UNIVERSITY AT STONY BROOK

CEAS Technical Report 722

NONUNIFORM NONLINEAR DISTRIBUTED RESISTIVE LINES AND NONUNIFORM LINEAR DISTRIBUTED RLGC LINES AS MEMBERS OF NETWORKS OF VLSI INTERCONNECTS

A. H. Zemanian

This work was supported by the National Science Foundation under Grant MIP-9423732.

April 3, 1996

NONUNIFORM NONLINEAR DISTRIBUTED RESISTIVE LINES AND NONUNIFORM LINEAR DISTRIBUTED RLGC LINES AS MEMBERS OF NETWORKS OF VLSI INTERCONNECTS *

A. H. Zemanian

Abstract — In a prior work, a remarkably rapid method was devised for analyzing networks of VLSI interconnects based upon their input-output mappings. However, the interconnects were modeled as lumped ladders, possibly nonuniform. In this work we extend that procedure by modeling the interconnects as (uniform or nonuniform) distributed lines, a more realistic representation. Two cases are considered. In the first, the interconnects can be nonlinear but must be purely resistive. In the second, the interconnects can be reactive as well as resistive but must be linear.

1 Introduction

The behavior of networks of VLSI interconnects is becoming an evermore important consideration in the design of semiconductor chips. In particular, the determination of the DC operating point and also the transient response of such a network is usually determined by applying various numerical procedures to find solutions of simultaneous, linear and nonlinear, differential equations. These have been incorporated into several versions of SPICE; but, SPICE is so robust, it is in fact inefficient for the special kind of networks that VLSI interconnections form. In a prior work [1] a method was devised that took advantage of the special structure of VLSI interconnection networks to radically reduce the number of unknown variables in the following way. The considered interconnection networks consisted of a number of lumped ladder networks or more generally of grounded two-port networks

*This work was supported by the National Science Foundation under Grant MIP-9423732.

joined together at their input and output terminals. Rather than treating the node voltages within each ladder or within each grounded two-port as unknowns — as SPICE would do, the ladders and grounded two-ports were described by input-output operators and only their input and output node voltages were taken as unknowns. This required an analysis of a network of operators, after which the internal-node voltages could be determined with very little additional computation.

However, that prior work was restricted to lumped parameter networks. Distributed parameter lines are nonetheless of considerable importance; in fact, they are more realistic representations of VLSI interconnects. The purpose of this paper is to show that distributed parameter lines also possess the required input-output characteristics needed for our operator-valued analysis, even when they are nonuniform and nonlinear in certain ways. In this fashion, the advantages of that analysis can be extended to a broader and more realistic class of interconnection networks.

In the following, R^n denotes n-dimensional real Euclidean space with its conventional norm $\|x\| = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}$ for $x = (x_1, \ldots, x_n) \in R^n$. $[a, b]$ denotes a compact interval in R^1 with endpoints a and b and $a < b$. $A \times B$ denotes the Cartesian product of two sets A and B. We shall take the adjectives "nonlinear" and "nonuniform" as subsuming "linear" and "uniform"; that is, "linear" is treated as a special case of "nonlinear," and "uniform" is treated as a special case of "nonuniform."

2 The Forward and Backward Mappings of a Grounded Two-Port

Consider the grounded two-port shown in Figure 1. The voltage-current pair $x_a = (v_a, i_a) \in R^2$ is its input, and $x_b = (v_b, i_b) \in R^2$ is its output voltage-current pair. The *forward mapping* $f : x_a \mapsto x_b$ and the *backward mapping* $b : x_b \mapsto x_a$ characterize that two-port. We need some restrictions on f and b if an analysis of a network of such two-ports is to be feasible. The following suffice for this purpose.

For $j = 1, 2, 3$, or 4, Q_j denotes the open jth quadrant in R^2, and $\overline{Q}_j$ denotes the closure of Q_j. Moreover, $\overline{Q}_j \backslash \{0\}$ denotes the closed jth quadrant but with the origin deleted.

Conditions 2.1.

(1) *f is a homeomorphism of R^2 onto R^2 with $f(0) = 0$ and $f(x) \neq x$ for $x \neq 0$. (That is, f is a bijection of R^2 onto R^2 and has exactly one fixed point, the origin; also, f and b are both continuous.)*

(2) **a.** *If $f(x) \in \overline{Q}_1 \backslash \{0\}$, then $x - f(x) \in Q_1$.*

b. *If $x \in \overline{Q}_2 \backslash \{0\}$, then $f(x) - x \in Q_2$.*

c. *If $f(x) \in \overline{Q}_3 \backslash \{0\}$, then $x - f(x) \in Q_3$.*

d. *If $x \in \overline{Q}_4 \backslash \{0\}$, then $f(x) - x \in Q_4$.*

(3) **a.** *If $x \neq y$, and if $y - x \in \overline{Q}_4$, then $f(y) - f(x) \in Q_4$ and $|y - x| < |f(y) - f(x)|$.*

b. *If $x \neq y$, and if $x - y \in \overline{Q}_1$, then $b(x) - b(y) \in Q_1$ and $|x - y| < |b(x) - b(y)|$.*

These are the slightly stronger versions of the conditions on nonlinear two-ports given in [1] that insure that networks of such two-ports have unique operating points.

3 A Nonuniform Nonlinear Resistive Distributed Line

Our first objective is to show that the nonuniform nonlinear resistive distributed line shown in Figure 2 has forward and backward mappings that satisfy Conditions 2.1 whenever Conditions 3.1, given below, are satisfied. In Figure 2, $l - a$ is the distance along the line from the input at $l = a$; the output of the line is at $l = b$, where $b > a$. Thus, $a \leq l \leq b$. The line voltage $v(l)$ and line current $i(l)$ depend upon l and are measured in accordance with the polarities shown in Figure 2. The notations for the input and output variables can be made to conform with those of Figure 1 by setting $v_a = v(a)$, $i_a = i(a)$, $v_b = v(b)$, and $i_b = i(b)$. Furthermore, we shall abuse notation by letting v and i be the values $v(l)$ and $i(l)$ of the line voltage and line current at an unspecified point l along the line; thus, v and i denote the range values $v(l)$ and $i(l)$ in volts and amperes respectively, instead of the corresponding mappings $l \mapsto v(l)$ and $l \mapsto i(l)$. (A more precise notation would encumber our equations unnecessarily.) Furthermore, $r(l, i)$ will denote the rate of voltage drop at the point l and current i, and $g(l, v)$ will denote the rate of current decrease at the point l and voltage v.

Thus, r and g represent the effects of distributed, nonuniform, and nonlinear series resistance and conductance-to-ground for the line. With the prime denoting differentiation with respect to l, we have the line equations:

$$v'(l) = -r(l, i(l)) \tag{1}$$

$$i'(l) = -g(l, v(l)) \tag{2}$$

They govern the variations of v and i in the forward direction from input to output. Moreover, they can be written more concisely by using a matrix-like notation. Set $x(l) = (v(l), i(l)) \in R^2$ for each $l \in [a, b]$. Then,

$$x'(l) = \begin{bmatrix} v'(l) \\ i'(l) \end{bmatrix} = \begin{bmatrix} 0 & -r(l, \cdot) \\ -g(l, \cdot) & 0 \end{bmatrix} \begin{bmatrix} v(l) \\ i(l) \end{bmatrix} = F(l, x(l)) \tag{3}$$

Thus, $F : R^2 \leadsto R^2$ is a mapping of R^2 into itself. Here too, we will abuse notation by letting x and $x(l)$ denote the same value in R^2.

A solution $x(l) = (v(l), i(l))$ to these equations for $a \leq l \leq b$ with the initial condition $x(a) = (v_a, i_a)$ will be called the *forward trajectory starting at the point* x_a. The variations in the backward direction from output to input are governed by the same equations (just shift the minus signs onto dl and use $x_b = (v_b, i_b)$ in place of x_a). In this case, we have the *backward trajectory starting at the point* x_b.

In the following, we let W represent some nonvoid open interval in R^1. We shall say that $r(l, i)$ *is Lipschitz in* i *on* W *and that* $g(l, v)$ *is Lipschitz in* v *on* W *both uniformly with respect to all* $l \in [a, b]$ if there exists a constant K_W not depending on l such that

$$|r(l, i) - r(l, \hat{i})| = K_W |i - \hat{i}| \tag{4}$$

and

$$|g(l, v) - g(l, \hat{v})| = K_W |v - \hat{v}| \tag{5}$$

for all $l \in [a, b]$ and for all $i, \hat{i}, v, \hat{v} \in R^1$. These conditions do not restrict the growth of $r(l, i)$ or of $g(l, v)$ as $|i|$ or $|v|$ tends to ∞; we need merely increase the constant K_W as W increases in size.

Henceforth, the following conditions are assumed.

Conditions 3.1.

(1) $r(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are continuous functions from $[a,b]\times R^1$ into R^1. Moreover, for each l, we have that $r(l,0)=g(l,0)=0$ and that $r(l,\cdot)$ and $g(l,\cdot)$ are strictly increasing functions of their second arguments.

(2) For each nonvoid open interval W in R^1, $r(l,i)$ is Lipschitz in i on W and $g(l,v)$ is Lipschitz in v on W both uniformly with respect to all $l \in [a,b]$.

Condition 3.1(1) implies that $F(\cdot,\cdot)$ is continuous from $[a,b]\times R^2$ into R^2. By Condition 3.1(2), $F(l,x)$ is Lipschitz with respect to x on $W\times W$ uniformly for all $l\in[a,b]$; that is,

$$||F(l,x)-F(l,\hat{x}|| \;\leq\; K_W||x-\hat{x}||. \tag{6}$$

for all $l\in[a,b]$ and all $x,\hat{x}\in W\times W$. These properties allow us to invoke a standard theorem (see, for instance, Theorem 1 on page 297 of [2]) to assert the following.

Theorem 3.2. *Let $l_0\in[a,b]$ and $x_0\in R^2$. Assume Conditions 3.1. Then, there exists a unique solution $x(l)$ to (3) defined for all $l\in[a,b]$ such that $x(l_0)=x_0$. Moreover, the mapping $l\mapsto x(l)$ is differentiable from $[a,b]$ into R^2.*

Proof. The cited theorem in [2] only asserts the unique existence of a differentiable x on an open interval $J\subset[a,b]$ with $l_0\in J$. However, since the conclusion holds for every $l_0\in[a,b]$, we can get the unique existence of the trajectory for all $l\in[a,b]$ by piecing together the open intervals (open with respect to $[a,b]$) around each point of $[a,b]$. Indeed, if J is one such interval and if c is one of its limit end points, then there will be another open interval $\hat{J}$ containing c, and the two solutions x on J and $\hat{x}$ on $\hat{J}$ must coincide on $J\cap\hat{J}$ if that cited theorem is to hold at any point of $J\cap\hat{J}$. ♣

Thus, given any input x_a for the line, the unique forward trajectory starting at x_a defines uniquely an output x_b. Moreover, the backward trajectory from output to input is governed by exactly the same differential equations (1) and (2) as is the forward trajectory. Hence, the forward mapping $f: x_a\mapsto x_b$ is a bijection from R^2 onto R^2, and f^{-1} is the corresponding backward mapping b.

Note also that, by Condition 3.1(1), the right-hand sides of (1) and (2) are simultaneously 0 only at the origin $(0,0)$ of the voltage-current plane. Hence, $f(x)\neq 0$ if $x\neq 0$, and $f(0)=0$; in short, f has only one fixed point, the origin of R^2. Thus, the trajectory starting at the origin remains at the origin, but all other trajectories satisfy $x(l) \neq x(\hat{l})$ if $l \neq \hat{l}$. Furthermore, by virtue of another standard theorem (see the theorem on page 169 of [2]), the forward and backward mappings are both continuous from R^2 onto R^2. (The cited theorem is stated for the autonomous (i.e., uniform) case, but its proof is given for the nonautonomous case.) All this shows that, under Conditions 3.1, the forward mapping f and backward mapping b satisfy all of Condition 2.1(1).

Our next objective is to verify that Condition 2.1(2) is satisfied whenever Conditions 3.1 hold. Again with the prime denoting d/dl, $x'(l) = (v'(l), i'(l))$ is the *tangent vector* for a trajectory $x(l)$ at the point l on the line. According to Condition 3.1(1), the right-hand side of (1) is negative (resp. positive) when $i(l)$ is positive (resp. negative), and similarly for the right-hand side of (2). Because of this, the tangent vector $x'(l)$ has certain directions. To specify them, we need some more notation. We have already defined Q_j and $\overline{Q}_j$ as the open and closed jth quadrants in R^2. In addition, set $V_+ = \{(v,0) \in R^2 : v > 0\}$, $V_- = -V_+ = \{(v,0) \in R^2 : v < 0\}$, $I_+ = \{(0,i) \in R^2 : i > 0\}$, and $I_- = -I_+ = \{(0,i) \in R^2 : i < 0\}$. Thus, V_+, V_-, I_+, and I_- denote sets in R^2, whereas the voltage v and the current i are one-dimensional variables. The directions of the tangent vector are as follows, whatever be l. If $x \in Q_1$, then $x' \in Q_3$. If $x \in Q_2$, then $x' \in Q_2$. If $x \in Q_3$,then $x' \in Q_1$. If $x \in Q_4$, then $x' \in Q_4$. If $x \in V_+$, then $x' \in I_-$. If $x \in I_+$, then $x' \in V_-$. If $x \in V_-$, then $x' \in I_+$. If $x \in I_-$, then $x' \in V_+$. These are illustrated in Figure 3. Since these results hold for every l and since trajectories cannot stay fixed except at the origin, an immediate consequence of all this is that the forward mapping of the the line satisfies Conditions 2.1(2).

Finally, to verify that Conditions 2.1(3) are fulfilled, let $y = (v_y, i_y)$ and $x = (v_x, i_x)$ be two different points in the voltage-current plane for a point l on the line, and consider the difference $y' - x'$ between the tangent vectors. By (1) and (2) again,

$$y'(l) - x'(l) \;=\; (r(l, i_x(l)) - r(l, i_y(l))\,,\, g(l, v_x(l)) - g(l, v_y(l)))\,.$$

According to the monotonicity properties asserted in Condition 3.1(1), we have the following results at each value of l. If $y - x \in Q_4$, then $y' - x' \in Q_4$. If $y - x \in V_+$, then $y' - x' \in I_-$. If $y - x \in I_-$, then $y' - x' \in V_+$. Again since these hold whatever be l and since trajectories cannot stay fixed except at the origin, it follows immediately once more that the forward mapping of the line satisfies Condition 2.1(3)a. A very similar argument shows that the line's backward mapping satisfies Conditions 2.1(3)b.

Altogether then, we have shown

Theorem 3.3. *A nonuniform, nonlinear, distributed resistive line whose parameters satisfy Conditions 3.1 has a forward mapping and a backward mapping that fulfill Conditions 2.1.*

Since the latter conditions are a somewhat stronger version of the Conditions B of [1], we can conclude that such distributed lines are allowable two-port elements for the nonlinear resistive transmission networks analyzed in [1].

4 Nonuniform Linear Distributed RLGC Lines

So far, we have been dealing only with purely resistive lines. We will now consider lines having distributed series inductances and capacitances-to-ground — in addition to distributed series resistances and conductances-to-ground — but will require that all these parameters be linear with respect to voltages and currents. On the other hand, nonuniformity with respect to l will still be permitted. The transient behavior of networks of such lines can be analyzed through the Laplace transformation in the following way.

For any fixed positive value of the complex variable s introduced by the Laplace transformation, the transformed line appears as a nonuniform linear resistive line. Thus, under an additional condition on the distributed linear parameters (namely, their continuity for all l), the transformed line will in fact satisfy Conditions 3.1, and therefore Theorems 3.2 and 3.3 can be invoked. Consequently, we can apply the analysis of [1] to analyze a Laplace-transformed network of such resistance-reactance lines for any fixed $s > 0$. This finally allows us to employ some standard algorithms that invert the Laplace transformation through a knowledge of the transform at finitely many points on the positive s axis. Whence, the transients of the said network can be determined numerically.

Let us be more specific. The line voltage $v(l,t)$ and line current $i(l,t)$ are now time-varying and depend upon the spatial distance l along the line and the time t. With the same polarities as those shown in Figure 2, the equations governing the line are now

$$\frac{\partial v(l,t)}{\partial l} = -L(l)\frac{\partial i(l,t)}{\partial t} - R(l)i(l,t), \tag{7}$$

$$\frac{\partial i(l,t)}{\partial l} = -C(l)\frac{\partial v(l,t)}{\partial t} - G(l)v(l,t). \tag{8}$$

The distributed parameters $L(l)$, $R(l)$, $C(l)$, and $G(l)$ are respectively the series inductance in henries per meter, the series resistance in ohms per meter, the capacitance-to-ground in farads per meter, and the conductance-to-ground in mhos per meter. Upon applying the distributional Laplace transformation [4], we convert these equations into the following.

$$\frac{dV(l,s)}{dl} = -(L(l)s + R(l))I(l,s) \tag{9}$$

$$\frac{dI(l,s)}{dl} = -(C(l)s + G(l))V(l,s) \tag{10}$$

(The use of the distributional Laplace transformation allows us to incorporate initial values into the transforms of derivatives.) In order to invoke the results of Section 3, we need merely assume the following.

Conditions 4.1. *L, R, C, and G are continuous positive functions of l on the compact interval $[a,b]$.*

Hence, there are two positive constants K and M such that $L(l)$, $R(l)$, $C(l)$, and $G(l)$ are all bounded above by K and bounded below by M. Again we abuse notation by letting V represent $V(l,s)$ and I represent $I(l,s)$, rather than the corresponding functions. Then, for fixed $s > 0$ again,

$$(l,I) \mapsto (L(l)s + R(l))I$$

and

$$(l,V) \mapsto (C(l)s + G(l))V$$

are continuous functions from $[a,b] \times R^1$ into R^1, which obviously satisfy all of Conditions 3.1 (with K replaced by $Ks + K$). Upon setting $X = X(l,s) = (V(l,s), I(l,s))$, we can therefore restate Theorem 3.2 for this case as follows.

Theorem 4.2. *Fix $s > 0$, and choose $l_0 \in [a,b]$ and $X_0(s) \in R^2$. Assume Conditions 4.1. Then, there exists a unique solution $X(l,s)$ to (9) and (10) defined for all $l \in [a,b]$*

such that $X(l_0, s) = X_0(s)$. *Moreover, the mapping* $l \mapsto X(l, s)$ *is differentiable from* $[a, b]$ *into* R^2.

Needless to say, all the other results of Section 3 also carry over to the present case. In particular, in the Laplace-transform domain, the line has an input-output representation with a forward mapping $F(s) : X(a, s) \mapsto X(b, s)$ and a backward mapping $B(s) = F(s)^{-1} : X(b, s) \mapsto X(a, s)$. Thus, we have the following.

Theorem 4.3. *A nonuniform linear distributed RLGC line whose parameters satisfy Condition 4.1 has a Laplace-transformed input-output representation whose forward mapping* $F(s)$ *and backward mapping* $B(s)$ *satisfy Conditions 2.1 (when* f *is replaced by* $F(s)$ *and* b *by* $B(s)$ *and where* s *is a fixed positive number.)*

As was mentioned above, any network of such lines can be analyzed by the method of [1] to get the Laplace transform of any voltage or current at finitely many positive values of s, and this data can then be used to numerically determine that voltage or current in the time domain. One such algorithm that works for node voltages when the inductances $L(l)$ are small as compared to the capacitances $C(l)$ is the Gaver-Stehfest algorithm [3].

References

[1] Yaw-Ruey Chan and A.H. Zemanian, *Analysis of Nonlinear Resistive Transmission Networks*, CEAS Technical Report 688, University at Stony Brook, Stony Brook, N.Y., May 16, 1994.

[2] M.W.Hirsch and S.Smale, *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic Press, New York, 1974.

[3] H.Stehfest, Numerical inversion of Laplace transforms, *Communications of the ACM*, **13** (1970), 47-49 and 624.

[4] A.H.Zemanian, *Generalized Integral Transformations*, Dover Publications, New York, 1987.

Legends for Figures

Figure 1. A grounded two-port showing the polarity conventions for the voltage-current pairs of its input (v_a, i_a) and it output (v_b, i_b).

Figure 2. A nonuniform nonlinear distributed resistive line. Here, l denotes a point on the line with $a \leq l \leq b$, and $v(l)$ and $i(l)$ are respectively the line voltage measured with respect to ground and the line current measured toward the output.

Figure 3. The directions of the tangent vector x' at various points of the voltage-current plane.

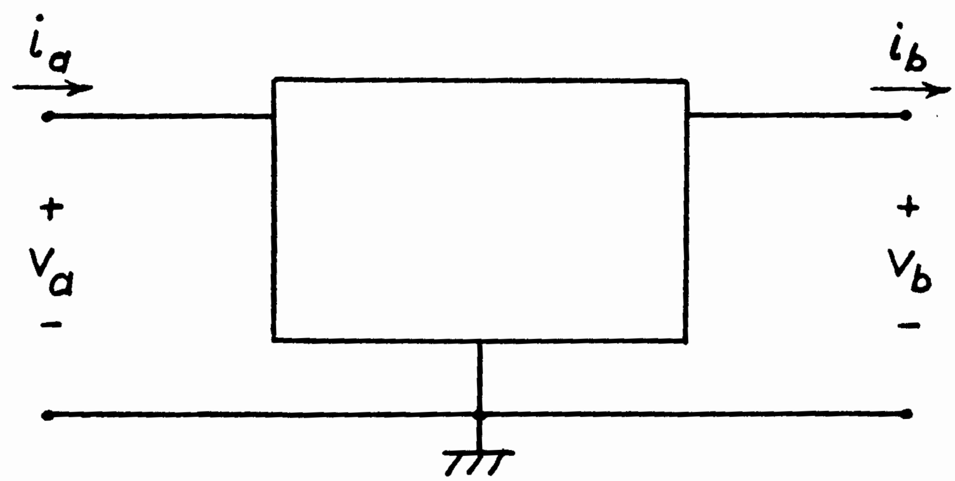

Fig. 1

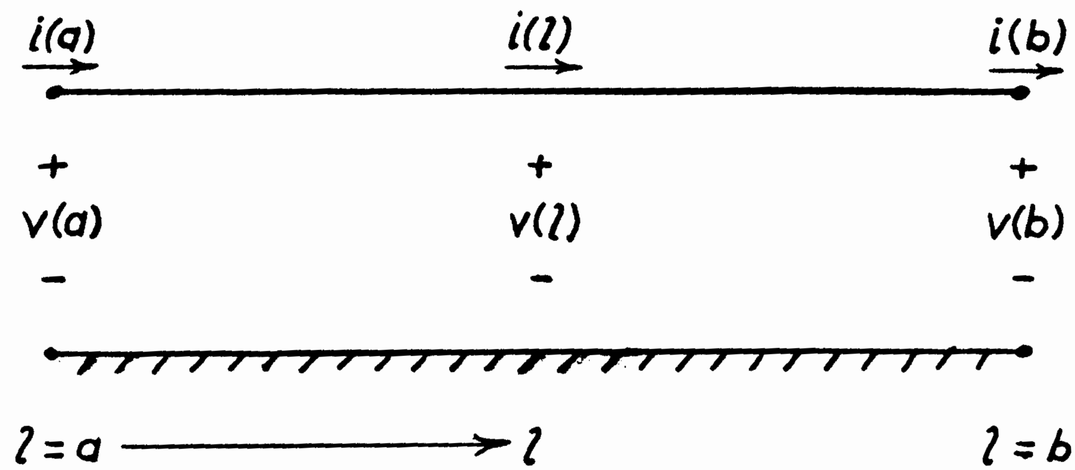

Fig. 2

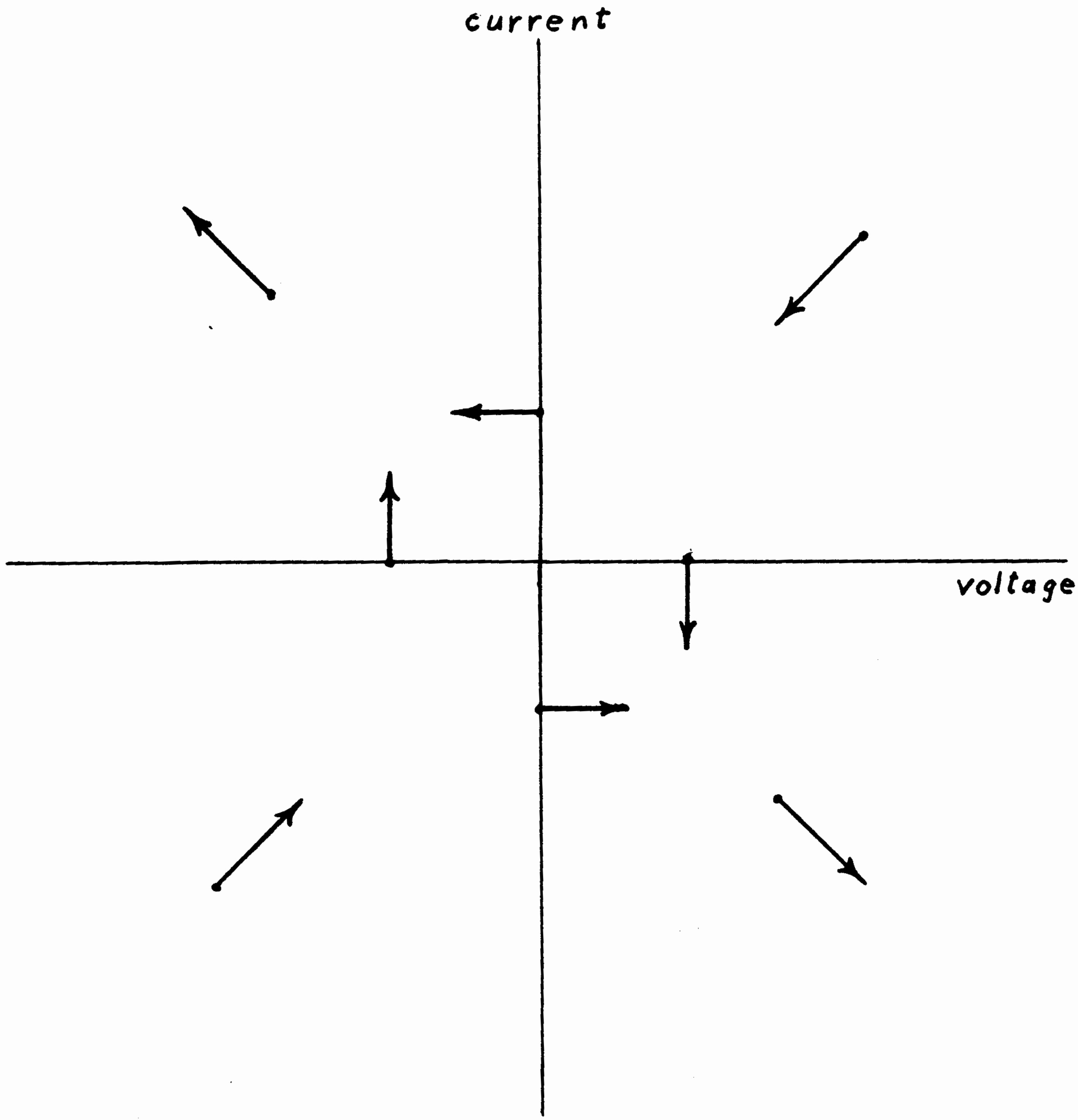

Fig. 3