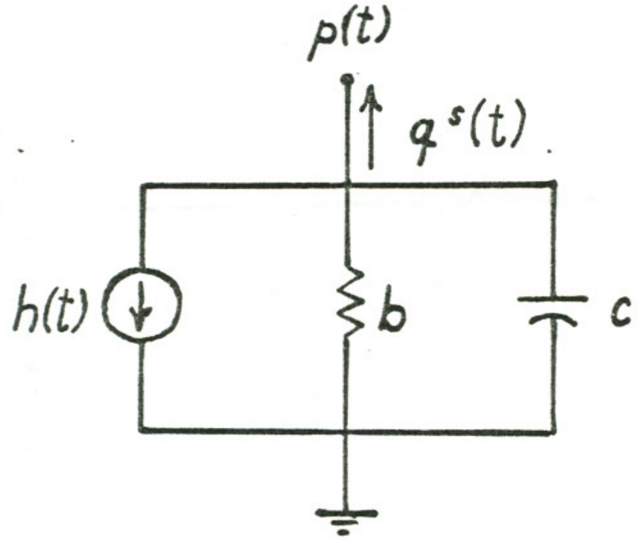


Figure 5

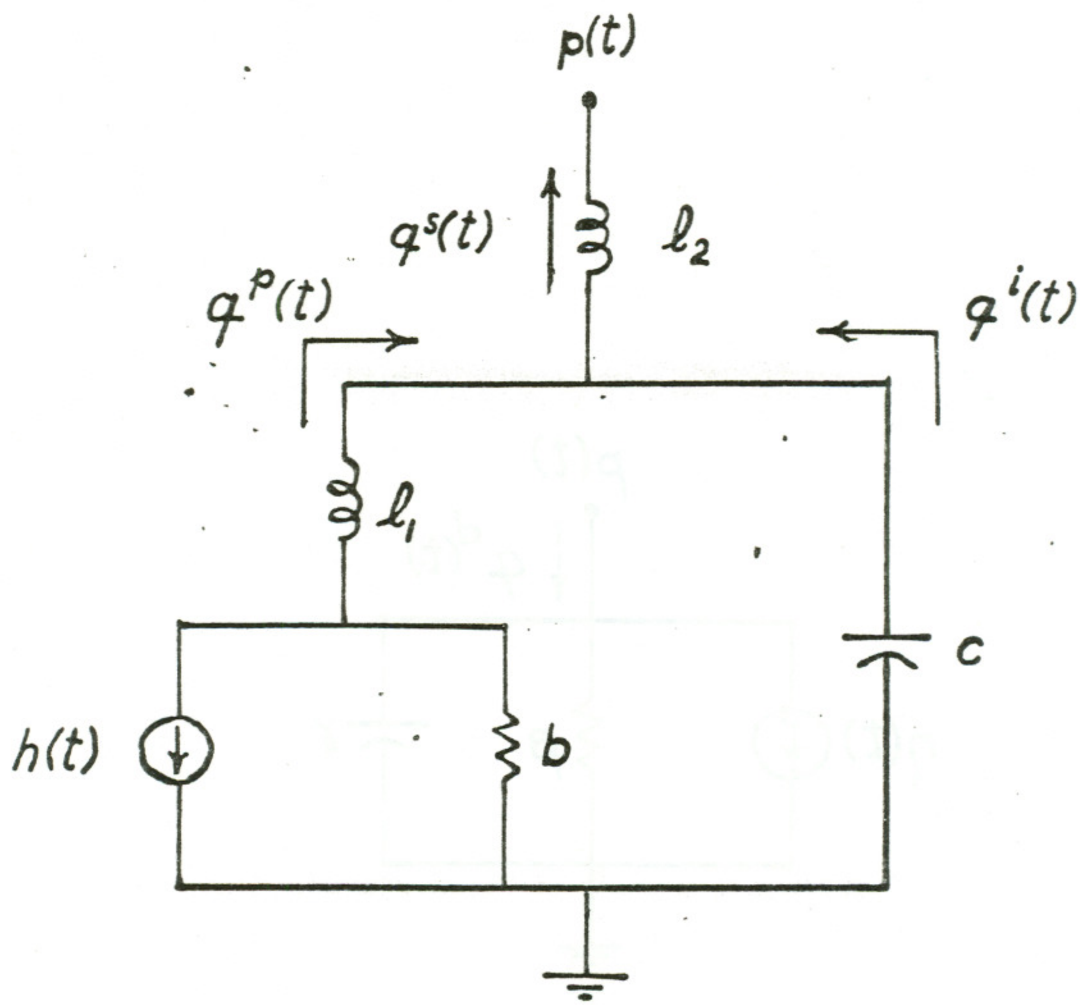


Figure 6

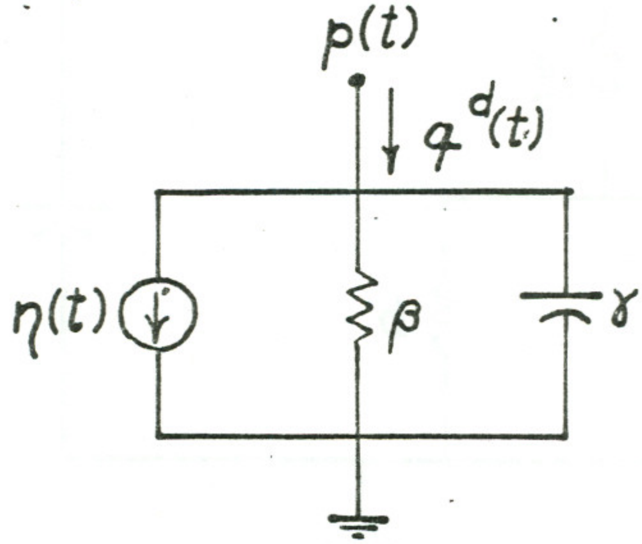


Figure 7

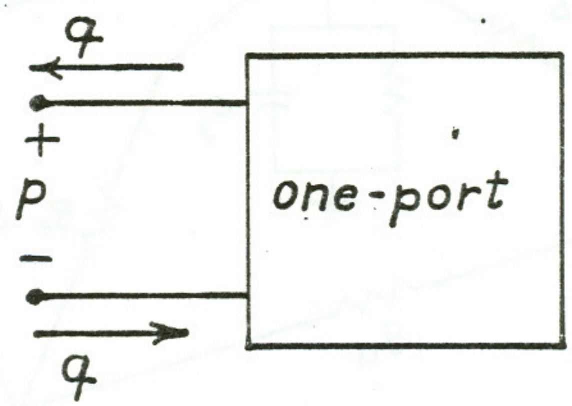


Figure 8

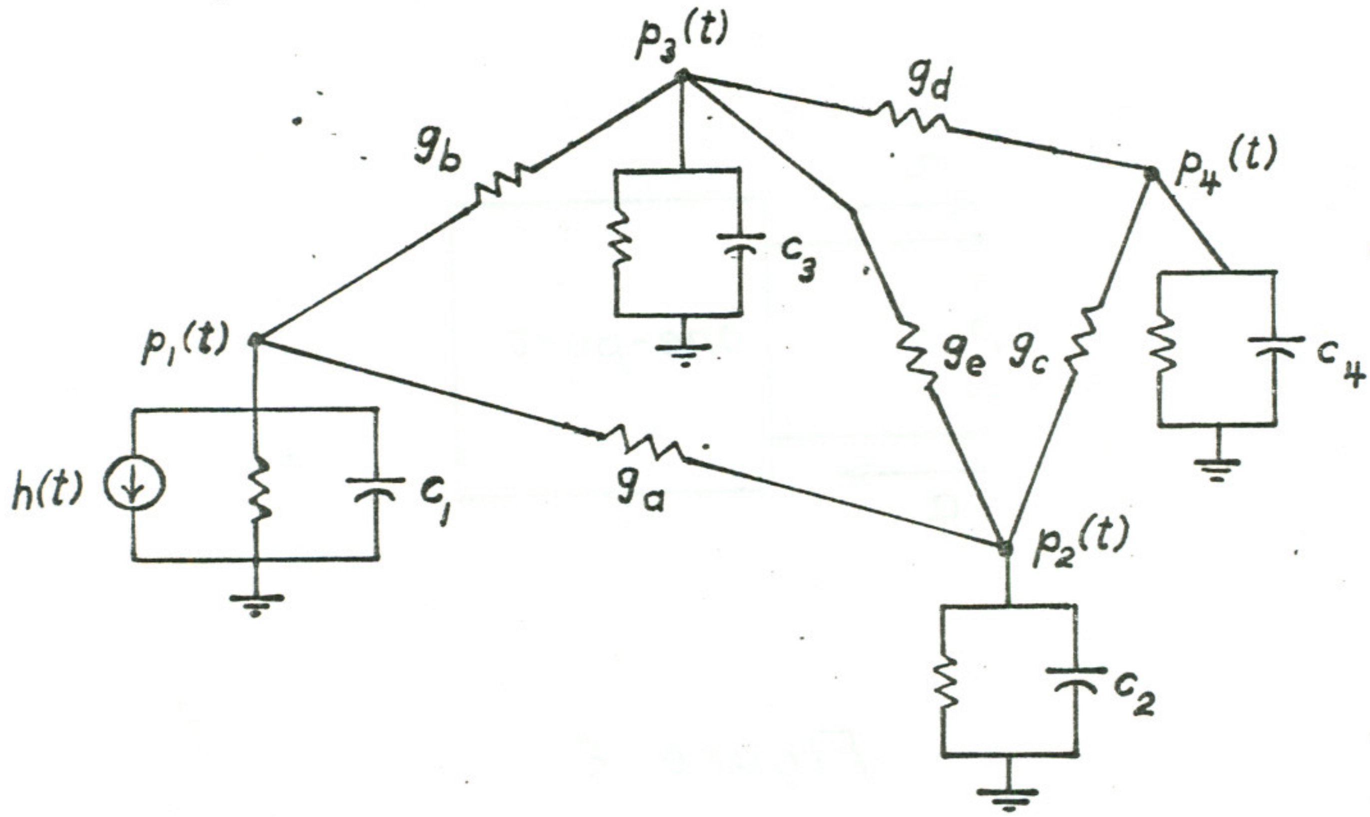


Figure 9

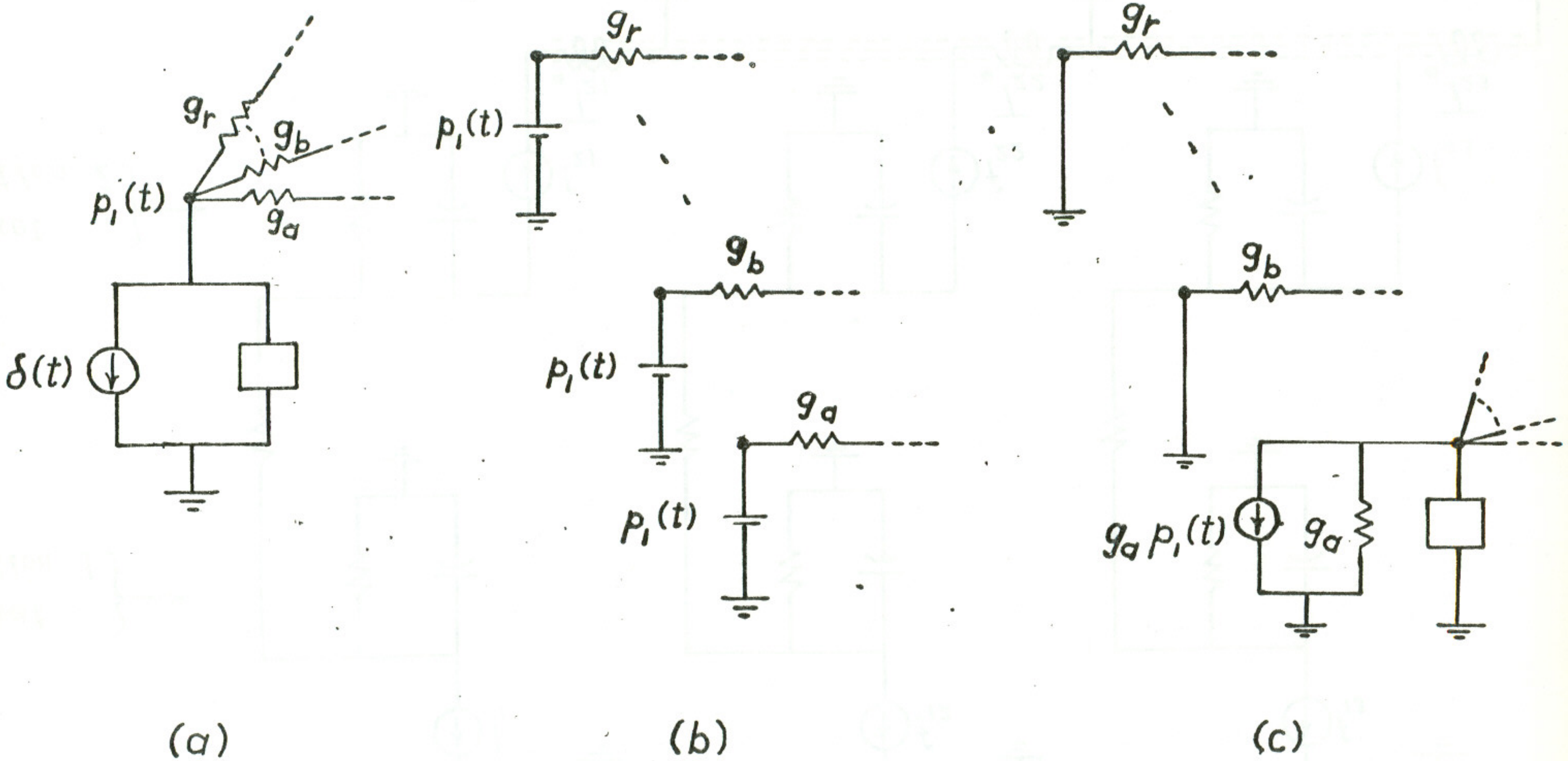
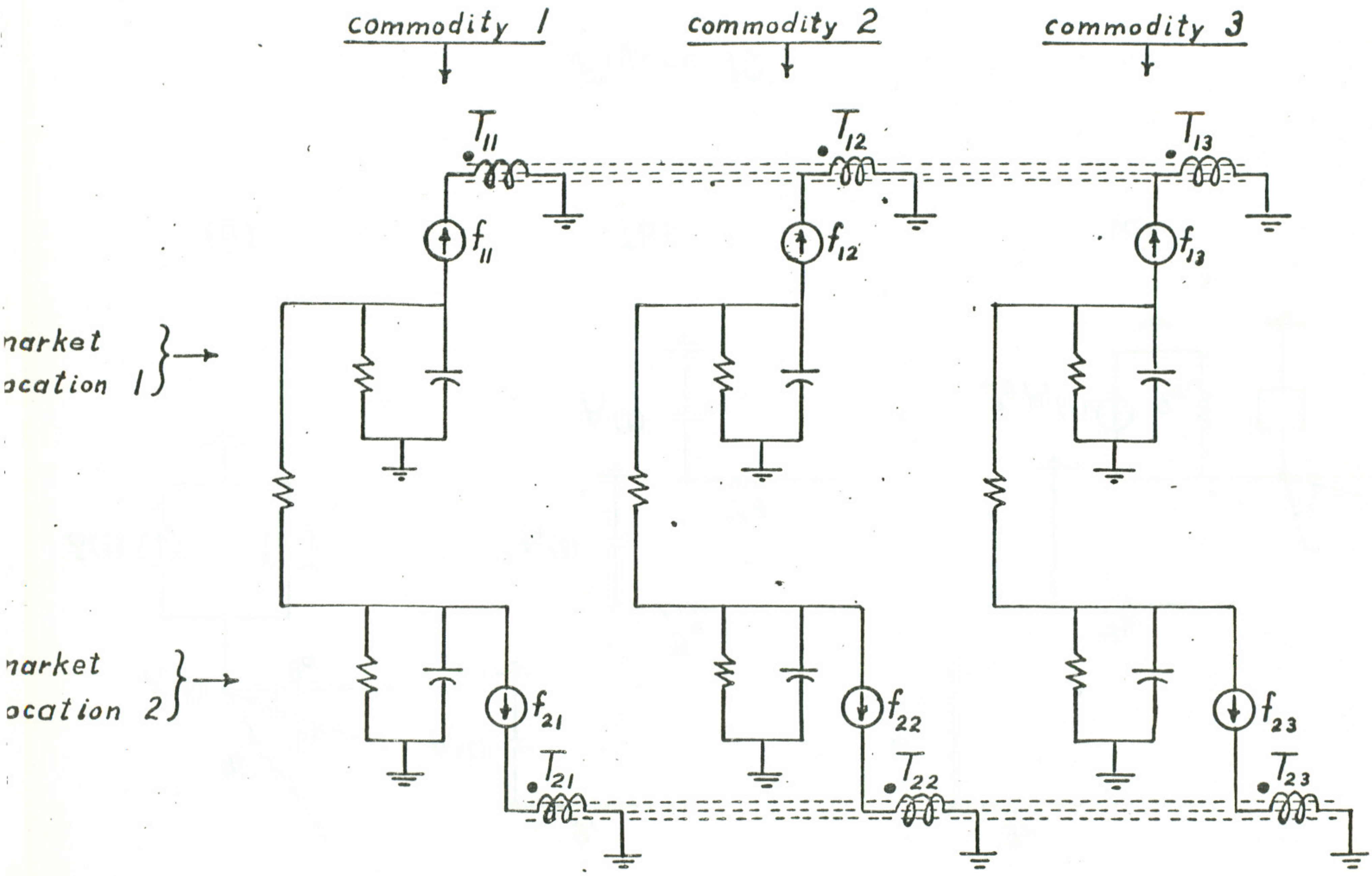


Figure 10



commodity 1
commodity 2
commodity 3
T11
T12
T13
f11
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narket
ocation 1
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f22
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Figure 11