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THE BALANCED STATES OF A PROPORTIONING NETWORK

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Abstract. Proportioning networks that need not be bipartite
arise as models of certain hypothetical marketing systems. This
work examines the balanced states of such networks. Necessary
and sufficient conditions for the existence of at least one
balanced state as well as for the existence of more than one
balanced state are given.

1. Bipartite proportioning networks model certain marketing
systems in underdeveloped economies [1]. They permit the
computation of time series in prices and quantities of a particular
commodity at the various markets by means of a system of nonlinear
difference equations that describe the flows of that commodity
along the branches of a bipartite network whose nodes represent
the markets. There does not seem to be any way of extending
the theory of [1] to marketing networks that are not bipartite.
Nevertheless, we can postulate a different kind of marketing
system involving many commodities which is modeled by a not
necessarily bipartite proportioning network. This is described
in the next section.

A balanced state in either of these two kinds of marketing
systems is one wherein the various time series in prices and
quantities maintain constant values and in addition the flow
in one direction along any branch always equals the flow in the
other direction. The objective of this work is to investigate
the balanced states of proportioning networks that need not be
bipartite. In Section 3 we extend Theorem 3 in [1] for the

existence of at least one balanced state to nonbipartite proportioning networks. In Section 4 we establish necessary and sufficient conditions under which there is more than one balanced state.

2. A hypothetical marketing system modeled by a possibly nonbipartite proportioning network. We now describe a multi-commodity marketing system that leads to a proportioning network that need not be bipartite. Although the assumptions employed herein are considerably less realistic than those used in [1], they do lead to at least one extreme case wherein nonbipartite proportioning networks arise.

Assume there exists a geographic region isolated from all outside trade. The region is partitioned into a finite number of localities, each of which produces a particular commodity both for domestic consumption within the locality and for trade with the other localities. The commodity produced in each region is different from those produced in all the other regions. Assume furthermore that there is a general shortage of all goods, as a result of which the governing body of the entire region has decreed rigid price controls by specifying the price of every commodity. For simplicity, we normalize the unit of quantity for each commodity in such a way that those units all have the same monetary value.

Markets meet periodically and on the same day in all localities. An authority in each locality decides on each market day how much of the goods produced in that locality is to be made available for domestic consumption and how much is to be traded for the goods of the other localities. Each trader

always operates between exactly two markets, trading in one market on one market day, proceeding to his other market on the next market day, returning to the first market on the third market day, and so forth. When two markets are so connected by a trader, we assume that there are at least two traders traveling between those two markets in opposite directions.

Since demand always exceeds supply everywhere, each trader sells all the goods he has brought with him to a particular market on a particular day and then acquires as many goods as the authority in that locality is willing to sell to him. He is willing to do so since, because of the price controls he makes a fixed monetary profit on each unit he transports. On the other hand, each local authority desires all the goods offered him by the traders and wishes to sell all of the supply of the local product, other than that set aside for domestic consumption, to the traders in order to maximize the amount of money received. Finally, in order to treat each trader equitably, the local authority allocates the local supply on any given market day in such a fashion that that day's relative prices of the traders' commodities measured with respect to the local commodity are all the same. This means that he allocates the local supply to the traders in proportion to the amounts of the goods provided by the traders. These assumptions allow the relative price of any good to vary from market day to market day even though its monetary price remains fixed. (We are also assuming that every agent always has enough money to buy whatever goods are offered to him at the fixed price.)

All this is represented by a proportioning network with
exogeneously given node values as illustrated in Figure 1.
Each local market is represented by a node. Two nodes,
between which at least one trader operates (and therefore at
least two traders operate in opposite directions), are connected
by two arcs oppositely directed and are said to be <u>adjacent</u>;
otherwise, there are no arcs between them. Throughout this
paper, every network will be assumed to be finite. The
integer values of the variable t represent the market days.
To the arc a_{ik} directed from node n_i to node n_k we assign the
value $s_{ik}(t)$, which denotes the amount of the commodity being
shipped from node n_i to node n_k between market day t and market
day t + 1. On the other hand, we denote by $s_i(t)$ the total
supply of the local product made available to all the traders
at the market n_i on market day t. Therefore, $s_i(t) = \sum_k s_{ik}(t)$.
A term $s_{pq}(t)$ in this summation is taken to be zero if n_p and
n_q are not adjacent. The proportioning principle under which
the local authorities allocate supplies leads to the following
equation.

$$(2.1) \qquad s_{ik}(t) = \frac{s_{ki}(t-1)}{\sum_k s_{ki}(t-1)} s_i(t)$$

Here, every quantity is a nonnegative number. Upon writing
(2.1) for each of the arcs a_{ik} in the proportioning network,
specifying the $s_i(t)$ as positive numbers for all i and all
t = 1, 2, 3, ... , and then assigning the initial values $s_{ki}(0)$
such that, for each i, $s_{ki}(0)$ is positive for at least one k,
we can solve the resulting system of equations recursively to
determine each time series $\{s_{ik}(t)\}_{t=1}^{\infty}$.

Although the single-branch structure of the proportioning networks of [1] looks quite different from the double-arc structure of Figure 1. the latter reduces to the former in the special case where the proportioning network is bipartite. Indeed, let M and N denote the node sets in the two parts of a bipartite proportioning network. Then, for every t, the $s_{ik}(t)$, where $n_i \in M$ and $n_k \in N$, depend only on the $s_{pq}(t - 1)$, where $n_p \in N$ and $n_q \in M$. Similarly, the $s_{ik}(t)$ for $n_i \in N$ and $n_k \in M$ depend only on the $s_{pq}(t - 1)$, where $n_p \in M$ and $n_q \in N$. Consequently, the system of equations (2.1) describing the behavior of the proportioning network can be decomposed into two systems, each of which does not depend on the other. Moreover, each one has precisely the form of the equations obtained in [1], where the signals $s_{ik}(t)$ flow from, say, M to N just after the odd values of t and from M to N just after the even values of t. Therefore, we no longer need to represent a transportation leg by two separate arcs and may use a single branch, as was done in [1].

3. Necessary and sufficient conditions for at least one balanced state. A balanced state for any proportioning network η_a of the form shown in Figure 1 is an infinite time series in the set of all arc and node values such that

(i) the value for any arc or node does not vary with time t (so that we may drop the argument notation in t),

(ii) $s_{ik} = s_{ki} \geq 0$ for every i and k, and

(iii) for each i, $s_{ki} > 0$ for at least one k.

When (i) holds, it follows from the governing equation (2.1) that the conjunction of (ii) and (iii) is equivalent to the

condition:

$$(3.1) \qquad s_i = \sum_k s_{ki} > 0$$

for every i. As before, we take the term s_{ki} in the summation of (3.1) to be zero if node n_k is not adjacent to node n_i. (3.1) will be called the <u>node condition at node</u> n_i. When it holds at all the nodes, we shall say that η_a <u>satisfies the node conditions</u>. #Since $s_{ik} = s_{ki}$ for every k and i, we may replace each pair of oppositely directed arcs by a single (undirected) branch. The resulting network has no branches in parallel and, as before, is finite. Throughout the rest of this paper, we shall always assume that these two conditions hold for every network, but we will not require in general that the network be either connected or bipartite. Any collection of components of a network η will be called a <u>disjoint subnetwork</u> of η. Let $\mathcal{K}$ be a disjoint subnetwork of η; $\eta - \mathcal{K}$ will denote the network obtained by deleting all the nodes (and therefore all the branches as well) of $\mathcal{K}$ from η.

Equivalent to our prior definition is the definition of a <u>balanced state</u> for η as a set of time-invariant positive node values and time-invariant nonnegative branch values such that the node condition (3.1) is satisfied for every i. Here, s_i denotes the value of node n_i and $s_{ik} = s_{ki}$ denotes the value of the branch connecting n_i to n_k. We shall somewhat relax this latter definition of a balanced state for η by merely requiring that all the s_i be nonnegative rather than positive.

A <u>gap set</u> G is a nonvoid set of nodes in η such that no two nodes in G are adjacent. (In particular, any single node of

$\mathcal{N}$ is a gap set.) The gap set G is said to be _in_ a subnetwork $\mathcal{M}$ of $\mathcal{N}$ if G is contained in the node set of $\mathcal{M}$. (In particular, $\mathcal{M}$ can be void in branches and therefore just a collection of nodes in $\mathcal{N}$.)

If N is a set of nodes in $\mathcal{N}$, the _adjacency_ A of N is the set of nodes such that each node in A is not in N but is adjacent to at least one node in N.

A _gap_ g is a value assigned to a gap set G and is defined as follows. Let H be the adjacency of G. Let w_G (and w_H) be the sum of the node values for all the nodes in G (respectively, H). Then, g is defined to be $w_H - w_G$. We shall say that g is the gap _for_ G. Given the subnetwork $\mathcal{M}$ of $\mathcal{N}$, g is said to be a gap _in_ $\mathcal{M}$ if it is the gap for some gap set in $\mathcal{M}$.

Theorem 1. Let there be given a network $\mathcal{N}$ with specified nonnegative node values. There exists a set of nonnegative branch values, which with the given node values comprise a balanced state for $\mathcal{N}$, if and only if all the gaps in $\mathcal{N}$ are nonnegative.

Proof. "Only if": Assume $\mathcal{N}$ has a balanced state, choose a gap set G in $\mathcal{N}$, and let H, w_H, w_G, and g be defined as above. Upon expanding w_H and w_G as sums of branch values by using (3.1), we see that the sum w_G appears as part of the sum w_H. Since all branch values are nonnegative, $g = w_H - w_G \geq 0$.

"If": Assume that all gaps in $\mathcal{N}$ are nonnegative. If $\mathcal{N}$ has any nodes whose values are zero, assign the value zero to all the branches incident to those nodes and then delete those nodes and branches. This does not alter the gap values for those gap sets not containing the deleted nodes. Thus,

all gaps in the resulting network n_1 are nonnegative.

Next, assume that a disjoint subnetwork m_1 of n_1 (m_1 might be all of n_1) is bipartite with respect to the gap set G (i.e., every branch of m_1 connects a node of G to a node not in G) and that the gap for G is zero. Since all other gaps in G are nonnegative, it follows from Theorem 3 of [1] that there exist values for the branches in m_1 which together with the node values comprise a balanced state for m_1.

Step A. Assign those values to the branches of m_1 and then delete m_1.

Let $n_2 = n_1 - m_1$. (When m_1 is void, $n_2 = n_1$.) Assume that n_2 has a gap set G whose gap is zero and let m_2 be the smallest disjoint subnetwork of n_2 that contains G. Assume that m_2 is not bipartite with respect to G. Let H be the adjacency of G, let I be the set of nodes in the adjacency of H that are not in G, and let K be the set of nodes that are not in $G \cup H \cup I$. (I and K may be void.)

Step B. Assign the value zero to every branch connecting two nodes of H and also to every branch connecting a node of H to a node of I. Then, delete all such branches.

This deletion does not alter the gaps in G, but it will decrease the gaps in I.

Let n_3 be the network that n_2 becomes after Step B is applied. Let m_3 be the disjoint subnetwork of n_3 having the node set $G \cup H$. m_3 is bipartite with respect to G. Moreover, the gap for G is zero and the other gaps in G are all nonnegative. So, again by Theorem 3 of [1], we can apply Step A to m_3.

Let $n_4 = n_3 - m_3$. (n_4 is the disjoint subnetwork of n_3

having the node set $I \cup K$.) We wish to show that all the gaps in $\mathcal{N}_4$ are nonnegative. Let J be any gap set in $\mathcal{N}_4$. Now, consider the prior network $\mathcal{N}_2$. In $\mathcal{N}_2$, $G \cup J$ is still a gap set since no two of its nodes are adjacent. By letting $g(X)$ denote the gap for the gap set X, we may write (for $\mathcal{N}_2$ still)

$$g(G \cup J) = g(G) + g(J) - w(M),$$

where M is the set of nodes in H each of which is adjacent to both a node in G and a node in J and $w(M)$ is the sum of the values of all the nodes in M. (When M is void, J is contained entirely in K and $w(M) = 0$.) Since $g(G) = 0$, $g(G \cup J) = g(J) - w(M)$. Also, we have again that $g(G \cup J)$ is nonnegative in $\mathcal{N}_2$. Since $g(J) - w(M)$ is the gap for J in $\mathcal{N}_4$, it follows that this gap is nonnegative, which is what we wished to show.

If possible, keep applying Step A or Step B or both to get a sequence $\mathcal{N}_2$, $\mathcal{N}_3$, $\mathcal{N}_4$, ... of reduced networks until we are left with either the void network or with a network $\mathcal{L}$ whose node values and gaps are all positive. In the former case, we will have obtained a balanced state for all of $\mathcal{N}$. In the latter case, we proceed as follows to reduce $\mathcal{L}$ to a network in which there is at least one node with a zero value or one gap equal to zero.

Step C. Choose any branch b in $\mathcal{L}$. Assign to it the value x, specified below, and decrease the values of b's nodes by x. (But, do not delete b from $\mathcal{L}$.)

Let $\mathcal{L}_1$ denote the network having the same graph as $\mathcal{L}$ but with the node values resulting after the application of Step C to $\mathcal{L}$. We shall say that a gap set G is <u>adjacent</u> to a node n if $n \notin G$ and n is adjacent to a node of G. During the transition

from $\mathcal{L}$ to $\mathcal{L}_1$, every gap whose gap set is adjacent to one node

of b and is neither adjacent to nor contains the other node

of b is decreased by x, and every gap whose gap set is adjacent

to both nodes of b is decreased by 2x. All other gaps remain

unchanged during that transition. Now, choose for x that unique

positive value which makes at least one node value or gap in

$\mathcal{L}_1$ equal to zero but does not render any node value or gap in

$\mathcal{L}_1$ negative. It follows that, if a balanced state can be found

for $\mathcal{L}_1$, then the addition of x to the found value for b and to the

two node values of b yields a balanced state for $\mathcal{L}$.

We now delete every node in $\mathcal{L}_1$ whose value is zero and

then apply Step A or Step B or both to $\mathcal{L}_1$ repeatedly until

either a void network is obtained or another network $\mathcal{K}$ having

only positive node values and positive gaps is found. In the

latter case, Step C is then applied to $\mathcal{K}$. Continuing this

procedure, we will eventually arrive at the void network since

the original network is finite and at each application of Step A

or Step B at least one branch is removed. The values assigned

to the branches of $\mathcal{N}$ as they are removed with the addition of

the values obtained while applying Step C constitute with the

originally given node values a balanced state for $\mathcal{N}$. This can

be seen by reversing the process by which the network $\mathcal{N}$ was

decomposed.

4. Necessary and sufficient conditions for more than one

balanced state. We will specify a tracing through a network

$\mathcal{N}$ along some nodes and branches by writing down the symbols

for those nodes and branches as they are met in the tracing.

For our purposes however it is important to distinguish between

any branch (or node) and its appearances in the sequence. To do so we shall refer to the former as a branch (respectively, node) of the sequence and to the latter appearances as branch terms (respectively, node terms) in the sequence. Thus, any branch or node may appear many times as branch terms or node terms in the sequence.

A walk is a (possibly infinite) alternating sequence of branch terms and node terms wherein the terms immediately adjacent to each branch term in the sequence are the node terms for the nodes of the corresponding branch. If the sequence is finite or semiinfinite, its terminal terms are node terms. The length of a walk is the number of branch terms in it, and the walk is called even (or odd) if its length is even (respectively, odd). A closed walk is a finite walk where the first and last node terms are the same.

We now turn to the idea of a unit weighting of a finite walk. Assign the value +1 or -1 to every branch term of a finite walk in such a fashion that the branch terms immediately adjacent to a branch term with the value +1 (or -1) have the value -1(respectively, +1). That is, the branch-term values +1 and -1 should alternate as one traces the walk. Then, assign to any branch that contributes branch terms to the walk the sum of all the values assigned to its branch terms in the sequence. Assign to every other branch the value zero. This assignment of an integer to every branch in η will be called a unit weighting of a finite walk, and the integer for any branch will be called the weight of the branch. Note that a finite walk has exactly two unit weightings, one of which can be obtained from the other by reversing the signs of all branch values.

A finite walk W will be said to be _degenerate_ _on_ _a_ _branch_ b if a unit weighting of W results in a weight of zero for b. W itself will be called _degenerate_ if it is degenerate on every branch. W will be called _nondegenerate_ if it is not degenerate on at least one branch. We shall say that W is _admissible_ if it is nondegenerate, closed, and even.

A _homogeneous_ _state_ for n is the assignment of real values to the branches of n such that for each node the sum of the values for the branches incident to that node equals zero. A homogeneous state is called _trivial_ if every branch value is zero and is called _nontrivial_ otherwise. A unit weighting of any closed even walk is an example of a homogeneous state.

Theorem 2. A network n with given nonnegative node values cannot have more than one balanced state if it does not contain an admissible walk.

Proof. Assume the proportioning network has two different balanced states B_1 and B_2 for the given node values. Subtract B_1 from B_2. That is, assign to each branch the value $u_2 - u_1$, where u_1 (respectively, u_2) is the branch's value with respect to B_1 (respectively, B_2). This yields a nontrivial homogeneous state $B_2 - B_1$.

Now, choose a branch b_1 having a nonzero value in $B_2 - B_1$. Start tracing a walk by proceeding from one node n_1 of b_1 along b_1 to the second node n_2 of b_1. Leave n_2 by proceeding along a branch b_2 whose value in $B_2 - B_1$ is not zero and of sign opposite to that of b_1; b_2 exists by virtue of the fact that the sum of the values in $B_2 - B_1$ of the branches incident to n_2 equals zero.

Continue this procedure indefinitely by leaving each node

along a branch whose nonzero value has a sign opposite to that
of the branch along which the node was approached. This generates
a semiinfinite walk W. Upon assigning +1 (or -1) to each branch
term in W if the corresponding branch has a positive (respectively,
negative) value, we get a proper unit weighting for any finite
subwalk F of W. Thus, every such F is nondegenerate on each
of its branches.

Since $\mathcal{N}$ is finite, there must be a branch term e that occurs
more than once in W. Indeed, there will be a finite subwalk
having e as its first and last branch terms and such that e
appears nowhere else in it. Trace that subwalk. Suppose that
both tracings of the branch corresponding to e are in the same
direction. Then, that subwalk with its last node and last branch
deleted is a closed walk W_1. It is also of even length since
the +1 and -1 branch terms alternate along W_1 and the values
of the first and last branch terms of W_1 are of opposite sign.
Thus, W_1 is an admissible walk.

If no branch term such as e exists in our one-ended walk W,
then there must be another branch term f having three consecutive
appearances in W such that a tracing of W passes through the
branch corresponding to f in the same direction for the first
and third appearances and in the opposite direction for the
second appearance. Let W_2 be the closed walk having the first
appearance of f and the branch term preceding the third appearance
of f as its first and last branch terms. Again, branch-term
values alternate in sign as W_2 is traced and the first and last
branches of W_2 have values of opposite sign. Hence, W_2 is even.
Thus, we have again found an admissible walk W_2. This proves the
theorem.

The network of Figure 2, wherein every node is assigned the value 1, shows that having an admissible walk is only a necessary but not a sufficient condition for a network to have more than one balanced state. It is easy to check that all gaps are nonnegative so that the network has at least one balanced state. But, in any balanced state, every end branch must have the value 1 and therefore every branch of the inner square must have the value 0. Thus, there is exactly one balanced state, even though the inner square comprises an admissible walk.

To obtain necessary and sufficient conditions for a network to have more than one balanced state we introduce still another definition. Let B be a balanced state in a network, and let W be a walk with a unit weighting. W is said to be admissible with respect to B if either (but not necessarily both) of the following conditions is satisfied:

(i) Every branch with a positive weight has a nonzero value in B.

(ii) Every branch with a negative weight has a nonzero value in B.

Theorem 3. A necessary condition for a network with given nonnegative node values to have more than one balanced state is that, for every balanced state B, there exists a walk W such that W is admissible with respect to B.

Proof. Let B_1 and B_2 be two different balanced states. Generate an admissible walk W as in the proof of Theorem 2. Assign a unit weighting to W such that +1 is assigned to the branch terms whose branches have positive values in $B_2 - B_1$. (This can be done because the branch values found in $B_2 - B_1$ as W is traced alternate in sign.) Then, every branch that

contributes terms to W and has a positive weight also has a positive value in $B_2 - B_1$. Since the branch values for B_1 are all nonnegative, W is admissible with respect to B_2. This completes the proof because B_2 can be any balanced state.

We define an _incremental state_ V of a network to be an assignment of values to the branches of the network in the following fashion: Choose an admissible walk with a unit weighting and choose some real number $\underline{a}$. The value of any branch b is $\underline{a}$ times the weight of b.

It follows that an incremental state is a homogeneous state.

Theorem 4. A sufficient condition for a network with given nonnegative node values to have more than one balanced state is that there exist a balanced state B and a walk W such that W is admissible with respect to B.

Proof. By multiplying the weights of a unit weighting of W by -1 if necessary, we can insure that the branches of W with positive weights have nonzero (and therefore positive) values in B. Let b denote any branch with a positive weight, let k_b be the weight of b, and let a_b be the value of b in B. Let $\varepsilon > 0$ be the minimum of the values $a_b / |k_b|$ for all such b. Finally, let V be the incremental state obtained by multiplying all the branch weights by $-\delta$, where $0 < \delta \leq \varepsilon$. Then, every branch in the state B + V is nonnegative. Moreover, B + V satisfies all the node conditions for the given node values of the network because V is a homogeneous state. Thus, B + V is another balanced state.

Upon combining Theorems 3 and 4, we get

Theorem 5. A necessary and sufficient condition for a network with given nonnegative node values to have more than one balanced state is that there exist a balanced state B and

a walk W such that W is admissible with respect to B.

Figure 3 shows an example of a network having a balanced state (the branch values are indicated with parentheses, the node values without) but for which the necessary condition of Theorem 3 is not satisfied. This can be seen by examining a unit weighting for the admissible walk of that network resulting from a tracing once around the hexagon.

Figure 4 shows a network for which the necessary and sufficient condition of Theorem 5 is satisfied. Figure 4(a) shows one balanced state, 4(b) an incremental state, and 4(c) the sum of those two states. In terms of the proof of Theorem 4, we have chosen an admissible walk that traces the entire network in a single "figure eight" and have chosen the unit weighting such that it is the two horizontal branches and the single vertical branch that have the postive weights. As a result, $\varepsilon = 2$ for Figure 4(a). We have then set $\delta = -1$ to get the incremental state of Figure 4(b).

Since the proofs of Theorems 1 and 4 are constructive, they provide a means for finding all the balanced states for a network with given nonnegative node values. The proof of Theorem 1 yields at least one balanced state, if any exist. Then, the proof of Theorem 4 yields many others; in fact, it allows all the other balanced states to be found because the difference between any two balanced states is a finite sum of incremental states, as we shall now show.

Theorem 6. Let B_1 and B_2 be any two balanced states of a network with given nonnegative node values. Then, $B_2 = B_1 + \sum_{i=1}^m V_i$, where each V_i is an incremental state and m is finite.

Note. This is equivalent to saying that every homogeneous

state is a finite sum of incremental states.

Proof. By proceeding as in the proof of Theorem 2, we can generate a walk W, all of whose branches have nonzero values in $B_2 - B_1$ which alternate in sign as W is traversed. Let b be any branch of W, let c_b be the value of b in $B_2 - B_1$, and let k_b be the weight of b under a unit weighting of W. All k_b are nonzero by virtue of the way W is generated. Let c be that value of $|c_b/k_b|$ which is smallest for all b in W. Thus, c > 0. Assign to b the value $\pm|k_b|c$ where the plus (minus) sign is chosen if b has a positive (respectively, negative) value in $B_2 - B_1$. Assign the value zero to all branches not in W. This yields an incremental state V_1 such that each branch has a value in $B_1 + V_1$ that does not lie outside the closed interval between its values in B_1 and B_2. Moreover, for each branch d for which $c = |c_d/k_d|$, the value of d in $B_1 + V_1$ equals its value in B_2. That is, $B_3 = B_1 + V_1$ is a balanced state that coincides with B_2 on all such branches d.

Repeat this procedure with B_1 replaced by B_3. This gives a new balanced state B_4, where at least one more branch has the same value in B_4 as it does in B_2. Moreover, the values of all the branches d are unchanged in going from B_3 to B_4. (Their values in $B_2 - B_3$ are zero so that they do not appear in the new walk.)

Continuing this procedure, we get a sequence of balanced states, where at each step at least one more branch has its value changed to the value it has in B_2 while all the branches whose values were previously converted to their values in B_2 are unaffected. Since there are only a finite number of branches, the procedure will terminate after m steps with the last balanced state being B_2.

Now, let V_i be the incremental state generated at the ith step. Then, $B_2 = B_1 + \sum_{i=1}^{m} V_i$.

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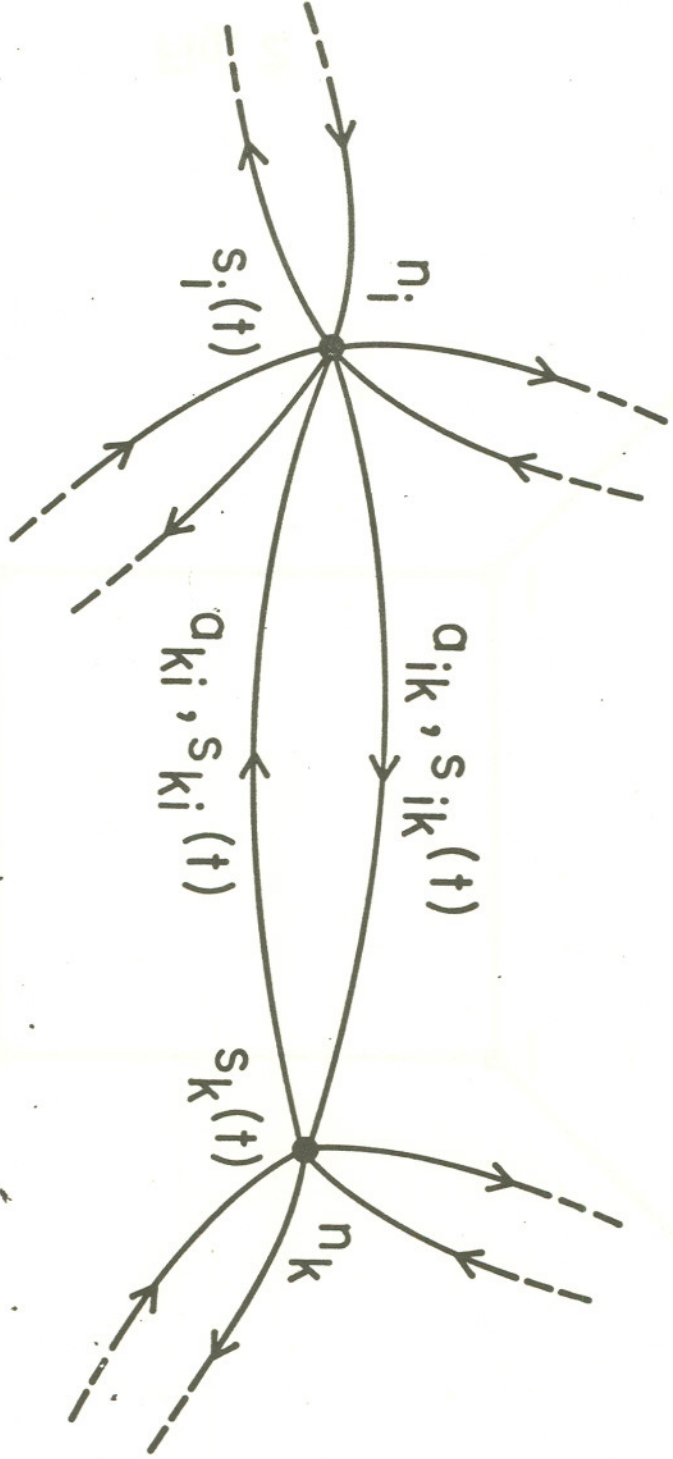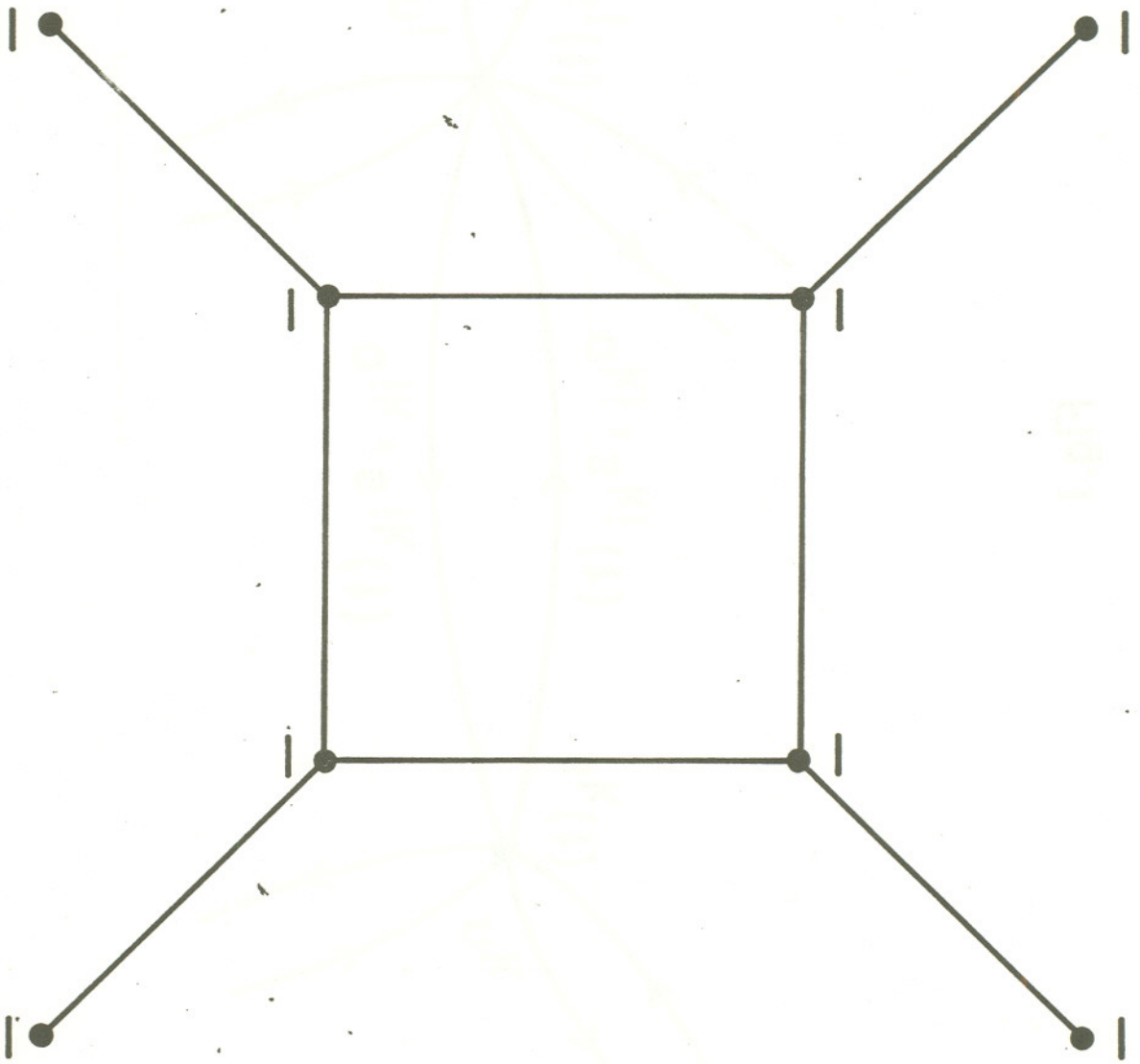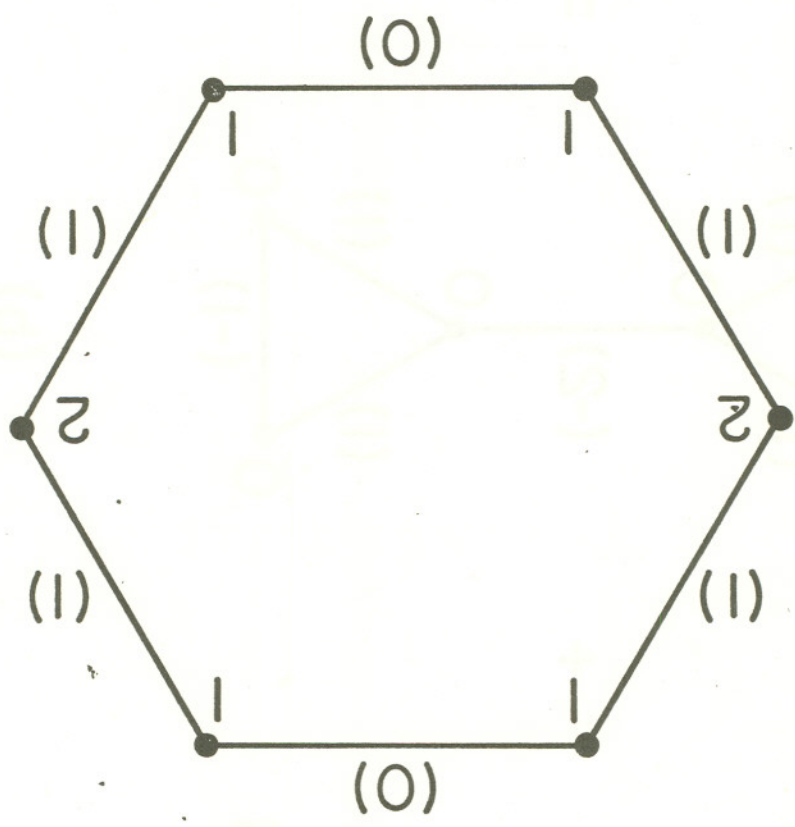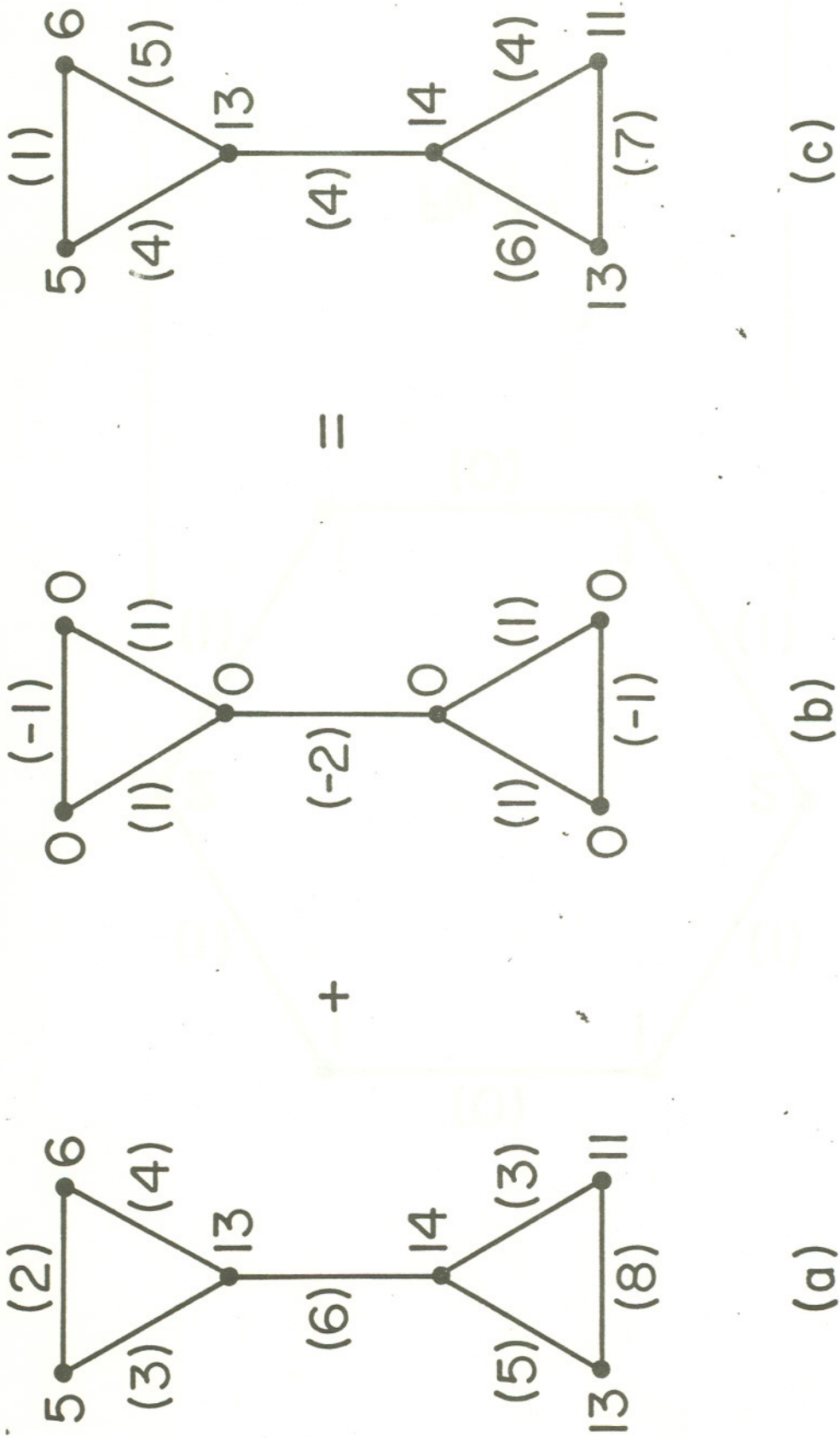
Fig. 1

Fig. 2

Fig. 4

(a) + (b) = (c)