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UNCOUNTABLE TRANSFINITE GRAPHS AND ELECTRICAL NETWORKS

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Abstract — Up to now, the theory of infinite and transfinite graphs and electrical networks has been established only for the case where the branch sets are countable. This work extends that prior theory to graphs and networks with uncountable branch sets. Moreover, conditions on the local structure of the transfinite graph are given which insure the countability of the branch set. Finally, it is shown that only countably many of the branches in an uncountable network can have nonzero voltages and currents, and therefore the restriction to countability in the prior theory is not essentially restrictive.

1 Introduction

Transfinite graphs and electrical networks were introduced in [4] and [5], but it was assumed in those works that such a graph had only countably many branches. This assumption can be relaxed, and doing so is the first objective of this work. We shall present a series of definitions which culminate in transfinite graphs and networks with possibly uncountable branch sets. The ranks of those graphs can be any countable ordinal. Furthermore, as another extension, we shall introduce a new kind of node, the "$\vec{\theta}$-*node*", where θ is any countable limit ordinal; such nodes had been omitted in our prior construction of transfinite graphs. Examples of such graphs are given.

A second objective of this work is to establish conditions under which transfinite graphs will have countable branch sets. In addition to the transfinite connectedness of the graphs, these conditions impose requirements on various kinds of extremities of the graphs. They are in fact generalizations of the standard result that an ordinary connected graph whose

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nodes are all of countable degrees has a countable branch set. But, the transfiniteness of the graphs considered herein complicates matters considerably.

Finally, the significance of uncountability for electrical networks is examined. It is shown that, even though the network may have an uncountable branch set, only countably many branches can have nonzero voltages and currents. This implies, for example, that an uncountable parallel circuit can be removed without affecting the voltages and currents in the rest of the network. It also implies that, so far as voltage-current regimes are concerned, no generality is lost by imposing countability on the branch sets.

A note about how we refer to paths: Instead of "finite transfinite paths," a terminology used in [4, pages 72 and 144] and [5], we now say "two-ended transfinite paths."

2 Countable and Uncountable Transfinite Graphs

The starting point in the construction of a transfinite graph is a finite or infinite set $\mathcal{T}^{\vec{0}}$. We require that $\mathcal{T}^{\vec{0}}$ have even cardinality if it is finite, but otherwise $\mathcal{T}^{\vec{0}}$ can be arbitrary. We will be primarily interested in the case where $\mathcal{T}^{\vec{0}}$ is an infinite set. The elements of $\mathcal{T}^{\vec{0}}$ will be called *elementary tips*. Later on, we will be introducing other kinds of "tips," categorized by their "ranks." The elementary tips are the tips of lowest rank, and $\vec{0}$ denotes that lowest rank. (For the sake of certain numerical formulas that will arise, it will be convenient to employ -1 as an alternative notation for $\vec{0}$.)

Next, $\mathcal{T}^{\vec{0}}$ is partitioned into two-element subsets, and each such subset is called a *branch*. $\mathcal{B}$ will denote the set of branches. If $\mathcal{T}^{\vec{0}}$ is either a countable (i.e., finite or denumerably infinite) set or alternatively an uncountable set, then $\mathcal{B}$ is countable or respectively uncountable, and the transfinite graphs we shall construct below will be said to be *countable* or *uncountable* accordingly. An $\vec{0}$-*path* is taken to be a single branch.

Furthermore, let us also partition $\mathcal{T}^{\vec{0}}$ in any arbitrary fashion. Each subset of this latter partition is called a 0-*node*, 0 being its *rank*. The set of 0-nodes is denoted by $\mathcal{N}^0$. $\mathcal{N}^0$ may be either finite, denumerable, or uncountable so long as $\mathcal{T}^{\vec{0}}$ is large enough. The cardinality of each 0-node is called its *degree*. If both elementary tips of a branch b are contained in the same 0-node, then b is called a *self-loop*. A branch b and a 0-node n^0 are said to be *incident* if $b \cap n^0 \neq \emptyset$. Thus, the degree of n^0 is the cardinality of the set of branches incident to n^0, except that a self-loop is counted twice when determining the cardinality of a finite 0-node.

A 0-*graph* is the pair

$$\mathcal{G}^0 = (\mathcal{B}, \mathcal{N}^0). \tag{1}$$

It is determined by $\mathcal{T}^{\vec{0}}$ and the two aforementioned partitions of $\mathcal{T}^{\vec{0}}$. The *rank* of $\mathcal{G}^0$ is designated as 0.

This construction implicitly determines a mapping from $\mathcal{B}$ into $\mathcal{N}^0$ through the incidences between branches and nodes. Thus, our 0-graphs are graphs in the conventional sense. All the definitions regarding conventional graphs can and will be transferred to 0-graphs. In contrast to conventional graphs, isolated nodes do not occur in 0-graphs.

Moreover, all the definitions pertaining to 0-graphs given in [4] and [5] again hold for the 0-graphs introduced here. The fact that we now allow $\mathcal{T}^{\vec{0}}$ and thereby $\mathcal{B}$ and possibly $\mathcal{N}^0$ to be uncountable does not alter those definitions. Thus, a 0-*path* in $\mathcal{G}^0$ is again an alternating sequence of branches and nodes and is the same as a path in a conventional graph. Furthermore, we can define the 0-*tips* of $\mathcal{G}^0$ as equivalence classes of one-ended 0-paths in $\mathcal{G}^0$, where two one-ended 0-paths are considered *equivalent* if they differ on no more than a finite number of elements. $\mathcal{G}^0$ may not have any 0-tips; this occurs when $\mathcal{G}^0$ has no one-ended 0-paths. But, if $\mathcal{G}^0$ does have 0-tips, we let $\mathcal{T}^0$ denote the nonvoid set of all 0-tips of $\mathcal{G}^0$.

Partition $\mathcal{T}^0$ in an arbitrary fashion to obtain the subsets $\mathcal{T}^0_\tau$, where τ denotes the indices of the partition; thus, $\mathcal{T}^0 = \cup_\tau \mathcal{T}^0_\tau$. Each $\mathcal{T}^0_\tau$ may be finite, denumerable, or uncountable (but never void). Furthermore, for each τ, let $\mathcal{N}^0_\tau$ be either the void set or a singleton containing one of the 0-nodes of $\mathcal{G}^0$; we require that $\mathcal{N}^0_{\tau_1} \cap \mathcal{N}^0_{\tau_2} = \emptyset$ if $\tau_1 \neq \tau_2$. We now define a 1-*node* for each τ as the set $\mathcal{T}^0_\tau \cup \mathcal{N}^0_\tau$. Thus, every 0-tip appears in exactly one of the 1-nodes. Also, every 0-node appears in either no 1-node or in exactly one 1-node; in the latter case, that 0-node is called the *exceptional element* of the 1-node. Conversely, every 1-node contains at least one 0-tip and no more than one 0-node. $\mathcal{N}^1$ will denote the set of all 1-nodes.

The 1-*graph* $\mathcal{G}^1$ is now defined as the triplet

$$\mathcal{G}^1 \; = \; \{\mathcal{B}, \mathcal{N}^0, \mathcal{N}^1\}. \tag{2}$$

Its *rank* as a transfinite graph is 1.

By continuing this construction recursively, we obtain the definition of a transfinite graph of rank μ for every natural number μ. (Henceforth, μ will always denote a natural number.) In short, assume that such graphs $\mathcal{G}^\alpha$ have been defined for each $\alpha = 1, \ldots, \mu - 1$. The α-paths and α-tips of $\mathcal{G}^\alpha$ are defined as in [4] and [5], and we assume that, for each α, $\mathcal{G}^\alpha$ has α-tips. If $\mathcal{G}^{\mu-1}$ has at least one one-ended $(\mu - 1)$-path and thereby at least one $(\mu - 1)$-tip, partition the set $\mathcal{T}^{\mu-1}$ of all $(\mu - 1)$-tips into subsets $\mathcal{T}_\tau^{\mu-1}$, where again τ denotes the indices for the partition. Furthermore, for each τ, let $\mathcal{N}_\tau^{\mu-1}$ be either the void set or a singleton containing some α-node n^α, where $0 \leq \alpha \leq \mu - 1$. As before, we require that, if $\mathcal{N}_{\tau_1}^{\mu-1}$ and $\mathcal{N}_{\tau_2}^{\mu-1}$ have nodes, those two nodes are different nodes and neither node embraces the other. Then, for each τ, we define a μ-*node* n^μ as the set $\mathcal{T}_\tau^{\mu-1} \cup \mathcal{N}_\tau^{\mu-1}$. The single element of $\mathcal{N}_\tau^{\mu-1}$, if it exists, is called the *exceptional element* of n^μ. Finally, we define the μ-*graph* $\mathcal{G}^\mu$ as the $(\mu + 2)$-tuplet

$$\mathcal{G}^\mu \; = \; \{\mathcal{B}, \mathcal{N}^0, \ldots, \mathcal{N}^\mu\}. \tag{3}$$

It is understood here that each $\mathcal{N}^\gamma$ $(\gamma = 0, \ldots, \mu)$ is nonvoid. Because the branch set $\mathcal{B}$ is allowed to be uncountable, we now have a more general definition of a μ-graph than that given in [4] and [5].

The next step of generalization is represented by a special kind of graph denoted by $\mathcal{G}^{\vec{\omega}}$, where $\vec{\omega}$ represents the *rank* of $\mathcal{G}^{\vec{\omega}}$. ($\vec{\omega}$ is the first *arrow rank* for a transfinite graph.) It will be possible to define such a graph so long as the constructions of the node sets $\mathcal{N}^0, \mathcal{N}^1, \ldots$ of progressively higher ranks can be continued without end. In other words, the sequence $\{\mathcal{G}^\mu\}_{\mu=1}^\infty$ of μ-graphs generated recursively must be such that each $\mathcal{G}^\mu$ has one-ended μ-paths (and thereby an infinity of μ-nodes), for otherwise there would be no μ-tips with which to define $\mathcal{G}^{\mu+1}$. So, let us assume that, for each natural number μ, nonvoid sets $\mathcal{N}^\mu$ have been constructed.

Let us also introduce a new kind of node, the $\vec{\omega}$-node, something that was omitted in [4] and [5]. Let $\{\mu_k\}_{k=0}^{\infty}$ be a strictly increasing, infinite sequence of natural numbers; thus, $0 \leq \mu_0 < \mu_1 < \mu_2 < \ldots$. For each k, let there be a μ_k-node n^{μ_k}. If, for each k, n^{μ_k} is the exceptional element of $n^{\mu_{k+1}}$, then the infinite sequence

$$n^{\vec{\omega}} = \{n^{\mu_0}, n^{\mu_1}, n^{\mu_2}, \ldots\} \qquad (4)$$

is called an *$\vec{\omega}$-node*, and $\vec{\omega}$ denotes its *rank*. The set of all $\vec{\omega}$-nodes will be denoted by $\mathcal{N}^{\vec{\omega}}$. It follows from this definition that two different $\vec{\omega}$-nodes are disjoint: that is, if a particular μ-node appears in both $\vec{\omega}$-nodes, then those $\vec{\omega}$-nodes must be the same [4, Lemma 5.1-2]. Note that an $\vec{\omega}$-node is automatically a nonsingleton.

This definition differs substantially from that of a μ-node n^{μ} with a natural-number rank μ in two ways. First, $n^{\vec{\omega}}$ does not contain any $\vec{\omega}$-tip, whereas n^{μ} always contains μ-tips. Secondly, $n^{\vec{\omega}}$ embraces an infinity of nodes, whereas n^{μ} embraces only finitely many nodes — perhaps none at all.

We now define the *$\vec{\omega}$-graph* $\mathcal{G}^{\vec{\omega}}$ of *rank* $\vec{\omega}$ as the well-ordered, infinite set of sets

$$\mathcal{G}^{\vec{\omega}} = \{\mathcal{B}, \mathcal{N}^0, \mathcal{N}^1, \ldots, \mathcal{N}^{\vec{\omega}}\}, \qquad (5)$$

where, for each natural number μ, the node set $\mathcal{N}^{\mu}$ is nonvoid. However, $\mathcal{N}^{\vec{\omega}}$ is allowed to be void. The node sets are ordered within (5) in accordance with the ranks of their nodes. We take the rank $\vec{\omega}$ to be larger than all natural-number ranks. (Later on, we will introduce still higher ranks; $\vec{\omega}$ will be the least rank larger than all the natural-number ranks and the largest rank less than all the transfinite-ordinal ranks $\omega, \omega+1, \ldots$.) This is a more general definition of an $\vec{\omega}$-graph than that given in [4] or [5] because now we allow $\vec{\omega}$-nodes as well as a possibly uncountable set $\mathcal{B}$ of branches.

Example 2.1. Here are three simple examples illustrating countable $\vec{\omega}$-graphs. The first of these is a star graph, illustrated in Figure 1(a). It consists of a branch b (i.e., a $\vec{0}$-path) and the μ-paths P^{μ} ($\mu = 0, 1, 2, \ldots$) all incident to the 0-node m^0 through branches. Thus, m^0 is of degree $\aleph_0$. Except for the incidences at m^0, these paths are totally disjoint. This is an $\vec{\omega}$-graph, but it has neither an $\vec{\omega}-$node nor an $\vec{\omega}$-path.

An $\vec{\omega}$-graph having an $\vec{\omega}$-node $n^{\vec{\omega}}$ but no $\vec{\omega}$-path is shown in Figure 1(b). The branch b and the μ-paths P^{μ} again meet through branches at the 0-node m^0, but now they also meet at the $\vec{\omega}$-node $n^{\vec{\omega}}$ through tips. More specifically, b meets the 0-node n^0, which is embraced by the 1-node n^1. Moreover, for each μ, the μ-tip of P^{μ} is one member of the two-element $(\mu - 1)$-node $n^{\mu+1}$, whose other element is the μ-node n^{μ}. Thus, each node of the sequence $\{n^0, n^1, n^2, \ldots\}$ embraces all the nodes of lower ranks in the sequence, and the sequence itself is an $\vec{\omega}$-node $n^{\vec{\omega}}$.

Finally, Figure 1(c) shows the beginning of an $\vec{\omega}$-path. (Such a path is defined in [4, page 147] and [5, Section 5].) This is the simplest example of an $\vec{\omega}$-graph with an $\vec{\omega}$-path but no $\vec{\omega}$-node. ♣

Following the $\vec{\omega}$-graphs, the next kind of transfinite graph of still higher rank is the "ω-graph," where the rank ω is the first transfinite ordinal. Such a graph can be constructed from a given $\vec{\omega}$-graph $\mathcal{G}^{\vec{\omega}}$ so long as $\mathcal{G}^{\vec{\omega}}$ has at least one $\vec{\omega}$-tip [4, page 148], [5, Section 5]. Partition the set $\mathcal{T}^{\vec{\omega}}$ of all $\vec{\omega}$-tips of $\mathcal{G}^{\vec{\omega}}$ into the subsets $\mathcal{T}_{\tau}^{\vec{\omega}}$; thus, $\mathcal{T}^{\vec{\omega}} = \cup_{\tau}\mathcal{T}_{\tau}^{\vec{\omega}}$, where as usual τ denotes the indices of the partition. Also, for each τ, let $\mathcal{N}_{\tau}^{\vec{\omega}}$ be either the void set or a singleton whose only member is either an $\vec{\omega}$-node of $\mathcal{G}^{\vec{\omega}}$ or a μ-node of $\mathcal{G}^{\vec{\omega}}$, μ being a natural number as always. Again, we require that, if $\mathcal{N}_{\tau_1}^{\vec{\omega}}$ and $\mathcal{N}_{\tau_2}^{\vec{\omega}}$ have nodes, those two nodes are different nodes and neither node embraces the other. Then, for each τ again, we define an *ω-node* n_{τ}^{ω} as the set $\mathcal{T}_{\tau}^{\vec{\omega}} \cup \mathcal{N}_{\tau}^{\vec{\omega}}$. Finally, an *$\omega$-graph* $\mathcal{G}^{\omega}$ is defined as the well-ordered, infinite set of sets

$$\mathcal{G}^{\omega} = \{\mathcal{B}, \mathcal{N}^0, \mathcal{N}^1, \ldots, \mathcal{N}^{\vec{\omega}}, \mathcal{N}^{\omega}\}, \tag{6}$$

where $\mathcal{N}^{\omega}$ denotes the set of ω-nodes. The only node set in (6) that is allowed to be void is $\mathcal{N}^{\vec{\omega}}$. Here too, we have a more general definition of an ω-graph than that given in [4] and [5].

With a view toward generating transfinite graphs of still higher ranks, let us note that the construction of the ω-graph $\mathcal{G}^{\omega}$ is like that of the 0-graph because the $\vec{\omega}$-tips assume the role played by the $\vec{0}$-tips previously. In particular, the limit ordinal ω has no immediately preceding ordinal, and neither does the ordinal 0; moreover, we have to introduce tip ranks, namely, $\vec{\omega}$ and respectively $\vec{0}$ that are not ordinals. So, it appears that the transfinite graphs of next higher rank, the $(\omega+1)$-graphs, can be constructed by mimicing how the 1-graphs were generated from the 0-graphs. In short, ω-tips are defined as equivalence classes of one-ended ω-paths, the set of ω-tips is partitioned into subsets, $(\omega+1)$-nodes are defined as the set of ω-tips in one of the partitioning subsets possibly accompanied by a single node of rank no larger than ω — with the sets of embraced elements of two $(\omega+1)$-nodes being disjoint, $\mathcal{N}^{\omega+1}$ is taken to be the set of all $(\omega+1)$-nodes, and finally the following definition is set up.

$$\mathcal{G}^{\omega+1} = \{\mathcal{B}, \mathcal{N}^0, \mathcal{N}^1, \ldots, \mathcal{N}^{\vec{\omega}}, \mathcal{N}^{\omega}, \mathcal{N}^{\omega+1}\} \tag{7}$$

Clearly, we can proceed in this way to define recursively any $(\omega+\mu)$-graph $\mathcal{G}^{\omega+\mu}$, where μ is any natural number again. Then, an $(\omega+\vec{\omega})$-graph $\mathcal{G}^{\omega+\vec{\omega}}$ can be defined by mimicing the construction of an $\vec{\omega}$-graph $\mathcal{G}^{\vec{\omega}}$; this entails the definition of $(\omega+\vec{\omega})$-nodes. Next, graphs of rank $\omega+\omega=\omega\cdot 2$ can be defined by appropriately modifying the definition of ω-graphs. And so forth.

In this way we recursively obtain transfinite graphs of all countable ordinal ranks, but in doing so we also acquire transfinite graphs of rank $\vec{\theta}$, whenever θ is a transfinite, countable, limit ordinal. $\vec{\theta}$ will be called an *arrow rank*; it precedes θ but lies beyond every ordinal smaller than θ. We also have $\vec{\theta}$-nodes, which are quite analogous to $\vec{\omega}$-nodes. In particular, a $\vec{\theta}$-node is a well-ordered infinite set of nodes, ordered according to their ranks with each node of the set embracing all the nodes preceding it. Every $\vec{\theta}$-node is automatically a nonsingleton.

Let us also note that, when ν is a rank no less than $\vec{\omega}$, any ν-node may embrace infinitely many nodes since there are infinitely many ranks less than ν. This stands in contrast to a μ-node, μ being a natural number, which can only embrace finitely many nodes.

3 Examples of Uncountable and Countable Transfinite Graphs

Example 3.1. Let $\mathcal{S}^0$ denote an uncountable star graph whose branch set is indexed by the real numbers and whose branches are joined only at a single central 0-node of degree $\mathbf{c}$, the cardinal number of the continuum. We can append a replicate of $\mathcal{S}^0$ to every end 0-node of $\mathcal{S}^0$ by shorting the central 0-node of each replicate to each end 0-node. Furthermore, we can indefinitely continue this process of appending replicates of $\mathcal{S}^0$ to every end 0-node that arises. The result is an uncountable tree $\mathcal{T}$. Its branch set also has the cardinality **c** of the continuum [1, pages 376 and 381].

Furthermore, if we replace every branch of $\mathcal{T}$ by an endless ν-path, where ν is any countable ordinal, and every 0-node of $\mathcal{T}$ by a $(\nu+1)$-node, we will obtain a $(\nu+1)$-graph with a branch set whose cardinality is again **c**. ♣

Example 3.2. The ladder $\mathcal{L}^0$ shown in Figure 2 is an example of a countable 0-graph having uncountably many 0-tips. To appreciate this uncountability, note first of all that any 0-tip of this graph can be uniquely identified with the set of just those representatives for it that start at the 0-node n^0. Any two such representatives for a given 0-tip differ on no more than finitely many branches. Moreover, every such representative for any 0-tip can be specified by a binary sequence $\{x_0, x_1, x_2, \ldots\}$, where $x_k = 0$ (or $x_k = 1$) if the representative passes through a_k (respectively, b_k). Of course, within the representative a vertical branch c_k appears between the branches corresponding to x_k and x_{k+1} whenever $x_k \neq x_{k+1}$, and c_0 precedes b_0 when $x_0 = 1$. Now, the set of one-ended binary sequences has the cardinality **c** of the continuum because every real number between 0 and 1 can be represented by such a sequence. However, in order to determine the cardinality of the set of 0-tips of the ladder, we must examine a partitioning of those binary sequences, where two sequences are considered equivalent if they differ on no more than finitely many elements. Note that every set S of equivalent sequences is countable. Indeed, choose any binary sequence $s_0 = \{x_0, x_1, x_2, \ldots\}$. Then, count the one binary sequence s_1 that differs from s_0 only in x_0, then count the two additional binary sequences s_2 and s_3 that differ from s_0 and s_1 only in x_1, then count the four additional binary sequences that differ from s_0, s_1, and s_2 only in x_2, and so forth. Thus, the cardinality of S is $\aleph_0$, the cardinal number of a denumerably infinite set. Consequently, the cardinal number of the set of 0-tips for the ladder is $\mathbf{c} \div \aleph_0 = \mathbf{c}$ [1, page 377].

We can construct an uncountable 1-graph $\mathcal{G}^1$ by appending branches to the 0-tips of the ladder $\mathcal{L}^0$, as is indicated in Figure 2(b). In particular, a 0-node n_x^0 of each appended branch is shorted to each 0-tip t_x^0 of the ladder by means of a 1-node $n_x^1 = \{n_x^0, t_x^0\}$. Note however that, except for the two 0-tips with representatives along the a_k branches alone and along the b_k branches alone, each 0-tip is nondisconnectable from every other 0-tip; that is, any representative of any 0-tip meets a representative of every other 0-tip infinitely often. Later on, we will at times restrict our attention to graphs in which pairs of nondisconnectable tips are either shorted or at least one of them comprises a singleton node (Condition 4.1). This 1-graph cannot be one of them. ♣

Example 3.3. Another example of a countable 0-graph with uncountably many 0-tips is the binary tree shown in Figure 3(a). In this case, every 0-tip can be uniquely represented by a one-ended 0-path that starts at the topmost 0-node. Each such 0-path can in turn be uniquely specified by a one-ended binary sequence. Hence, the cardinality of the set of 0-tips for the binary tree is also **c**.

Once again we can construct an uncountable 1-graph by appending branches bijectively to the 0-tips as before. See Figure 3(b). Now, however, no two 0-tips are nondisconnectable. Thus, this uncountable 1-graph is not eliminated by Condition 4.1. ♣

Example 3.4. It may be worth noting at this point that every one of the examples of ν-graphs ($\nu \geq 1$) given above possesses maximal nonsingleton nodes of all ranks up to and including ν. This need not be the case. In fact, there are ν-graphs whose maximal nonsingleton nodes are all of rank ν — all the maximal nodes of lower ranks being singletons. It is the nonsingleton nodes that are of importance in electrical network theory; the singleton nodes play no role with regard to the flow of current. An example of such a transfinite graph requires a rather complicated construction, as follows.

Start with a single branch b. Then, introduce two one-ended 0-paths P_1^0 and P_2^0, whose 0-tips are shorted to the two nodes of b through two 1-nodes; this is shown in Figure 4(a). Next, short the 0-nodes of P_1^0 and P_2^0 to the 0-tips of other one-ended 0-paths in a bijective fashion, as shown in Figure 4(b). Do the same thing to the 0-nodes of the newly introduced one-ended 0-paths. Continuing in this way indefinitely, we obtain a 1-graph $\mathcal{G}^1$, all whose maximal nodes are of rank 1. The result of the first four steps of this construction is shown in Figure 4(c).

$\mathcal{G}^1$ has 1-tips. For example, a representation for a 1-tip t^1 can be traced in Figure 4(c) by starting at the 1-node n_1^1, proceeding along a 0-path to n_2^1, then along another 0-path to n_3^1, and so forth in $\mathcal{G}^1$. In fact, $\mathcal{G}^1$ has at least a continuum of 1-tips, for a binary tree of one-ended 0-paths can be embedded in $\mathcal{G}^1$. Every 1-tip of $\mathcal{G}^1$ will comprise the sole member of a singleton 2-node in our subsequent constructions.

Now, consider a replicate of $\mathcal{G}^1$ and let t_0^1 be the 1-tip of the replicate corresponding to the 1-tip t^1 introduced above. Short t_0^1 to the 1-node n_1^1 of Figure 4(c), creating thereby a nonsingleton 2-node. Do the same thing for all the other 1-nodes of $\mathcal{G}^1$, using another replicate of $\mathcal{G}^1$ for each 1-node of $\mathcal{G}^1$. This yields an embracing 2-node for each 1-node of $\mathcal{G}^1$. The 1-nodes of the appended replicates of $\mathcal{G}^1$ are also treated in the same way to embrace them as well by 2-nodes, and this process is continued indefinitely. Moreover, all the 1-tips of $\mathcal{G}^1$ and its replicates — other than the 1-tips of the replicates used in constructing nonsingleton 2-nodes — are taken to comprise singleton 2-nodes. All this yields a 2-graph $\mathcal{G}^2$, whose maximal nodes are all of rank 2.

Furthermore, $\mathcal{G}^2$ has 2-tips. One of them, say, t_0^2 can be found by starting at the 1-node n_1^1, shown in Figure 4(c), passing through the 1-tip of the replicate of $\mathcal{G}^1$ appended to n_1^1 and proceeding through that replicate to reach the 1-node that corresponds to n_1^1, continuing through the 1-tip of another replicate of $\mathcal{G}^1$, and so forth. All those 2-tips will comprise singleton 3-nodes in our later constructions.

We now use replicates of $\mathcal{G}^2$ and in particular their 2-tips corresponding to t_0^2 to construct 3-nodes out of the nonsingleton 2-nodes of $\mathcal{G}^2$. We then repeat this for the replicates, and then again for the newly introduced replicates, and so on. This gives a 3-graph whose maximal nonsingleton nodes are 3-nodes. That 3-graph will also have singleton 2-nodes as well as 3-tips.

This process can be continued indefinitely to obtain, for each natural number μ, a μ-graph $\mathcal{G}^\mu$, all of whose maximal nonsingleton nodes are of rank μ and whose maximal nodes of lower ranks are all singletons.

We can proceed still further by never terminating at any rank μ. The result is an $\vec{\omega}$-graph $\mathcal{G}^{\vec{\omega}}$, whose maximal nonsingleton nodes are $\vec{\omega}$-nodes and whose maximal nodes of all lower ranks are singletons. $\mathcal{G}^{\vec{\omega}}$ also has $\vec{\omega}$-tips, and so the process can be continued to obtain an ω-graph $\mathcal{G}^\omega$. In fact, for each countable-ordinal rank $\nu = 1, 2, \ldots, \vec{\omega}, \omega, \omega+1, \ldots$, we can generate a ν-graph $\mathcal{G}^\nu$ whose maximal nonsingleton nodes are all of rank ν and whose maximal nodes of all lower ranks are singletons.

Every such $\mathcal{G}^\nu$ has a countable branch set. This can be shown by repeatedly using the principle that a countable collection of countable sets is countable. Moreover, since every maximal nonsingleton node of $\mathcal{G}^\nu$ is incident to a branch, those nodes comprise a countable set too. ♣

4 Countability Conditions on Sections

Henceforth, we let $\mathcal{G}^\nu$ be any countable or uncountable, transfinite graph. Its rank ν is either a countable ordinal or an arrow rank other than $\vec{0}$. Henceforth, ρ will denote any rank no larger the ν; thus, ρ is either an ordinal rank or an arrow rank such as $\vec{0}$. For the sake of a more concise notation, we augment conventional symbolism regarding the ordinals as follows. If the rank ρ is an ordinal, we let $\rho - 1$ denote the rank immediately preceding ρ; thus, $\rho - 1 = \vec{\rho}$ if ρ is either 0 or a limit ordinal. However, $\rho - 1$ will never be used when ρ is an arrow rank. On the other hand, $\rho + 1$ will always denote the rank immediately succeeding ρ; thus, if ρ is an arrow rank $\vec{\theta}$, then $\rho + 1 = \theta$.

Two branches b_1 and b_2 of $\mathcal{G}^\nu$ are said to be *ρ-connected* when the following condition holds: If ρ is an ordinal rank (alternatively, arrow rank other than $\vec{0}$), then there exists a finite γ-path P^γ, where $\gamma \leq \rho$ (respectively, $\gamma < \rho$), such that P^γ meets b_1 and b_2.

A *ρ-section* of $\mathcal{G}^\nu$ is a reduction of $\mathcal{G}^\nu$ induced by a maximal set of branches that are pairwise ρ-connected. Since a finite ρ-path is also a finite η-path whenever $\rho < \eta \leq \nu$, a ρ-section $\mathcal{S}^\rho$ is entirely contained in an η-section $\mathcal{S}^\eta$, and $\mathcal{S}^\eta$ may in fact coincide with $\mathcal{S}^\rho$. Furthermore, we allow the special case where $\rho = \vec{0}$; now, $\mathcal{S}^{\vec{0}}$ is defined to be a single branch.

ρ-connectedness is a binary relation between branches that is reflexive and symmetric, but it need not be transitive [6, Section 3]. As a result, the ρ-sections may not partition $\mathcal{G}^\nu$; in fact, two different ρ-sections may overlap, that is, may have common branches. We wish to avoid this pathology and can do so as follows.

We first need to define "nondisconnectable tips." Recall that, for an ordinal rank γ, a representative of a γ-tip is a one-ended γ-path which in turn is a one-way infinite alternating sequence of γ-nodes n_i^γ and $(\gamma-1)$-paths $P_i^{\gamma-1}$ of the form:

$$P^\gamma = \{n_0^\eta, P_0^{\gamma-1}, n_1^\gamma, P_1^{\gamma-1}, n_2^\gamma, P_2^{\gamma-1}, \cdots\} \tag{8}$$

where the first node n_0^η has a rank $\eta \leq \gamma$ and certain conditions are satisfied [5, Section 4]. (If $\gamma = 0$, $P^{\gamma-1}$ denotes a branch.) Similarly, for an arrow rank $\vec{\delta}$ $(\delta > 0)$, a representative of an $\vec{\delta}$-tip is a one-ended $\vec{\delta}$-path which in turn is an alternating sequence of the form:

$$P^{\vec{\delta}} = \{n_0^\eta, P_0^{\gamma_0-1}, n_1^{\gamma_1}, P_1^{\gamma_1-1}, n_2^{\gamma_2}, P_2^{\gamma_2-1}, \cdots\} \tag{9}$$

where η and the γ_k are ordinal ranks with $\eta \leq \gamma_0 < \gamma_1 < \gamma_2 < \cdots$ and $\delta = \lim \gamma_k$ [1, page 334] and again certain conditions are satisfied [5, Section 5].

Now consider an infinite sequence of nodes $\{m_1, m_2, m_3, \ldots\}$ of possibly differing ranks. We shall say that the m_l *approach* a γ-tip t^γ (alternatively, an $\vec{\delta}$-tip $t^{\vec{\delta}}$) if there is a representative (8) for t^γ (respectively, (9) for $t^{\vec{\delta}}$) such that, for each natural number i, all but finitely many of the m_l are shorted to nodes embraced by the members of (8) (respectively, (9)) lying to the right of n_i^γ (respectively, $n_i^{\gamma_i}$). We also say that those nodes lie *beyond* n_i^γ (respectively, $n_i^{\gamma_i}$) and that the m_l *approach* any node that embraces t^γ (respectively, $t^{\vec{\delta}}$).

Let t_a and t_b be two tips, not necessarily of the same rank. We say that t_a and t_b are *nondisconnectable* if there is an infinite sequence of nodes that approach both t_a and t_b.

Condition 4.1. *If two tips (of possibly differing ranks but both ranks no larger than* $\nu - 1$*) are nondisconnectable, then either the two tips are shorted together (i.e., are both embraced by some node) or at least one of them is open (i.e., is the sole member of a singleton node).*

Under this condition, ρ-connectedness becomes a transitive relation and ρ-sections partition $\mathcal{G}^\nu$, whatever be the rank ρ [6, Section 6].

A ρ-section $\mathcal{S}^\rho$ is said to *traverse a tip* t if $\mathcal{S}^\rho$ embraces a representative of t (i.e., all the branches of the representative are in $\mathcal{S}^\rho$). The rank of the tip may be larger than ρ; this will be illustrated in Figure 5 below.

Given a ρ-section $\mathcal{S}^\rho$, a node n is called a *boundary node of* $\mathcal{S}^\rho$ if n embraces a tip traversed by $\mathcal{S}^\rho$ and also embraces a tip not traversed by $\mathcal{S}^\rho$.

Lemma 4.2. *If n is a boundary node of a ρ-section $\mathcal{S}^\rho$, then n satisfies at least one of the following two conditions:*

(i) *All the tips embraced by n and traversed by $\mathcal{S}^\rho$ have ranks no less than ρ.*

(ii) *All the tips embraced by n and not traversed by $\mathcal{S}^\rho$ have ranks no less than ρ.*

Moreover, every boundary node of $\mathcal{S}^\rho$ has a rank larger than ρ, and any path P that meets a branch in $\mathcal{S}^\rho$ and a branch not in $\mathcal{S}^\rho$ must have a rank larger than ρ.

Proof. If n satisfies neither (i) nor (ii), then n embraces a node n^α of rank α no larger than ρ, and there is an α-path passing through n^α and embracing a branch of $\mathcal{S}^\rho$ and a branch not in $\mathcal{S}^\rho$. This violates the definition of $\mathcal{S}^\rho$ as a reduction induced by a maximal set of ρ-connected branches. The rest of the lemma follows directly. ♣

Two ρ-sections $\mathcal{S}_1^\rho$ and $\mathcal{S}_2^\rho$ will be said to be $(\rho+1)$-*adjacent* if there is a $(\rho+1)$-node $n^{\rho+1}$ that embraces a tip of $\mathcal{S}_1^\rho$ and a tip of $\mathcal{S}_2^\rho$ with one of those tips having a rank of ρ and the other tip having a rank no larger than ρ. It follows that $\mathcal{S}_1^\rho$ and $\mathcal{S}_2^\rho$ are $(\rho+1)$-connected through $n^{\rho+1}$ and are both embraced by some $(\rho+1)$-section.

Given a ρ-section $\mathcal{S}^\rho$, the $(\rho+1)$-*adjacency of* $\mathcal{S}^\rho$ is the set $\mathcal{J}$ of all ρ-sections that are $(\rho+1)$-adjacent to $\mathcal{S}^\rho$. If $\mathcal{J}$ consists of only countably many ρ-sections, we say that $\mathcal{S}^\rho$ has a *countable* $(\rho+1)$-*adjacency*; this does not necessarily mean that the ρ-sections of $\mathcal{J}$ are themselves countable. Note that, when $\rho = \vec{0}$, $\mathcal{S}^\rho$ is a single branch b, and its $(\rho+1)$-adjacency is the set of branches that share nodes with b. Also note that, if $\mathcal{S}^\rho$ has a countable set of boundary $(\rho+1)$-nodes and if each such node is incident to only countably many ρ-sections (in which case we may say that that boundary $(\rho+1)$-node is ρ-*sectionally of countable degree*), then $\mathcal{S}^\rho$ has a countable $(\rho+1)$-adjacency; but, these two conditions are only sufficient — not necessary — for that countable $(\rho+1)$-adjacency.

Example 4.3. Consider the $\vec{\omega}$-graph of Figure 1(a) and let $0 \le \rho < \vec{\omega}$. For each μ with $\rho \le \mu < \vec{\omega}$, let $m_\mu^{\rho+1}$ be the first $(\rho+1)$-node in the path P^μ with respect to a tracing from m^0 and let $n_\mu^{\rho+1}$ be the second $(\rho+1)$-node in P^μ. Then, the branches in the paths P^η ($\eta = \vec{0}, 0, \ldots, \rho$) along with the branches between m_0 and $m_\mu^{\rho+1}$ in the paths P^μ ($\mu = \rho+1, \rho+2, \ldots$) induce a ρ-section $\mathcal{S}^\rho$. The boundary nodes of $\mathcal{S}^\rho$ are the $m_\mu^{\rho+1}$ ($\mu = \rho+1, \rho+2, \ldots$). The $(\rho+1)$-adjacency of $\mathcal{S}^\rho$ consists of the ρ-sections between $m_\mu^{\rho+1}$ and $n_\mu^{\rho+1}$ for all those μ. ♣

Example 4.4. Figure 5 illustrates a 4-graph. n^4 is the 4-node $\{n^3, t^3\}$, where t^3 is the 3-tip of the one-ended 3-path P^3. Also, n^3 is the 3-node $\{n^1, t^2\}$, where t^2 is the 2-tip of the one-ended 2-path P^2. Finally, n^1 is the singleton 1-node $\{t^0\}$, where t^0 is the 0-tip of the one-ended 0-path P^0. Every 0-node embraced by P^3 (or P^2) is 0-adjacent to the 0-node n_1^0 (respectively, n_2^0); that is, there is a branch that is incident to that 0-node in the said path and to n_1^0 (respectively, n_2^0) as well. The reduced graphs $\mathcal{S}_1$ and $\mathcal{S}_2$ to the left and to the right of n^4 are 0-connected in themselves and are connected to each other by a 3-path such as the one passing along P^0, then through n^3, and then along P^2. No path of rank lower than 3 connects $\mathcal{S}_1$ and $\mathcal{S}_2$. Therefore, $\mathcal{S}_1$ and $\mathcal{S}_2$ are ρ-sections for each of $\rho = 0, 1, 2$, but not for $\rho = 3, 4$. On the other hand, $\mathcal{S}_1 \cup \mathcal{S}_2$ is the entire 4-graph and is a 3-section as well. Note that, even though $\mathcal{S}_1$ is a 0-section, it traverses a 3-tip — a situation we promised to illustrate. n^3 is a boundary node for both $\mathcal{S}_1$ and $\mathcal{S}_2$, and so too is n^4. But, n^1 is not for either $\mathcal{S}_1$ or $\mathcal{S}_2$. Finally, as 2-sections, $\mathcal{S}_1$ and $\mathcal{S}_2$ are 3-adjacent because of the 3-path that passes along P^0, through n^3, and then along P^2. However, as 0-sections (or as 1-sections), $\mathcal{S}_1$ and $\mathcal{S}_2$ are not 1-adjacent (respectively, 2-adjacent). ♣

Theorem 4.5. *Let the ν-graph $\mathcal{G}^\nu$ satisfy the following conditions.*

(a) *$\mathcal{G}^\nu$ is ν-connected.*

(b) *$\mathcal{G}^\nu$ satisfies Condition 4.1.*

(c) *For each rank ρ less than ν, every ρ-section has a countable $(\rho+1)$-adjacency.*

Then, $\mathcal{G}^\nu$ has a countable branch set.

Proof. We shall use transfinite induction, working with ranks rather than ordinals. First, choose any 0-section $\mathcal{S}^0$ and let b be any branch in $\mathcal{S}^0$. Since a $\vec{0}$-section is the same thing as a branch, hypothesis (c) asserts that there are only countably many branches adjacent to b. Set $\mathcal{H}_0 = \{b\}$. For each positive natural number k, let $\mathcal{H}_k$ be the set of branches each of which is adjacent to a branch of $\mathcal{H}_{k-1}$. Since, $\mathcal{S}^0$ is 0-connected, every two branches of $\mathcal{S}^0$ is connected by a finite 0-path in $\mathcal{S}^0$. Furthermore, no 0-path starting at b will leave $\mathcal{S}^0$. It follows that the branch set of $\mathcal{S}^0$ is $\cup_{k=0}^{\infty}\mathcal{H}_k$. Since a countable union of countable sets is countable, $\mathcal{S}^0$ is countable. (So far, we have argued a standard result [3, page 39].)

Next, let ρ be any ordinal rank and let $\mathcal{S}^\rho$ be any ρ-section. By hypothesis (b) and by [6, Corollary 6.4], $\mathcal{S}^\rho$ is partitioned by $(\rho-1)$-sections of $\mathcal{G}^\nu$. (Remember that $\rho - 1 = \vec{\rho}$ if ρ is a limit ordinal.) For the inductive hypothesis, assume that every $(\rho-1)$-section in $\mathcal{S}^\rho$ is countable. Let $\mathcal{S}^{\rho-1}$ be one of them and set $\mathcal{H}_0 = \{\mathcal{S}^{\rho-1}\}$. For each positive natural number k, let $\mathcal{H}_k$ be the set of $(\rho-1)$-sections, each of which is ρ-adjacent to a $(\rho-1)$-section of $\mathcal{H}_{k-1}$. Since $\mathcal{S}^\rho$ is ρ-connected, every two branches of $\mathcal{S}^\rho$ is connected by a finite ρ-path in $\mathcal{S}^\rho$. Moreover, every finite ρ-path meets only finitely many $(\rho-1)$-sections, and consecutively met $(\rho-1)$-sections are ρ-adjacent. Also, no ρ-path that meets $\mathcal{S}^{\rho-1}$ will ever leave $\mathcal{S}^\rho$. It follows that $\cup_{k=0}^{\infty}\mathcal{H}_k$ is the set of all $(\rho-1)$-sections in $\mathcal{S}^\rho$. By hypothesis (c) and the fact that a countable union of countable sets is countable, each $\mathcal{H}_k$ consists of only countably many $(\rho-1)$-sections, and so too does $\cup_{k=0}^{\infty}\mathcal{H}_k$. By the inductive hypothesis, each $(\rho-1)$-section in $\mathcal{S}^\rho$ is countable (i.e., has only countably many branches). Therefore, $\mathcal{S}^\rho$ is countable too.

Finally, let ρ be any arrow rank other than $\vec{0}$. Thus, $\rho = \vec{\theta}$, where θ is a limit ordinal. Let $\mathcal{S}^{\vec{\theta}}$ be any $\vec{\theta}$-section in $\mathcal{G}^\nu$. Let $\vec{\gamma}$ be the last arrow rank before $\vec{\theta}$; thus, γ is a limit ordinal and every rank larger than γ and less than $\vec{\theta}$ is a successor ordinal. Also, let η denote any rank such that $\gamma \leq \eta < \vec{\theta}$. Given any two branches b_1 and b_2 of $\mathcal{S}^{\vec{\theta}}$, there is for some η a finite η-path that meets them. This implies that, for $\gamma \leq \eta_1 < \eta_2 < \vec{\theta}$, any $\mathcal{S}^{\eta_1}$ in $\mathcal{S}^{\vec{\theta}}$ is contained in some $\mathcal{S}^{\eta_2}$ in $\mathcal{S}^{\vec{\theta}}$, and moreover

$$\mathcal{S}^{\vec{\theta}} = \bigcup_{\gamma \leq \eta < \vec{\theta}} \mathcal{S}^\eta, \tag{10}$$

where $\mathcal{S}^\gamma$ is any arbitrarily chosen γ-section in $\mathcal{S}^{\vec{\theta}}$ and $\mathcal{S}^{\eta_1} \subset \mathcal{S}^{\eta_2}$ for $\gamma \leq \eta_1 < \eta_2 < \vec{\theta}$. As our inductive hypothesis, we now assume that every $\vec{\gamma}$-section in $\mathcal{S}^{\vec{\theta}}$ is countable. This is trivially so for $\vec{\gamma} = \vec{0}$. By Hypothesis (c) and the preceding paragraph, every $\mathcal{S}^\gamma$ in $\mathcal{S}^{\vec{\theta}}$

is countable. In fact, we may successively apply the result of the preceding paragraph to assert that, for each η with $\gamma \leq \eta < \vec{\theta}$, every $\mathcal{S}^\eta$ in $\mathcal{S}^{\vec{\theta}}$ is countable. Hence, by (10), $\mathcal{S}^{\vec{\theta}}$ is countable too.

The last two paragraphs taken together provide a transfinitely inductive argument establishing the theorem. ♣

Actually, the last proof has also established the following result.

Corollary 4.6. *Let $\mathcal{S}^\gamma$ be a γ-section in a ν-graph $\mathcal{G}^\nu$, where γ is any rank no larger than ν. Assume that all tips traversed by $\mathcal{S}^\gamma$ satisfy Condition 4.1 and that, for each rank ρ less than γ, every ρ-section in $\mathcal{S}^\gamma$ has a countable $(\rho+1)$-adjacency. Then, $\mathcal{S}^\gamma$ has a countable branch set.*

5 Implications for Electrical Networks

A theory for transfinite electrical networks whose graphs are uncountable can be constructed exactly as it was for the case of countable graphs [4], [5]. Hilbert spaces for the current vectors are constructed exactly as before, but now inner products are summable series over uncountable branch-index sets. A discussion of summability for uncountable series is given in [2]. All the results of [4, Chapters 3 and 5] and [5] extend to uncountable networks with this simple change.

Let $\mathbf{N}^\nu$ be an electrical network with an uncountable graph and let J be the index set for its uncountably many branches. Consider the case where every branch b_j $(j \in J)$ is in the Thevenin form; that is, b_j is a series connection of a positive resistance r_j and a real-valued pure voltage source e_j. If $e_j = 0$, b_j is considered to consist only of r_j and is called *purely resistive.* The branch current i_j, branch voltage v_j, and branch voltage source e_j are measured in accordance with the polarities shown in Figure 6. Thus, $v_j = i_j r_j - e_j$. Also, $g_j = r_j^{-1}$ is the conductance of the branch b_j.

In the aforementioned theory, several restrictions are imposed upon $\mathbf{N}^\nu$ in order to obtain a unique voltage-current regime. For one thing, $\mathbf{N}^\nu$ is assumed to be ν-connected. This means that every two branches are connected by a two-ended ν-path P^ν. But, by definition of a ν-path, P^ν has no more than countably many branches. Thus, for example,

$\mathbf{N}^{\nu}$ cannot contain a series circuit having uncountably many branches.

But, ν-connectedness does not eliminate parallel circuits with uncountably many branches. Those are effectively removed by two other restrictions of the theory; namely, the total isolated source power is required to be finite:

$$\sum_{j \in J} e_j^2 g_j \; < \; \infty, \tag{11}$$

and the total dissipated power is also required to finite:

$$\sum_{j \in J} i_j^2 r_j \; < \; \infty. \tag{12}$$

When these two conditions hold, we say that we have a *finite-power voltage-current regime.*

These two condition imply that only countably many of the e_j and i_j can be nonzero. Indeed, choose a sequence $\{\epsilon_k\}_{k=1}^{\infty}$ of positive numbers tending to 0. For each k, the number of e_j for which $|e_j| > \epsilon_k$ must be finite if (11) is to hold, and similarly for the i_j. Hence, the asserted countability.

It follows that only countably many of the branch voltages $v_j = i_j r_j - e_j$ can be nonzero. So, in any uncountable parallel circuit at least one branch voltage must be 0. Since all branch voltages of the parallel circuit are the same, every branch voltage is 0. Hence, the two nodes of the parallel circuit can be shorted. This makes every one of its branches into a self-loop and effectively removes the parallel circuit so far as the voltage-current regime of the rest of the network is concerned. In particular, Tellegen's equation [4, page 79], which is the fundamental equation upon which the theory of infinite networks is based, remains satisfied after that removal.

But, how does one account for the currents in the countably many source branches of the parallel circuit as the parallel circuit is shorted? For example, all the voltage sources may be oriented the same way and may therefore feed currents all in the same direction from one node of the parallel circuit to the other node; these currents no longer impinge upon those nodes after the parallel circuit is removed. Does this not disturb the voltage-current regime in the rest of the network? The answer is that Kirchhoff's current law need not be satisfied at the two nodes of the parallel circuit because of the presence of the uncountably many purely resistive branches of the parallel circuit. In fact, an uncountable sum of positive conductances cannot be finite, and therefore both nodes of the parallel circuit cannot be restraining [4, page 81].

We can infer something more from (11) and (12). Except for countably many branches of $\mathbf{N}^\nu$, all other branches will be purely resistive with 0 branch currents. Those branches will be in a state of balance — such as the central branch of a balanced wheatstone bridge. Hence, they can be removed either by opening them or by shorting them. Whatever choices are made, the result will be a countable network whose branches carry the same voltages and currents as they do in the uncountable network $\mathbf{N}^\nu$. In fact, there will be an uncountable collection $\mathcal{C}$ of countable networks that are equivalent to $\mathbf{N}^\nu$ in this sense. This result also implies that *no generality is lost by confining our attention to countable electrical networks so far as the nonzero parts of the finite-power voltage-current regimes in uncountable networks are concerned.* But, doing so would not reveal the equivalences between the countable networks in $\mathcal{C}$ and the uncountable network $\mathbf{N}^\nu$ from which they are derived. Moreover, since we have made a binary choice between opening and shorting each of the removed branches and since the cardinality card$\mathcal{B}$ of the branch set $\mathcal{B}$ of $\mathcal{N}^\nu$ is equal to the cardinality of the set of removed branches (the two sets being uncountable and differing only by a countable set), the cardinality card$\mathcal{C}$ of $\mathcal{C}$ is higher than card$\mathcal{B}$. In fact, we have $\mathrm{card}\mathcal{C} = 2^{\mathrm{card}\mathcal{B}}$.

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Figure Legends

Figure 1. In this diagram, heavy dots denote 0-nodes, small circles denote 1-nodes, and double circles denote 2-nodes.

(a) A star graph consisting of a branch b and one-ended μ-paths ($\mu = 0, 1, 2, \ldots$) terminally incident through branches at the 0-node m^0 but otherwise totally disjoint. This is an $\vec{\omega}$-graph having no $\vec{\omega}$-node and no $\vec{\omega}$-path.

(b) A parallel connection of b and the P^μ ($\mu = 0, 1, 2, \ldots$) meeting terminally at m^0 through branches and at an $\vec{\omega}$-node $n^{\vec{\omega}}$ through tips. This is also an $\vec{\omega}$-graph. It has no $\vec{\omega}$-path.

(c) An $\vec{\omega}$-path. This too is an $\vec{\omega}$-graph. It has no $\vec{\omega}$-node.

Figure 2. (a) A ladder network $\mathcal{L}^0$. The a_k, b_k, and c_k denote branches, and n^0 is a 0-node. $\mathcal{L}^0$ has an infinite set of 0-tips, whose cardinality $\mathbf{c}$ is that of the continuum. The 0-tips are pairwise nondisconnectable.

(b) An uncountable 1-graph wherein a branch is appended to each 0-tip of $\mathcal{L}^0$.

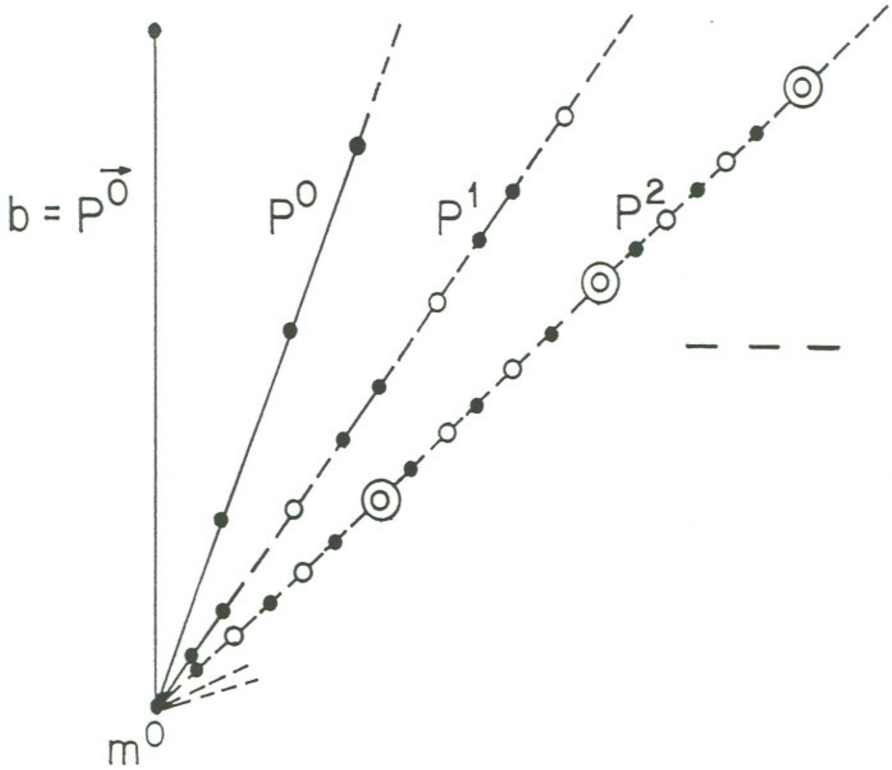
Figure 3. (a) The binary tree. The set of its 0-tips has the cardinality $\mathbf{c}$ of the continuum. The 0-tips are pairwise disconnectable.

(b) An uncountable 1-graph having a branch appended to each 0-tip of a binary tree.

Figure 4. The first three steps in the construction of a 1-graph all of whose maximal nodes are 1-nodes.

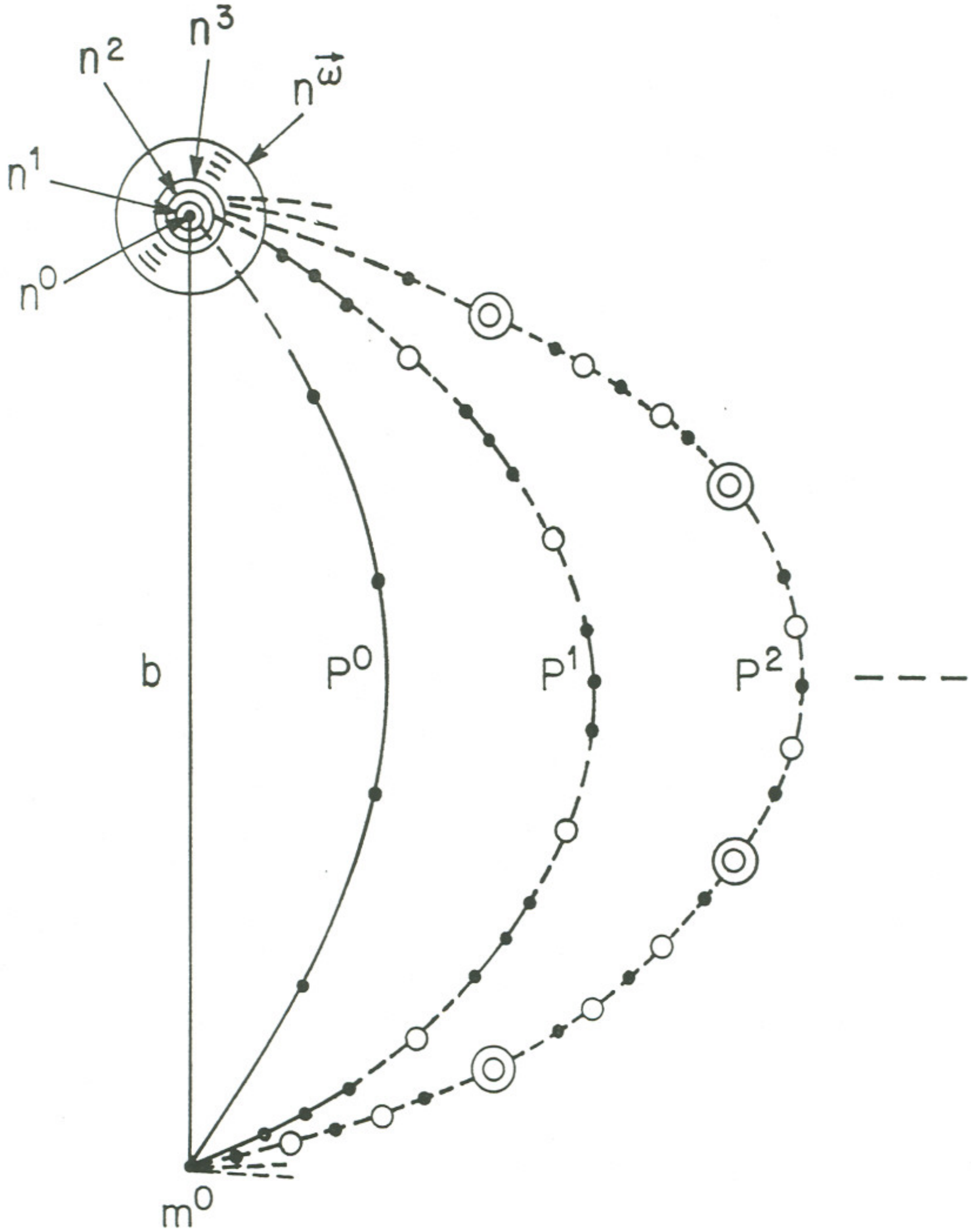
Figure 5. A 4-graph. The dots denote 0-nodes, the smallest circles are 1-nodes, the x's are 2-nodes, the double circles are 3-nodes, and the triple circle is a 4-node. P^μ ($\mu = 0, 2, 3$) denotes a one-ended μ-path. $\mathcal{S}_1$ and $\mathcal{S}_2$ are ρ-sections for each $\rho = 0, 1, 2$.

Figure 6. The Thevenin form of a branch.



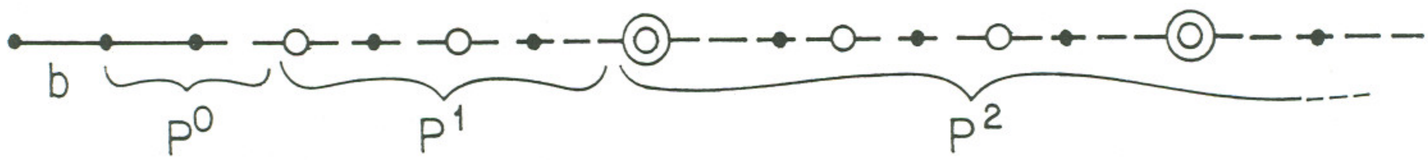
(a)

FIG. 1



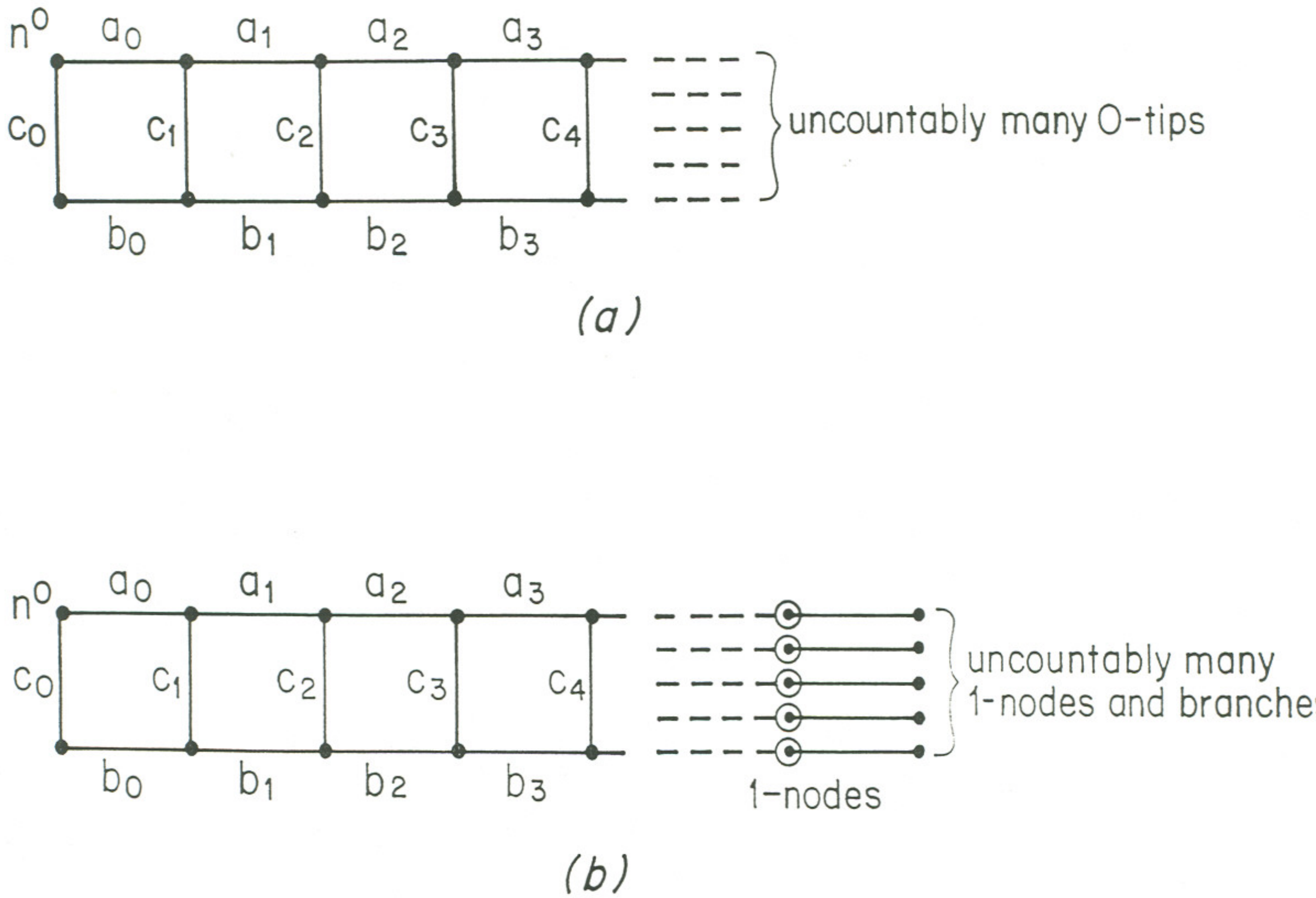
(b)

FIG. 1



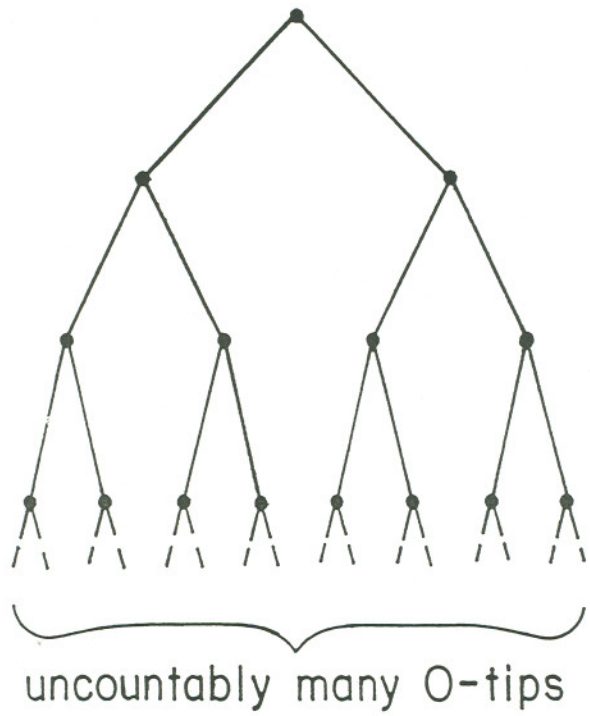
(c)

FIG. 1



n^0
a_0 a_1 a_2 a_3
c_0 c_1 c_2 c_3 c_4
uncountably many 0-tips
b_0 b_1 b_2 b_3
(a)
n^0
a_0 a_1 a_2 a_3
c_0 c_1 c_2 c_3 c_4
uncountably many
1-nodes and branches
b_0 b_1 b_2 b_3
1-nodes
(b)

FIG. 2


uncountably many 0-tips

(a)

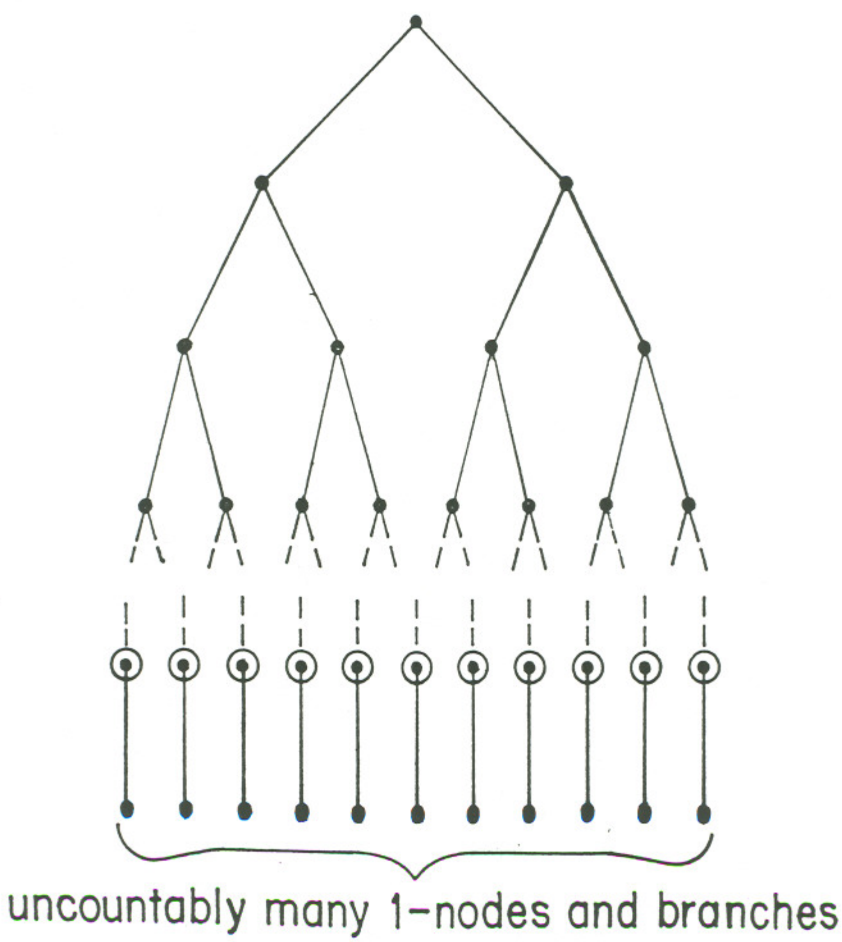

uncountably many 1-nodes and branches

FIG. 3

(b)

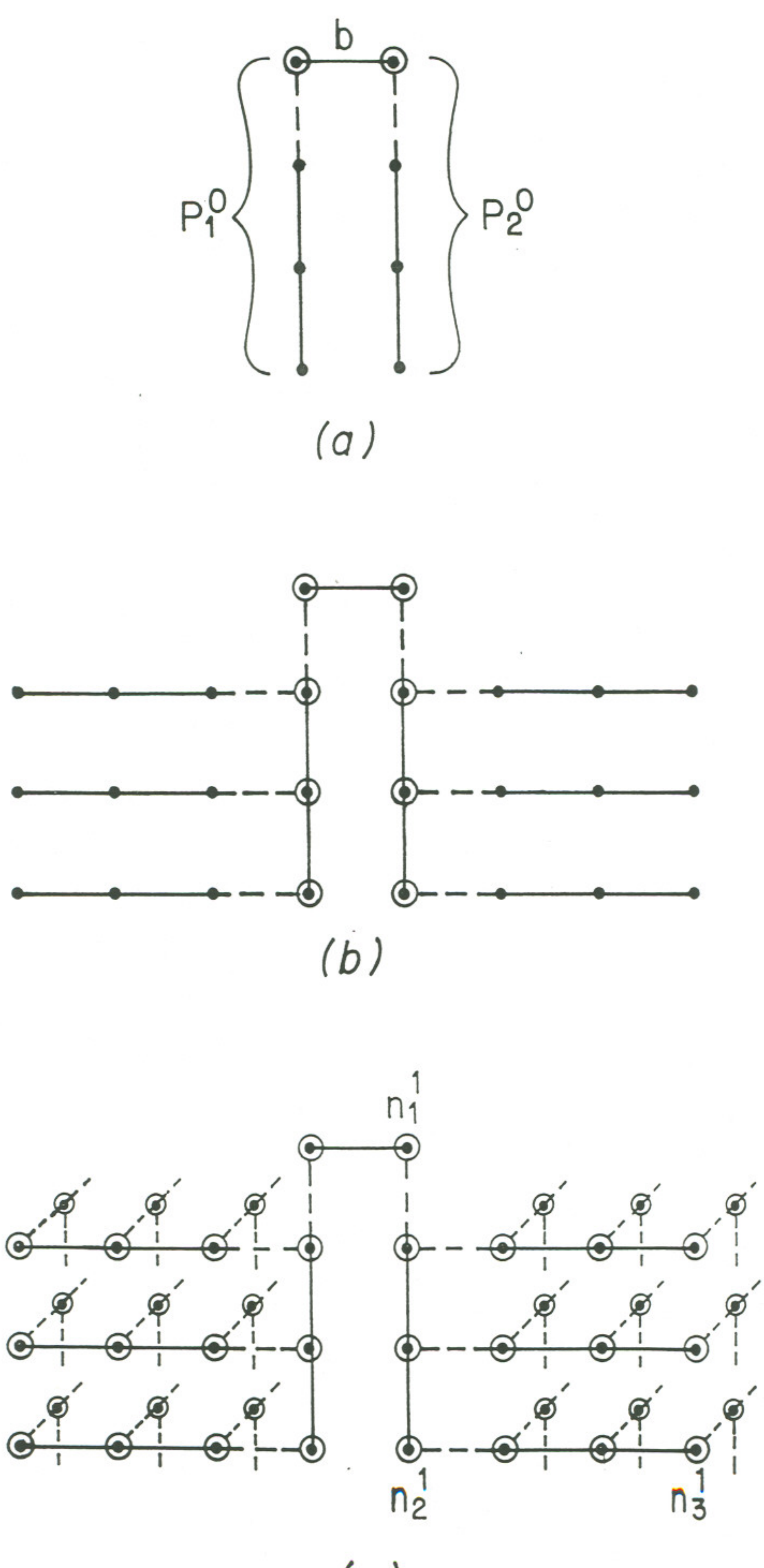


Fig. 4

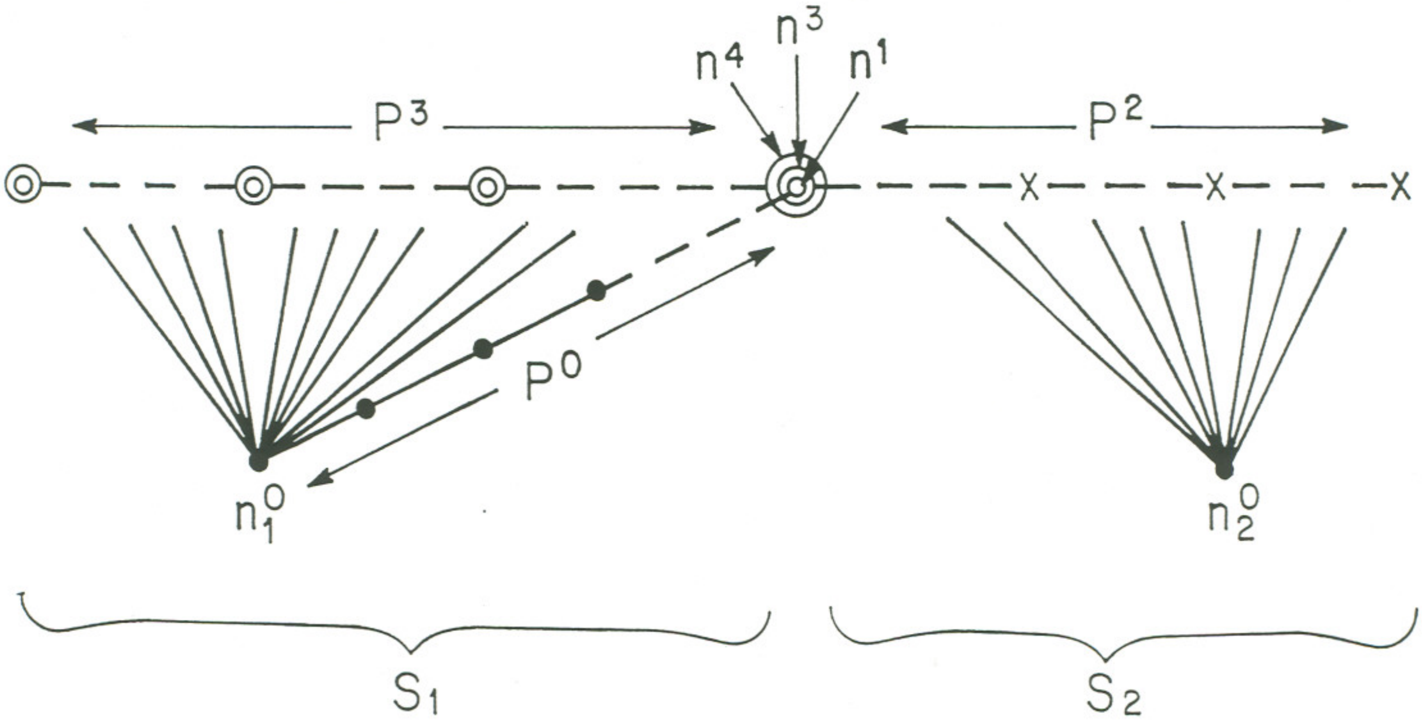


FIG. 5

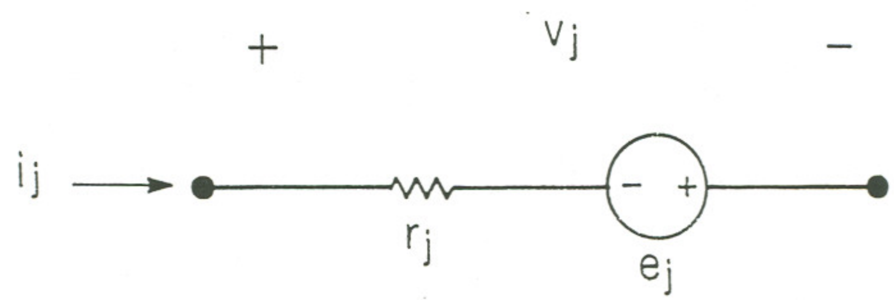


FIG. 6