CEAS #295

TWO-LEVEL PERIODIC MARKETING NETWORKS WITHOUT MARKET NEWS

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Abstract. A model is constructed for a two-level periodic marketing network wherein traders buy goods from farmers in a number of spatially separated markets and transport them to urban centers for sale to consumers. It is assumed that there is no market news dissemination. As a result, market disturbances are transmitted throughout the network at a rate limited by the trading activity. Moreover, distant markets may even be isolated from local disturbances. This model possesses one and only one equilibrium state. That equilibrium state is asymptotically stable if the slopes of the various supply and demand functions are sufficiently restricted in the vicinity of the equilibrium state.

1. Introduction

In certain African countries such as Nigeria, Kenya, and Sierra Leone there exist two-level marketing systems having essentially no market news [4]. Figure 1 is a schematic representation of such a system. The nodes m_j, where $j = 1, 2, 3, \ldots$, denote farmers' markets and the nodes n_k, where $k = 1, 2, 3, \ldots$, denote consumers' markets for a particular commodity. The farmers' markets open periodically, perhaps one day out of every four days. We shall assume that the market days for every m_j occur at the odd integer values of the time variable t: $t = \ldots, -3, -1, 1, 3, \ldots$. At each t farmers in the rural areas of some geographic region bring their supplies of the commodity to the m_j and sell them to traders. The traders in turn transport those supplies to the urban centers of the region and sell them to consumers between the times t and t + 2. As a notational convenience we shall assume that these sales to urban consumers occur at the even integer time values t + 1. Each urban center comprises one consumers' market n_k. We assume that pure competition occurs in every m_j and every n_k. In addition there is local demand in each m_j. Thus, in m_j traders are competing not only among themselves as buyers but also against domestic consumers.

Furthermore, there is essentially no market news for the dissemination of price and quantity conditions in the various markets. In fact, Jones [4, p. 116] reports the tendency of traders "to buy regularly in the same producing centers, and not to know what prices are elsewhere, or if they do, to buy there. Further than this, traders in most of the commodities included in our study confess to knowing very little about prices in nearby towns that are also consuming centers." We shall idealize this situation by assuming that each trader restricts his activities to one farmers' market and one consumers' market and is ignorant of the conditions of the markets in which he does not trade.

In this paper we develop a mathematical analysis of such a two-level marketing system. It is based upon some assumed behavior for the agents in the system as expressed by various supply and demand curves. These lead to a system of nonlinear difference equations from which time series in the prices and quantities exchanged can be recursively computed. We then show that the qualitative behavior described by Jones [4; p. 119] for the propagation of a disturbance throughout the marketing system is reproduced by our model. Furthermore, we prove that every marketing system satisfying our assumptions has one and only one equilibrium state for a given set of supply curves in the farmers' markets. That equilibrium state is asymptotically stable if the supply and demand curves satisfy certain sufficiently strong restrictions on their slopes.

We have previously given another analysis [8], [9] of this kind of marketing system, but the present analysis adopts a radically different approach. The prior work assumes that traders maintain conceptions of normal prices in the markets and adjust the amounts they buy in the farmers' markets in accordance with the deviations of prices from their normal values. Moreover, the resulting analysis has points of similarity with the classical cobweb phenomenon. (See. for example, [2], [3], [6], and [7].) The present work assumes that each trader has a normal capacity for shipment (e.g., the size of his truck) and adjusts the price he is willing to pay in m_j for that amount of goods.

Other differences between these works are as follows. The basic equations governing the dynamic behavior of the system are somewhat simpler in the former work, although their derivation is more complicated. Also, the assumptions imposed there seem to be less realistic than those of the present work. In fact, the most severe assumption of the former work is that the price elasticity of demand (of anticipated supply) in m_j (respectively, in n_k) for the aggregate of all traders operating between m_j and n_k is independent of k (respectively, j). One result of this is that all the traders in a consumers' market might cut off their activities if price deviations become too large. But, if they do, they do so simultaneously and can never recover activity once it has stopped. Consequently, the former work is restricted to moderate price deviations around normal values. No such assumption of common elasticities is imposed in this work, and, although trading can cutoff, it will restart once price conditions return to more normal values. Still another fundamental difference is that the former work requires the supply curves of the farmers in the m_j to be perfectly inelastic, whereas the present work takes those curves to be sloping. On the other hand, it should be pointed out that the former work, having more structure, leads to certain interesting mathematical questions that do not arise in the present work. So, although the present work is perhaps more realistic and allows stronger conclusions to be drawn, the former work remains of interest from the mathematical point of view.

2. The Model

We assume throughout that the two-level marketing network N, illustrated in Figure 1, has a finite connected simple graph, which is bipartite with respect to the partition of its nodes into the set of farmers' markets and the set of consumers' markets. There exists a branch joining a farmers' market and a consumers' market if and only if there are traders available for transporting goods between those two markets. In each farmers' market m_j at time t, the farmers supply the quantity $S_j(p, t)$ if the price is p, as shown in Figure 2. Thus, the supply curves are allowed to shift from one market day to another. On the other hand, we take domestic demand from local consumers at m_j to be determined by the fixed demand curve $L_j(p)$ shown in Figure 2. (Essentially the same analysis would hold if we allowed L_j to shift from one market day to another.)

To model the aggregate behavior of all the traders between two given markets, say, m_j and n_k, we first postulate the behavior of the ith individual trader between those markets as follows. That trader restricts his activities to the markets m_j and n_k and may or may not transport goods between them depending on the prices. In particular, at time t he views the the last received price $P_k(t-1)$ in n_k as the most likely value of $P_k(t+1)$. Moreover, we assume that his cost T^i_{jk} of transporting one unit of the commodity from m_j to n_k and selling it in n_k remains fixed with respect to time (but we allow T^i_{jk} to vary as i varies). Thus, at time t that trader's anticipated profit in selling one unit in N_k at time t+1 is

$$P_k(t-1) - T^i_{jk} - R_j(t), \tag{1}$$

where $R_j(t)$ is the price in m_j at t. Finally, we assume that each trader possesses a limited capacity for the transportation of goods. He refuses to acquire any goods in m_j at t if (1) is negative, but, as (1) increases into positive values, the amount he buys rapidly increases until his capacity is filled. In short, we assume that the demand curve of the ith individual trader looks like the curve of Figure 3.

Since there can be many traders between m_j and n_k each having perhaps a different T^i_{jk}, the aggregate demand curve in m_j for all those traders will look like that of Figure 4, where $T_{jk} = \min_i T^i_{jk}$ is the largest price difference $P_k(t-1) - R_j(t)$ below which all traders between m_j and n_k will refuse to acquire goods in m_j. In accordance with the preceding paragraph, we shall assume that the aggregate demand curve in m_j at t of all the traders between m_j and n_k is given by

$$V_{jk}[P_k(t-1) - T_{jk} - p] \tag{2}$$

where $V_{jk}(x)$ is a continuous function on $-\infty < x < \infty$ that equals zero for $x \leq 0$, is strictly increasing for $x > 0$, and tends to a finite limit $V_{jk}(\infty)$ as $x \to \infty$. The stipulation that the monotonicity of V_{jk} be strict on the positive real line is imposed to avoid certain technical problems of a mathematical nature; it can be supported by noting that traders will tend to stretch their transportation capacities (perhaps by overloading their trucks) as anticipated profits become more and more lucrative.

Given any node v of a graph, we define the adjacency of v as the set of all nodes that are adjacent to v.

The aggregate demand curve $D_j(p, t)$ in m_j at t is the sum of the demand curves for the local consumers and the various traders at m_j. Thus,

$$D_j(p, t) = L_j(p) + \sum_{k \in K_j} V_{jk}[P_k(t-1) - T_{jk} - p] \tag{3}$$

where K_j is the index set for the adjacency of m_j. The price $R_j(t)$ at which the market m_j clears at t is the value of p for which

$$D_j(p, t) = S_j(p, t). \tag{4}$$

Under the assumptions stated below there will be one and only one such value $R_j(t)$. Consequently, the amount acquired by the local consumers is $L_j[R_j(t)]$, and the amount transported by the traders from m_j to n_k for sale at t+1 is

$$S_{jk}(t) = V_{jk}[P_k(t-1) - T_{jk} - R_j(t)]. \tag{5}$$

We next assume that traders do not store goods; they sell the goods they have on hand at time t+1 for whatever price they can get. Thus, the supply curve in n_k at time t+1 is completely inelastic; that is, it is a vertical line intersecting the quantity axis at the value

$$C_k(t) = \sum_{j \in J_k} S_{jk}(t) \tag{6}$$

where J_k is the index set of the adjacency of n_k. On the other hand, we take the demand curve $H_k(p)$ forthe consumers in n_k to be fixed with time and of the shape shown in Figure 5. Thus, the market n_k will clear at a unique price $P_k(t+1)$ if the inverse G_k of the demand curve H_k is a well-defined function on $0 \leq q < \infty$. We shall relax this condition a bit by allowing $G_k(0)$ to be either finite or infinite. Thus, we may write

$$P_k(t+1) = G_k[C_k(t)]. \tag{7}$$

In the event that $C_k(t-2) = 0$ and $P_k(t-1) = \infty$, the value used for (5) in (3) and (6) (as well as in (8) and (9) below) is the finite number $V_{jk}(\infty)$.

Upon combining equations (3), (4), (5), (6), and (7), we obtain the two basic equations for our model: For every j,

$$S_j[R_j(t), t] = L_j[R_j(t)] + \sum_{k \in K_j} V_{jk}[P_k(t-1) - T_{jk} - R_j(t)] \tag{8}$$

and, for every k,

$$P_k(t+1) = G_k\left\{\sum_{j \in J_k} V_{jk}[P_k(t-1) - T_{jk} - R_j(t)]\right\} \tag{9}$$

We now list all the assumptions that we shall impose on the various supply and demand functions of the marketing network N throughout this paper.

Conditions I:

I_1. $L_j(p)$ is a continuous nonnegative function on $0 < p < \infty$ and strictly decreasing on $0 < p < R_j^* \leq \infty$. If $R_j^* < \infty$, $L_j(p) = 0$ on $R_j^* \leq p < \infty$. If $R_j^* = \infty$, $L_j(p) \to 0$ as $p \to \infty$.

I_2. For each fixed t, $S_j(p, t)$ is a continuous nonnegative function on $0 \leq p < \infty$, equal to zero on $0 \leq p \leq R_j^s(t)$, and strictly increasing on $R_j^s(t) < p < \infty$, where $R_j^s(t) < R_j^*$.

I_3. $V_{jk}(x)$ is a continuous nonnegative function on $-\infty < x < \infty$ such that $V_{jk}(x) = 0$ for $x \leq 0$, $V_{jk}(x)$ is strictly increasing on $0 < x < \infty$, and $V_{jk}(x)$ tends to a finite limit $V_{jk}(\infty)$ as $x \rightarrow \infty$. Moreover, it satisfies the Lipschitz condition

$$|V_{jk}(x) - V_{jk}(y)| \leq M|x - y|$$

everywhere. (Since the marketing network N is finite, we can take the constant M to be independent of j and k.)

I_4. $G_k(q)$ is a continuous positive strictly decreasing function on $0 < q < \infty$. As $q \rightarrow \infty$, $G_k(q) \rightarrow 0$. As $q \rightarrow 0+$, $G_k(q)$ tends to P_k^*, where P_k^* can be either a finite number or ∞. We set $G_k(0) = P_k^*$.

These conditions and a specification of the initial prices $P_k(0)$ for all k such that $0 \leq P_k(0) \leq P_k^*$ allow us to compute recursively the price series $\{R_j(t): t = 1, 3, 5, \ldots\}$ and $\{P_k(t+1): t = 1, 3, 5, \ldots\}$ for all j and k. Indeed, since $R_j^s < R_j^*$, the continuities and strict monotonicities of $S_j(p, 1)$ and $D_j(p, 1)$ dictate that for each j there is a unique positive value $R_j(1)$ for which (8) is satisfied. The corresponding values for the $S_{jk}(1)$ are all nonnegative and finite, and therefore (9) yields the nonnegative values $P_k(2)$ for all k. The computations proceed in this fashion.

It will be understood throughout this work that the initial value of $P_k(t+1)$ will be restricted according to $P_k(t+1) \leq P_k^*$. All subsequent values of $P_k(t+1)$ satisfy this restriction. Note that, when $P_k^* = \infty$, the time series in the $P_k(t+1)$ may have ∞ as some of its terms. Indeed, when $C_k(t) = 0$, we have $P_k(t+1) = P_k^*$. In this case we set $S_{jk}(t+2)$ equal to the limiting value $V_{jk}(\infty)$. But, since the latter is finite, the computations continue without difficulty.

Also, note that

$$C_k(t) \le C^* = \sum_{j \in J_k} V_{jk}(\infty) < \infty. \tag{10}$$

Hence,

$$P_k(t+1) \ge G_k(C^*) > 0 \tag{11}$$

for $t \geq 1$.

3. The Propagation of Disturbances

W.O.Jones [4; p. 119] has pointed out that a two-level marketing network without market news responds only sluggishly to disturbances in supplies at the farmers' markets. In particular, suppose a shortage of supplies occurs at some farmers' market m_j. The resulting price disturbances will be propagated through the marketing system only by means of the trading activities. As a result, a rise in price due to the supply shortage can appear at a distant market no sooner than the minimum time it takes for a signal to be transmitted in a step-by-step fashion along the shortest path connecting m_j to that distant market. We shall show in this section that our model possesses this qualitative behavior and moreover that the time lag before the price rise occurs in the distant market may actually be longer than the aforementioned minimum transmittal time.

The <u>length</u> of a path joining two nodes of N is the number of edges in the path. The <u>distance</u> between those two nodes is the smallest length among all the paths joining those nodes. It follows from the governing equations (8) and (9) that it takes at least d units of time for a disturbance to travel a distance d. We will assume that a disturbance occurs at the farmers' market m_1. For $t = 3, 5, 7, \ldots$, let Y_t (or Z_{t-1}) denote the set of indices of those m_j (respectively, n_k) that are at a distance t-1 (respectively, t-2) from m_1. Y_1 will denote the set consisting only of the index 1. Because N is bipartite, no two nodes in Y_t are adjacent, and the same is true for Z_{t-1}.

Let y denote either a farmers' market or a consumers' market. Choose a shortest path between m_1 and y and let $\{b_s\}_{s=1}^{n}$ be the sequence of branches met as one traces the path from m_1 to y. In the following we will be considering two different dynamic processes in N and will denote the variables of each dynamic process by the superscript i, where i = 1, 2. If s is odd (even), we let $F_s^i(s)$ (respectively, $F_s^i(s+1)$) denote the amount of goods flowing along b_s that were acquired in a farmers' market at time s (respectively, s+1) for sale in a consumers' market at time s+1 (respectively, s+2). The resulting sequence

$$\{F_1^i(1), F_2^i(3), F_3^i(3), F_4^i(5), F_5^i(5), \ldots\} \tag{12}$$

will be called an m_1 _to_ y _passage_ for the ith process. We shall say that a _coincident cutoff_ occurs for the two passages if there exists a term in the sequence that is zero for both values of i.

Theorem 1. Let the superscript 1 and 2 denote two different dynamic processes in N. Assume that

(α) $P_k^1(0) = P_k^2(0) \leq P_k^*$ for all k,

(β) $S_1^1(p, 1) < S_1^2(p, 1)$ for all p such that $S_1^2(p, 1) > 0$, and

(γ) $S_j^1(p, t) = S_j^2(p, t)$ for all $p > 0$, for all $j \neq 1$, and for all $t = 1, 3, 5, \ldots$.

Then, for every odd positive integer t, we have the following conclusions:

(a) For $j \in Y_t$, $R_j^1(t) \geq R_j^2(t)$. Strict inequality holds if and only if there exists at least one m_1 to m_j passage having no coincident cutoff. (Thus, for $j = 1$, strict inequality holds.)

(b) For $j \in Y_{t+2} \cup Y_{t+4} \cup \cdots$, $R_j^1(t) = R_j^2(t)$.

(c) For $k \in Z_{t+1}$, $P_k^1(t+1) \geq P_k^2(t+1)$. Strict inequality holds if and only if there exists at least one m_1 to n_k passage having no coincident cutoff.

(d) For $k \in Z_{t+3} \cup Z_{t+5} \cup \cdots$, $P_k^1(t+1) = P_k^2(t+1)$.

Note: We interpret hypotheses (α), (β), and (γ) to mean that the conditions at all the markets other than m_1 are the same for both processes and that a shortfall in supply has occurred in m_1 at $t = 1$ for the first process but not for the second.)

Proof. It follows from (α) that $D_1^1(p, 1) = D_1^2(p, 1)$. So, by virtue of ($\beta$), the strict monotonicities of the D_1^i and S_1^i functions, and the condition that $R_j^s < R_j^*$ for both processes, we have $R_1^1(1) > R_1^2(1)$. Now, let $k \in Z_2$. By the strict monotonicity of V_{1k}, we also have $S_{1k}^1(1) < S_{1k}^2(1)$ except when $R_1^2(1) > P_k^2(0) - T_{jk}$. In the latter case, $S_{1k}^1(1) = S_{1k}^2(1) = 0$; thus, coincident cutoff occurs on the m_1 to n_k passage, that is, at time $t = 1$ on the branch connecting m_1 to n_k. Moreover, it is clear that $R_j^1(1) = R_j^2(1)$ for $j \in Y_3 \cup Y_5 \cup \cdots$.

Since conditions are the same at all the other farmers' markets at $t = 1$, we have $S_{jk}^1(1) = S_{jk}^2(1)$ for $j \in Y_3$ and $k \in Z_2$. But, $C_k^i(1) = \sum_{j \in J_k} S_{jk}^i(1)$, where $i = 1, 2$. Consequently, for $k \in Z_2$, $C_k^1(1) \leq C_k^2(1)$ and therefore, by the strict monotonicity of G_k, $P_k^1(2) \geq P_k^2(2)$. We have strict inequalities here if and only if coincident cutoff does not occur on the m_1 to n_k passage. On the other hand, for $k \in Z_4 \cup Z_6 \cup \cdots$, we have $C_k^1(1) = C_k^2(1)$

and therefore $P_k^1(2) = P_k^2(2)$.

We now construct an inductive proof. Assume that, for some $t \geq 3$, (c) and (d) both hold with t replaced by t-2. That is:

(i) For $k \in Z_{t-1}$, $P_k^1(t-1) \geq P_k^2(t-1)$. The inequalities are strict if and only if there exists at least one m_1 to n_k passage having no coincident cutoff.

(ii) For $k \in Z_{t+1} \cup Z_{t+3} \cup \cdots$, $P_k^1(t-1) = P_k^2(t-1)$.

We have shown above that these conditions hold for $t = 3$. Now, upon combining (i) and (ii), we have that for $j \in Y_t$, where $t \geq 3$, $D_j^1(p, t) \geq D_j^2(p, t)$ with strict inequality holding for at least some p under the condition stated in (i). Since $S_j^1(p, t) = S_j^2(p, t)$ by (γ), it follows from the monotonicities of the D_j^i and S_j^i that $R_j^1(t) \geq R_j^2(t)$ where $j \in Y_t$. Strict inequality will occur if and only if the supply curve $S_j^1(p,t) = S_j^2(p, t)$ intersects the $D_j^1(p, t)$ curve at a point different from that for $D_j^2(p, t)$ curve. But, this occurs if and only if there is at least one $k \in Z_{t-1}$ for which the condition of (i) for strict inequality holds and in addition $S_{jk}^1(t) > 0$. In other words, $R_j^1(t) > R_j^2(t)$ if and only if there is some m_1 to m_j passage having no coincident cutoff. Thus, all of assertion (a) has been obtained.

Assertion (b) follows directly from (ii) and (γ).

Next, for $j \in Y_t$ and $k \in Z_{t+1}$, where $t \geq 3$, we have from (ii) that

$$V_{jk}\left[P_k^1(t-1) - T_{jk} - p\right] = V_{jk}\left[P_k^2(t-1) - T_{jk} - p\right]$$

for all p. Hence, if $R_j^1(t) > R_j^2(t)$ and if $R_j^2(t) < P_k^2(t-1) - T_{jk}$, we have from the strict monotonicity of the $V_{jk}(x)$ for $x > 0$ that $S^1_{jk}(t) < S^2_{jk}(t)$. On the other hand, $S^1_{jk}(t) = S^2_{jk}(t)$ if either $R^1_j(t) = R^2_j(t)$ or $R^2_j(t) \geq P^2_k(t-1) - T_{jk}$. The former case holds if and only if there is coincident cutoff in every m_1 to m_j passage, and the latter case holds if and only if $S^1_{jk}(t) = S^2_{jk}(t) = 0$ (i.e., coincident cutoff occurs in the m_j to n_k branch). Thus, we have shown that $S^1_{jk}(t) \leq S^2_{jk}(t)$ and that strict inequality holds if and only if there is at least one m_1 to n_k passage having no coincident cutoff. On the other hand, for $j \in Y_{t+2} \cup Y_{t+4} \cup \cdots$ and $k \in Z_{t+1} \cup Z_{t+3} \cup \cdots$, we clearly have from (ii) and (γ) that $S^1_{jk}(t) = S^2_{jk}(t)$.

By virtue of the strict monotonicities of the G_k, these results on the S_{jk}'s immediately imply (c) and (d), which completes our inductive proof.

4. The Existence of an Equilibrium State

We now wish to determine whether it is possible for the various prices and quantities in our marketing network to be fixed with respect to time. We say that N is in <u>equilibrium</u> if $S_j(p, t)$, $R_j(t)$, and $P_k(t)$ are independent of t for every j and k. By an <u>equilibrium state</u> for N we mean the corresponding set of all R_j and P_k. The governing equations for an equilibrium are obtained by deleting the time variable from the equations (8) and (9). That is, for every j

$$S_j(R_j) = L_j(R_j) + \sum_{k \in K_j} V_{jk}(P_k - T_{jk} - R_j), \qquad (13)$$

and for every k

$$P_k = G_k\Big[\sum_{j\in J_k} V_{jk}(P_k - T_{jk} - R_j)\Big]. \tag{14}$$

Note that, under Conditions I and for any assumed values of the P_k such that $0 \le P_k \le P_k^*$, (13) has a unique positive solution R_j for every j. Therefore, the question at hand is whether (14) has a nonnegative solution P_k when the R_j are determined by (13).

In the following we shall use the demand functions $H_k(p)$ in the n_k. In accordance with Condition I_4, each H_k is a continuous positive strictly decreasing function on $0 < p < P_k^*$. It tends to ∞ as $p \to 0+$. If P_k^* is finite, $H_k(P_k^*) = 0$. If $P_k^* = \infty$, $H_k(p) \to 0$ as $p \to \infty$. In the latter case, we also write $H_k(P_k^*) = H_k(\infty) = 0$.

In this and the next two sections we assume that each m_j has a fixed supply function S_j.

<u>Theorem 2</u>. The marketing network N has at least one equilibrium state.

<u>Proof</u>. For each k we will construct a finite or infinite nonincreasing sequence of values for each of the P_k which will either end at or tend toward a value for that P_k corresponding to an equilibrium state for N.

We start by setting all $P_k = P_k^*$, which we also denote by $P_{k,0}$. Let $R_{j,0}$ be the corresponding unique solution of (13) for each j. Since $H_k(P_{k,0}) = 0$ whereas

$$\sum_{j\in J_k} V_{jk}(P_{k,0} - T_{jk} - R_{j,0}) \ge 0,$$

we have

$$\sum_{j \in J_k} V_{jk}(P_{k,s} - T_{jk} - R_{j,s}) \geq H_k(P_{k,s}) \qquad (15)$$

for all k when s = 0. If equality holds in (15) for all k, then we have found an equilibrium state, namely, the set of all $R_{j,0}$ and $P_{k,0}$. (In this case, all $S_{jk} = 0$.)

If not, then there is strict inequality in (15) for at least one k. Let h be the number of consumers' markets. Choose the first integer k in the ordered set (1, 2, ... , h), say k_1 for which there is strict inequality in (15). As P_{k_1} decreases from the value $P_{k_1,0} = P^*_{k_1}$ and all other $P_{k,0}$ ($k \neq k_1$) are held fixed, the curve $V_{jk_1}(P_{k_1} - T_{jk_1} - p)$ shifts (downward in Figure 2) for each j whereas all the other $V_{jk}(P_k - T_{jk} - p)$ remain fixed. Therefore, the value $D_j(p)$ decreases or stays fixed at each value of $p \geq 0$. By the strict monotonicity of the S_j, the values of R_j and Q_j both decrease so long as $D_j(p)$ actually decreases for every p in a neighborhood of the initial R_j. Because of the continuities and strict monotonicities of the V_{jk} for positive values of their arguments, $D_j(p)$ decreases in that neighborhood for every $j \in J_{k_1}$ for which the corresponding term in the right-hand side of (15) is greater than zero. But, for $k \neq k_1$, $V_{jk}(P_k - T_{jk} - R_j)$ will increase in value or stay fixed at zero as R_j decreases, and the same is true for $L_j(R_j)$. Since Q_j equals the right-hand side of (13) and actually decreases, we can conclude that $V_{jk_1}(P_{k_1} - T_{jk_1} - R_j)$ decreases as P_{k_1} decreases and R_j takes on the values for which equality holds in (13). This process will continue until $V_{jk_1}(P_{k_1} - T_{jk_1} - R_j)$ reaches the value zero, which occurs before P_{k_1} becomes negative.

On the other hand, $H_{k_1}(P_{k_1})$ increases as P_{k_1} decreases from $P_{k_1,0}$. It follows that there will be some positive value for P_{k_1} for which equality is acheived in (15) (with s deleted) for $k = k_1$. Furthermore, for $k \neq k_1$ the inequality (15) will continue to hold and possibly get larger when $P_{k,s} = P_{k,0}$ and the $R_{j,s}$ are replaced by the values corresponding to equality in (13). This is because, for $k \neq k_1$, $V_{jk}(P_k - T_{jk} - R_j)$ increases in value or stays fixed as R_j decreases and P_k remains fixed. Let $P_{k,1} = P_{k,0}$ for $k \neq k_1$, but let $P_{k_1,1}$ be the found value of P_{k_1} for which equality is acheived in (15). Also, let $R_{j,1}$ denote the corresponding values that satisfy (13).

Next, choose the first value, say, k_2 in the ordered set $(k_1+1, k_2+2, \ldots, h, 1, 2, \ldots, k_1-1)$ for which strict inequality holds in (15) with $s = 1$. (If no such k_2 exists, we will have found an equilibrium state. So, assume a k_2 does exist.) Repeat the argument of the preceding two paragraphs, decreasing now P_{k_2} instead of P_{k_1} and holding all other P_k fixed. We will obtain a new set of values for the R_j and P_k, which we will denote with the subscript $s = 2$, such that (13) is satisfied for all j and, for $s = 2$, (15) holds for all k with equality occurring for $k = k_2$.

Once again, choose the first value, say, k_3 in the ordered set $(k_2+1, k_2+2, \ldots, h, 1, 2, \ldots, k_2-1)$ for which strict inequality holds in (15) with $s = 2$, and repeat the above procedure. Continue this process as long as possible. Two possibilities arise:

(i) The process terminates after a finite number of steps because equality is acheived in (15) for all k.

(ii) The process continues indefinitely.

Under (i), an equilibrium state will have been found. We will now show that, under (ii), the sequence $\{P_{k,s}\}_{s=1}^{\infty}$ converges for every k to a value $P_{k,\infty}$ such that equality holds in (14) when $P_k = P_{k,\infty}$ and the R_j are determined by (13).

First note that the above process generates a nonincreasing sequence $\{P_{k,s}\}_{s=1}^{\infty}$ for each k. Since $P_{k,s} \geq G_k(C^*) > 0$ by virtue of (11), the sequence converges to a positive limit $P_{k,\infty}$. Thus, $\{H_k(P_{k,s})\}_{s=1}^{\infty}$ is a nondecreasing convergent sequence tending to the finite limit $H_k(P_{k,\infty})$.

Let $\{H_{k,s}\}_{s=1}^{\infty}$ and $\{C_{k,s}\}_{s=1}^{\infty}$ be the sequences of values taken by $H_k(P_k)$ and $C_k = \sum_{j \in J_k} V_{jk}(P_k - T_{jk} - R_j)$ respectively at each step of the above process. We have already noted that, as $s \to \infty$, $H_{k,s} \to H_k(P_{k,\infty})$, the $H_{k,s}$ are nondecreasing, and $H_{k,s} \leq C_{k,s}$. Moreover, for each k there are subsequences such that, for n = 1, 2, ... , $H_{k,s_n} = C_{k,s_n}$: also, for $n \geq 2$ and $s_{n-1} \leq s \leq s_n - 1$, $H_{k,s} = H_{k,s_{n-1}}$ whereas $C_{k,s}$ is nondecreasing. (See Figure 6.) Here, the s_n depend on the choice of k. We wish to show that, for each k, $C_{k,s} \to H_k(P_{k,\infty})$. By virtue of the continuity of the functions H_k, V_{jk}, L_j, and S_j, this will prove that, for $P_k = P_{k,\infty}$, equality holds in (14) when the R_j assume the unique values for which equality holds in (13).

In the following we will be choosing subsequences of subsequences. Out of consideration for the typesetter, we introduce the following notation: $s{:}n = s_n$. Thus, $\{X_{s:n:\nu:\mu}\}_{\mu=1}^{\infty}$ denotes a subsequence indexed by μ of a subsequence indexed by ν of a subsequence indexed by n of a sequence indexed by s.

Suppose $C_{k,s}$ does not tend to $H_k(P_{k,\infty})$ as $s \to \infty$. Then, there exists an $\varepsilon > 0$ and an infinite subsequence $\{C_{k,(s:n:\nu)-1}\}_{\nu=1}^{\infty}$ such that

$$C_{k,(s:n:\nu)-1} - H_k(P_{k,\infty}) > \varepsilon.$$

Since $C_{k,s:n:\nu} = H_{k,s:n:\nu} \leq H_k(P_{k,\infty})$, we have

$$C_{k,(s:n:\nu)-1} - C_{k,s:n:\nu} > \varepsilon.$$

Let $|J_k|$ denote the (finite) number of farmers' markets adjacent to the consumers' market n_k. Let $\{V_{j,k,s}\}_{s=1}^{\infty}$ be the sequence of values assumed by the $V_{jk}(P_k - T_{jk} - R_j)$ in the aforementioned process. Since the sum over J_k of the $V_{j,k,s}$ equals $C_{k,s}$, there exists a subsequence $\{V_{j,k,s:n:\nu:\mu}\}_{\mu=1}^{\infty}$ such that

$$V_{j,k,(s:n:\nu:\mu)-1} - V_{j,k,s:n:\nu:\mu} > \frac{\varepsilon}{|J_k|}.$$

Consider now the sequence of increments $\Delta P_{k,\mu}$ by which the value of P_k drops for the step in the aforementioned process corresponding to the increase in our index s from $(s:n:\nu:\mu) - 1$ to $s:n:\nu:\mu$. By virtue of the Lipschitz condition satisfied by the V_{jk}, we must have that

$$\Delta P_{k,\mu} \geq \frac{\varepsilon}{|J_k| M}.$$

It follows that $P_{k,s}$ must tend to $-\infty$, in contradiction to the fact that $P_{k,s} > 0$. Hence, our supposition at the beginning of this paragraph is false. This completes the proof.

5. The Uniqueness of the Equilibrium State

We now show that N has at most one equilibrium state.

Theorem 3. N has one and only one equilibrium state.

Proof. Suppose there exist two different equilibrium states (i.e., in at least one consumers' market the price under one state is different from the price under the other state.) We shall denote the prices and quantities for one state with the superscript 1 and those for the other state by the superscript 2. We can choose that notation and a value k_0 for k such that $P^1_{k_0} > P^2_{k_0}$ and

$$P^1_{k_0} - P^2_{k_0} \geq P^1_k - P^2_k$$

for every k. Starting with the first equilibrium state, we shall alter the prices in the consumers' markets in two steps:

Step 1: Let $\Delta P_{k_0} = P^1_{k_0} - P^2_{k_0}$. Thus, $\Delta P_{k_0} > 0$. Decrease every P^1_k by ΔP_{k_0} to get the value $P^a_k = P^1_k - \Delta P_{k_0}$. (These P^a_k may not correspond to an equilibrium state.) Set

$$C^a_k = \sum_{j \in J_k} V_{jk}(P^a_k - T_{jk} - R^a_j)$$

where the R^a_j are the values determined by (13) when $P_k = P^a_k$. Since the L_j and the V_{jk} are continuous and strictly decreasing and the S_j are continuous and strictly increasing whenever their range values are positive, it follows that

$$V_{jk}(P^a_k - T_{jk} - R^a_j) \leq V_{jk}(P^1_k - T_{jk} - R^1_j)$$

for every j and k. Therefore, $C^a_k \leq C^1_k$.

Step 2: Keep $P^a_{k_0} = P^2_{k_0}$ fixed, but for every $k \neq k_0$ change the value of P_k from P^a_k to P^2_k. Thus, $P^2_k - P^a_k \geq 0$. Consequently, at this step R_j either stays fixed or increases in value so that $V_{jk_0}(P^2_{k_0} - T_{jk_0} - R_j)$ either stays fixed or decreases in value for every j. As a result, $C^2_{k_0} \leq C^a_{k_0}$.

Upon combining the results of these two steps, we get $C^1_{k_0} \geq C^2_{k_0}$. But, since $H_{k_0}(p)$ is strictly decreasing on $0 < p \leq P^*_{k_0}$ and $0 < P^2_{k_0} < P^1_{k_0} \leq P^*_{k_0}$, we have

$$H_{k_0}(P^2_{k_0}) > H_{k_0}(P^1_{k_0}) = C^1_{k_0} \geq C^2_{k_0} .$$

Thus, the second state cannot be an equilibrium state, in contradiction to our assumption.

6. The Asymptotic Stability of the Equilibrium State

We continue to assume that the supply functions $S_j(p)$ are independent of t. Let $Z_j(q)$ be the inverse function of $S_j(p)$. Thus, $Z_j(q)$ is a continuous strictly increasing function on $0 < q < S_j(\infty)$, where $S_j(\infty) = \lim_{p \to \infty} S_j(p)$. The right-hand side of (8) will lie in the interval $0 < q < S_j(\infty)$ for every j. Thus, (8) can be rewritten as

$$R_j(t) = Z_j\left\{L_j[R_j(t)] + \sum_{k \in K_j} V_{jk}\left[P_k(t-1) - T_{jk} - R_j(t)\right]\right\}. \tag{16}$$

In order to examine the asymptotic **stability** of the equilibrium state, we linearize the dynamic equations (9) and (16) around the equilibrium state. This can be done by taking total derivatives of (9) and (16) with respect to the P_k and R_j and then solving the two resulting equations for $dP_k(t+1)$ in terms of $dP_k(t-1)$ by eliminating $dR_j(t)$. We also assign to the derivatives of the functions G_k, V_{jk}, Z_j, and L_j the values that they assume in the equilibrium state. We denote these values by G_k', V_{jk}', Z_j', and L_j' respectively. This yields

$$dP_k(t+1) = G_k' \sum_{j \in J_k} \left[V_{jk}' \, dP_k(t-1) - \frac{V_{jk}' Z_j' \sum_{l \in K_j} V_{jl}' \, dP_l(t-1)}{1 - Z_j'(L_j' - \sum_{l \in K_j} V_{jl}')} \right].$$

Let there be h consumers' markets. In terms of the vector $w(t) = [dP_1(t), \ldots , dP_h(t)]^T$ (the superscript T denotes the transpose), the last equation can be written in matrix form as $w(t+1) = Aw(t)$. A is a constant $h \times h$ matrix whose elements tend to zero as the values G_k', V_{jk}', Z_j', and L_j' tend to zero.

Now, the equilibrium state is asymptotically stable if all the eigenvalues of A have absolute values less than one [1; p. 126]. The latter will certainly be the case if the absolute values of every entry of A is less than K^{-1} [5; p. 211]. This result can be written out explicitly as bounds on the values G_k', V_{jk}', Z_j', and L_j' which insure asymptotic stability, but those bounds will be in general much more restrictive than need be. In any case, we have the qualitative result that the equilibrium state is asymptotically stable if the slopes of the functions Z_j, G_k, L_j, and V_{jk} are small enough in neighborhoods of the equilibrium-state values of their arguments. This means that, with respect to the axes of Figures 2 and 5, the functions $Z_j = S_j^{-1}$ and $G_k = H_k^{-1}$ should be flat enough whereas L_j and V_{jk} should be steep enough in the vicinity of the equilibrium state.

7. More Complicated Models of Anticipated Prices

The model we have discussed so far assumes that at time t traders between m_j and n_k take the last received price $P_k(t-1)$ in n_k as the anticipated value for $P_k(t+1)$. This is similar to the classical cobweb model [3]. A number of authors have modified the cobweb model by using more complicated formulas for expected prices. (See, for example, [2], [6], and [7].) Thus, one might assume that the m_j to n_k traders take at time t some weighted average of many prior prices in n_k as the anticipated value of $P_k(t+1)$. We denote that weighted average by

$$B_{jk}[P_k(t-1), P_k(t-3), P_k(t-5), \dots]. \tag{17}$$

This idea can be incorporated into the governing equations for the dynamic behavior of our model simply by replacing the term $P_k(t-1)$ in the right-hand sides of (8) and (9) by (17). The recursive computation of the various time series in prices and quantities proceeds as before, but now initial conditions must be given on enough prior prices to determine (17) for some t. In general, (17) serves to smooth out the effects of disturbances.

If B_{jk} is a strictly increasing function of $P_k(t-1)$ when the $P_k(t-3)$, $P_k(t-5)$ ... are all held fixed, then Theorem 1 and its proof extend directly to our present model. If in addition (17) assumes a certain value whenever all the $P_k(t-1)$, $P_k(t-3)$, ... assume that same value, then Theorems 2 and 3 also hold once again with no changes in their proofs. Finally, the equilibrium state will again be asymptotically stable under all the assumptions stated so far if we add the requirement that every first partial derivative of B_{jk} with respect to each of its arguments as not too large in the vicinity of the equilibrium state.

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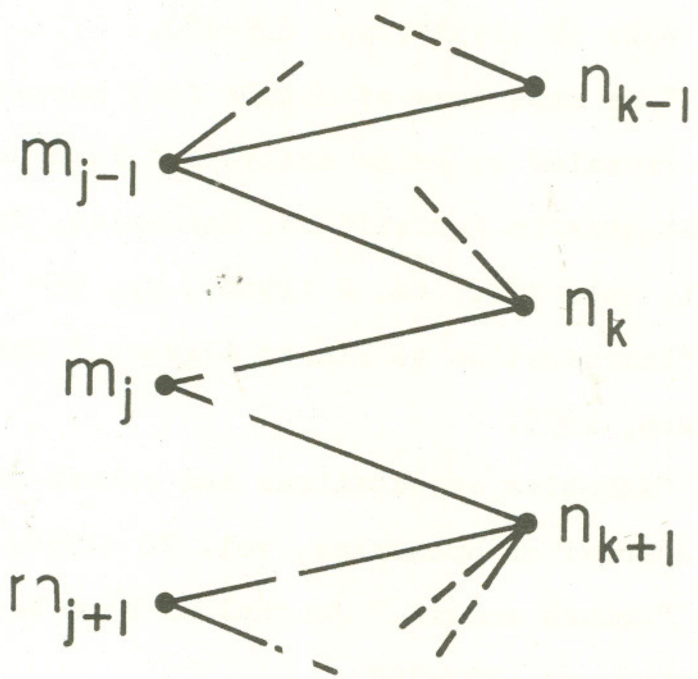

Fig. 1

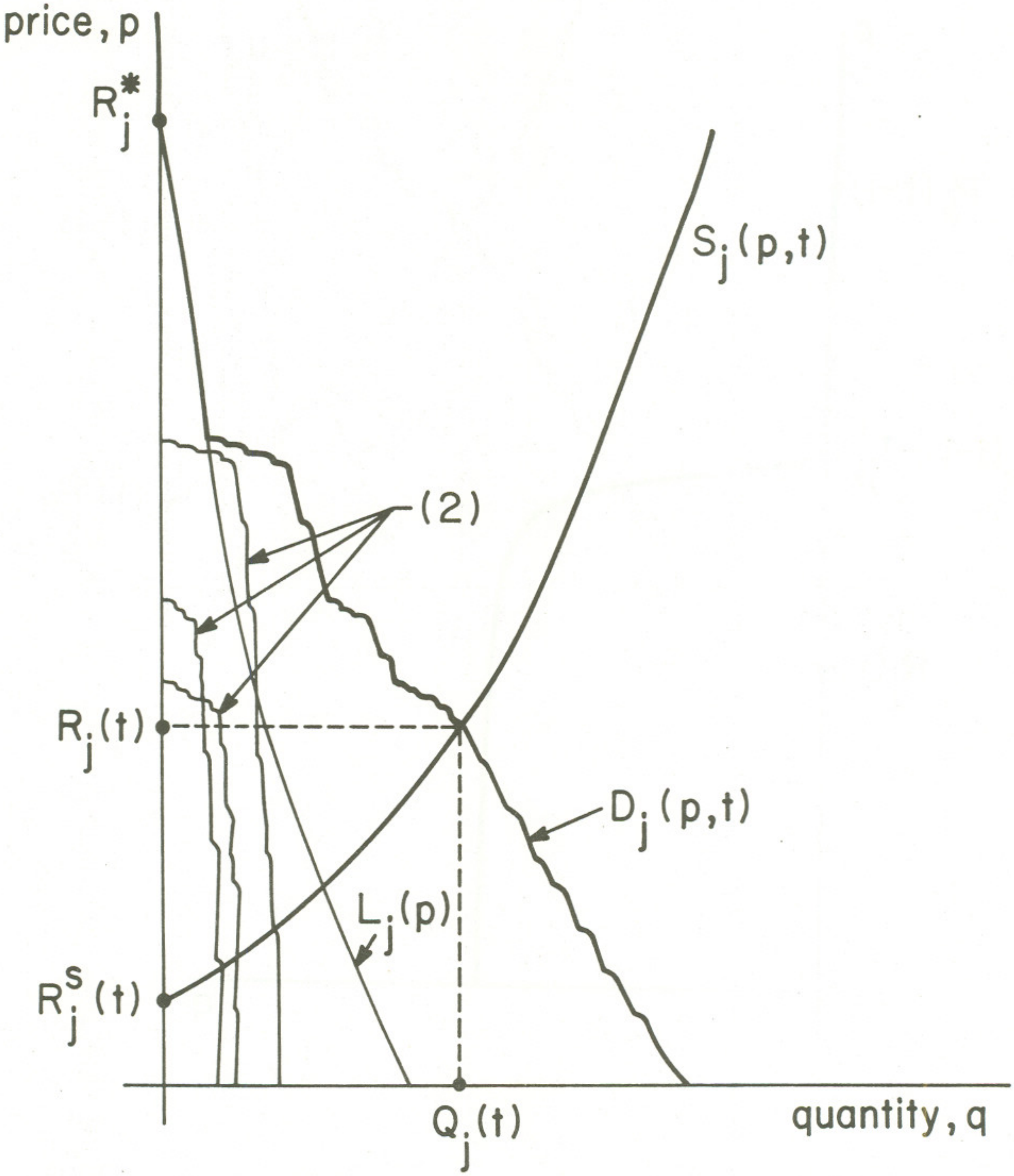

Fig. 2

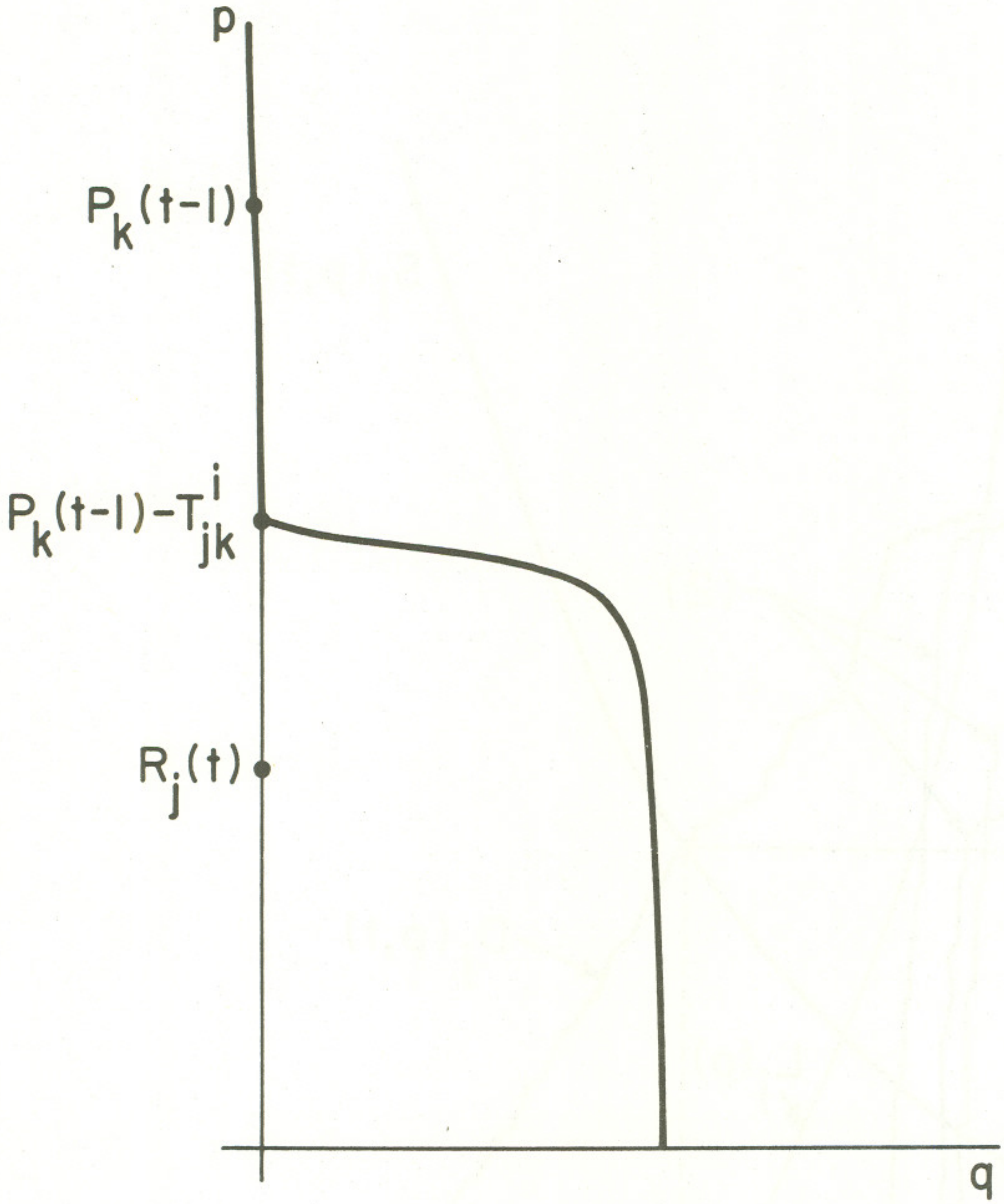

Fig. 3

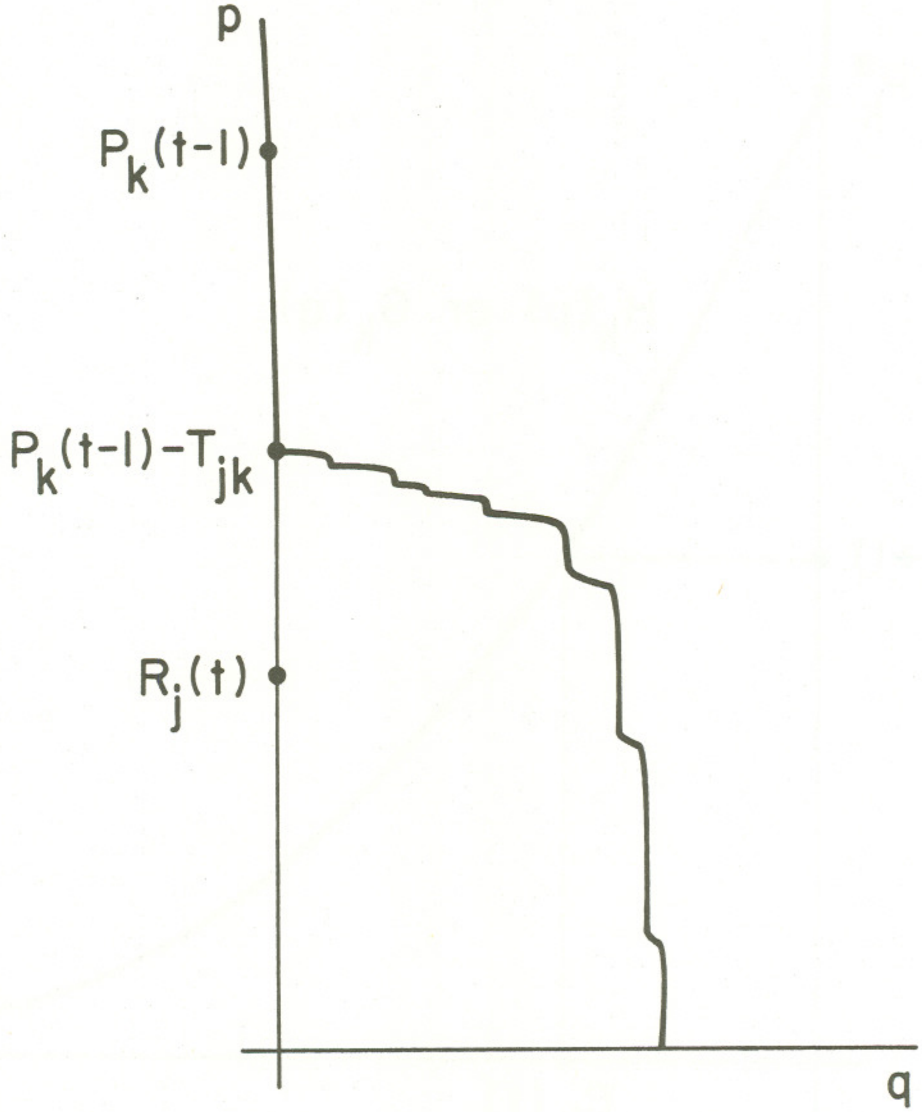

Fig. 4

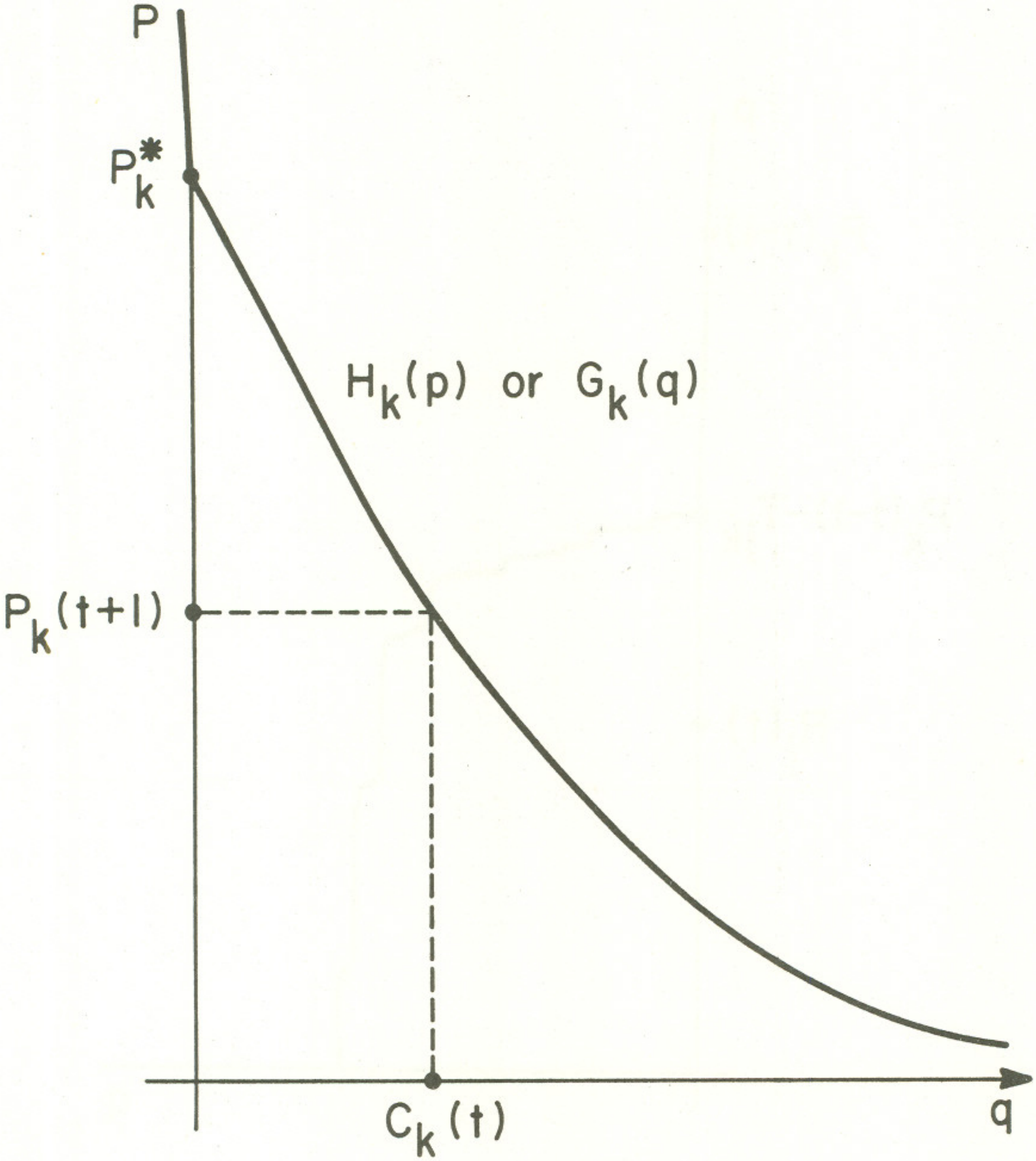

Fig. 5

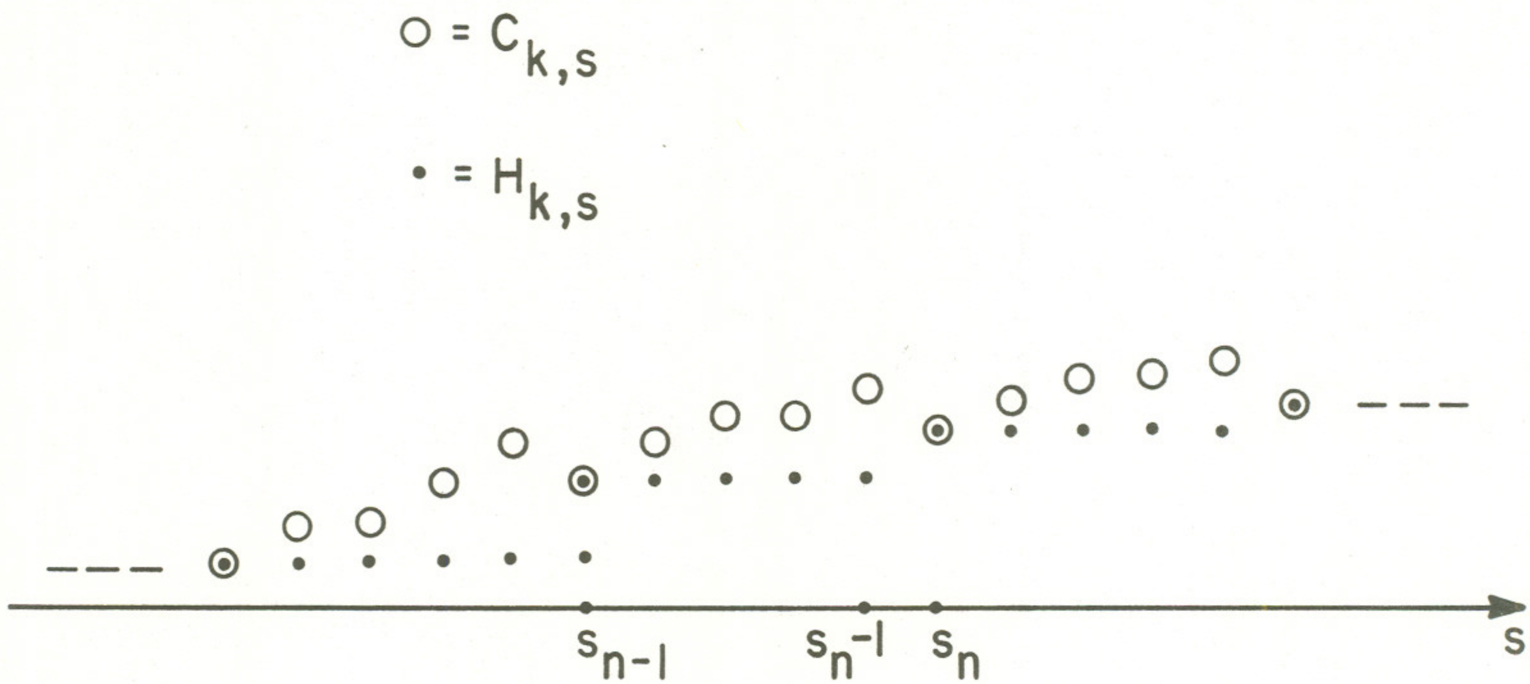

Fig. 6