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NONSTANDARD TRANSFINITE GRAPHS AND RANDOM WALKS ON THEM

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Abstract — It is shown that transfinite graphs can be generated through two simple operations, called "appending a branch" and "inserting a branch," applied to finite graphs infinitely many times. In this way, transfinite graphs are natural extensions of conventional graphs. Certain pathological transfinite graphs cannot be so obtained. For those transfinite graphs that can, a specific procedure for obtaining them from an expanding sequence of finite subgraphs is established. That sequence of finite subgraphs determines a "nonstandard graph," the sequence being a representative of an equivalence class of sequences of finite graphs modulo a chosen nonprincipal ultrafilter. This mimics a definition of the hyperreal numbers as equivalence classes of sequences of real numbers. With such a nonstandard transfinite graph in hand, random walks on it can be obtained by considering random walks on a representative sequence of finite subgraphs. This allows us to lift many standard results into a nonstandard setting, but now probabilities will be hyperreals. A typical result is that we can now quantify and compare infinitesimal escape probabilities from different nodes, thereby comparing probabilities of recurrence. All the many restrictions needed to establish random walks on standard transfinite graphs are no longer required.

Key Words: Transfinite graphs, nonstandard graphs, nonstandard random walks, nonstandard analysis.

1 Introduction

The relatively recent theory of transfinite graphs was inspired by several problems concerning infinite electrical networks that led to the question of how connections can be made to their infinite extremities [18], [19, Sec. 1.6], [20, Sec. 1.4], [22, Sec. 1.3]. A primary objective of this paper is to point out how transfinite graphs arise in a strictly graphical context from two simple operations on finite graphs repeated infinitely often. These produce not merely the conventionally infinite graphs but far more generally a hierarchy of "buildable" transfinite graphs. We argue that transfinite graphs are not just artificial constructs introduced merely for the sake of generalization but rather are inevitable consequences of introducing branches sequentially by "appending" and "inserting" them into finite graphs. From this point of view, one might claim that transfinite graphs have simply been overlooked until a dozen years ago.

A consequence of this way of generating transfinite graphs is that the natural language for transfinite electrical networks and transfinite random walks is nonstandard analysis—rather than standard analysis, and this in turn leads to more powerful theories for those subjects. Indeed, standard analysis has required the imposition of a variety of restrictions in order to ensure the convergence of expressions for voltages, currents, and probabilities. Finite total electrical power and local finiteness for the graphs are commonly imposed, but other conditions are also employed. See, for example, [3] - [10], [12], [15] — [17], [20], [22]. Even then, difficulties continue to arise such as the occasional collapse of Kirchhoff's laws or the zero probability of a random walker leaving a transfinite node—leading to the conditioning of probabilities on the exceptional cases where the random walker does leave transfinite nodes [20, Sec. 7.3], [22, Sec. 8.3]. All such restrictions are no longer needed under nonstandard analysis because answers are now obtained in terms of hyperreal, rather than real, numbers. Moreover, basic principles are restored, as for example Kirchhoff's laws expressed in terms of hyperreals. Similarly, random walks need no longer be restricted to those that leave transfinite nodes; they will now always do so—albeit with perhaps infinitesimal probabilities. Furthermore, a standard random walk on a conventionally infinite network is either recurrent or not. The nonstandard approach allows recurrent walks to be compared as being more recurrent or less recurrent by comparing infinitesimal escape probabilities. Other such advantages are explored in Sec. 12.

Nonstandard electrical networks were introduced in [21] as connections of finitely many, internally infinite one-ports, whose hyperreal terminal parameters were determined from sequences of finite subnetworks. Our present approach is more general and subsumes those prior results. With regard to the graph-structure alone, a nonstandard graph is constructed in a strictly graphical way as the result of an expanding sequence of finite graphs, and it depends upon the choice of that sequence.

To conform with our prior works on transfinite graphs, we use electrical terminology, for example, writing "branches" instead of "edges" and "nodes" instead of "vertices."

Throughout this work, we allow self-loops (i.e., branches that are incident to only one node) and parallel branches (i.e., multiple edges). We have to do so because, when shorting a branch—as we shall do later on, any other branch in parallel with it will become a self-loop. Similarly, two branches that were not in parallel may become parallel branches.

To make this work more accessible, we very briefly present the basic concepts and definitions for transfinite graphs in Sec. 2 and for nonstandard analysis in Sec. 3. Also, we augment this with additional brief explanations and references as still other concepts arise later on.

2 Transfinite Graphs

Let us first indicate heuristically how transfiniteness might be introduced into graph theory by examining a simple example. Then, we will summarize the definitions concerning transfinite graphs that will be used herein.

The first challenge is to define in some way the extremities of a conventionally infinite graph, for connections at those extremities will produce the first rank of graphical transfiniteness. There are presently four ways of specifying such extremities—by using paths [20, Chap. 2], more generally by using walks [23], by using a hierarchy of metrics based upon branch resistances [5], [22], and by adapting nonstandard analysis to graphs—as is done herein. However, the last approach exploits the first, as so we now explain the first.

Consider the one-way infinite ladder of Fig. 1(a) consisting of the branches b_k but not branches β_1 and β_0. z^0 is a conventional node of infinite degree. We wish to define precisely the extremity reached by the one-ended (i.e., one-way infinite) path P^0 through the branches $b_1, b_3, b_5, \ldots$. However, that extremity should be independent of how P^0 proceeds through finite portions of the graph. So, any other path that is eventually identical to P^0 will be considered to be equivalent to P^0. That extremity is then taken to be the equivalence class of all one-ended paths that are eventually identical to P^0. We call that extremity a 0-tip and refer to all conventional nodes such as x^0, y^0, and z^0 as 0-nodes. Moreover, we associate that 0-tip with one end of branch β_1 to obtain a node of higher rank, a 1-node w^1. In this way, the one-way infinite ladder is connected at this extremity to β_1 through w^1. This yields the transfinite graph of rank 1 indicated in Fig. 1(a).

Moreover, we may replace the branches β_1 and $\beta 0$ by another one-way infinite ladder and can repeat this procedure infinitely often to obtain an infinite cascade of infinite ladders having a transfinite path along its upper branches that reaches an extremity of higher rank, a 1-tip, which in turn can be associated with another branch end to get a node of rank 2, a 2-node. More repetitions of this kind yield transfinite ladders of still higher ranks.

In general and precisely, a *transfinite graph* $\mathcal{G}^\nu$ *of rank* ν is a set

$$\mathcal{G}^\nu = (\mathcal{B}, \mathcal{N}^0, \mathcal{N}^1, \ldots, \mathcal{N}^\nu) \tag{1}$$

defined as follows: The rank ν is either a natural number or a countable transfinite ordinal. In [19], [20], or [22], we only presented in detail the cases where $0 \leq \nu \leq \omega$, ω being the first transfinite ordinal. That presentation extends to many higher ranks without any substantial alterations. Here too, we will restrict ν to $0 \leq \nu \leq \omega$, $\nu \neq \vec{\omega}$.[1] The ellipses ... in (1) represent finitely many $\mathcal{N}^\rho$ if $\nu < \omega$ and infinitely many $\mathcal{N}^\rho$ if $\nu = \omega$. $\mathcal{B}$ is a set of branches, and each branch is a pair of tips of rank -1, called *elementary tips*. In this paper, we assume that $\mathcal{B}$ is a countably infinite set. Each $\mathcal{N}^\rho$ $(0 \leq \rho \leq \nu)$ is a set of nodes of rank ρ (i.e., ρ-nodes). Each *ρ-node* x^ρ is a set consisting of tips of rank $\rho - 1$ (i.e., $(\rho - 1)$-tips and possibly a single node x^α of lower rank $(0 \leq \alpha < \rho)$). For $\rho > 0$, a *$(\rho - 1)$-tip* is an equivalence class of one-ended (i.e., one-way infinite) paths that proceed through an infinite sequence of $(\rho - 1)$-nodes (and nodes of lower ranks as well if $\rho > 0$), two such paths being considered *equivalent* if they are identical after finitely many $(\rho - 1)$-nodes. A $(\rho - 1)$-tip is called *open* if it is the sole member of a singleton ρ-node; otherwise, it is called *nonopen*. In

[1] In this paper, we do not allow $\vec{\omega}$-graphs. The construction of nonstandard $\vec{\omega}$-graphs can be accompished by making each $\vec{\omega}$-tip the sole member of singleton ω-node and then proceeding with the resulting ω-graph as indicated below.

this way, $\mathcal{G}^\nu$ is determined from its tips of ranks $-1, \ldots, \nu-1$ and how they are assigned to nodes. See [19, Chap. 5], [20, Chap. 2], [22, Chap. 2] for more specifics. If $\nu = 0$, then $\mathcal{G}^\nu = \mathcal{G}^0$ is a conventional graph. Throughout this work, we assume that $\mathcal{G}^\nu$ is *connected*; that is, for any two nodes of any ranks in $\mathcal{G}^\nu$, there is a two-ended[2] path of some rank in $\mathcal{G}^\nu$ that terminates to those two nodes.

Furthermore, each ρ-node is said to *embrace* its $(\rho-1)$-tips, all the $(\alpha-1)$-tips of its single α-node x^α $(\alpha < \rho)$, if it has such a node, all the $(\beta-1)$-tips of the single β-node x^β $(\beta < \alpha)$ in x^α, if such a node exists, and so on down through nodes of decreasing ranks. Also, a node is called *maximal* if it is not contained in a node of higher rank. Thus, every tip of $\mathcal{G}^\nu$ is embraced by some maximal node, and the set of maximal nodes is a partition of the set of all tips of all ranks for $\mathcal{G}^\nu$. Furthermore, two tips (not necessarily of the same rank) are said to be *shorted together* if they are embraced by the same node. Altogether, $\mathcal{G}^\nu$ is uniquely determined by a specification of which pairs of tips are shorted together. In fact, we have the following proposition, whose precise proof is derived from the definitions given in either [19], [20], or [22].

Proposition 2.1. *Two ν-graphs are the same (i.e., isomorphic) if there is a bijection between their sets of tips such that two tips in one ν-graph are shorted together if and only if the corresponding tips in the other ν-graph are shorted together.*

3 Some Elements of Nonstandard Analysis

Our construction of nonstandard graphs mimics the construction of the hyperreal numbers. Moreover, the hyperreal numbers will be employed in Sec. 12 when we examine nonstandard random walks. Furthermore, nonstandard analysis comes in various forms with divergent notations. So, let us summarize the definitions we shall use. For the most part, we adopt the terminology and symbolism in [11].

Let $I\!N = \{0, 1, 2, \ldots\}$ be the set of all the natural numbers. A *nonprincipal ultrafilter $\mathcal{F}$ on $I\!N$* is a collection of nonempty subsets of $I\!N$ satisfying the following axioms:

1. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

[2]The nodes of highest rank in the path are finite in number.

2. If $A \in \mathcal{F}$ and $A \subset B \subset \mathbb{N}$, then $B \in \mathcal{F}$.

3. For any $A \subset \mathbb{N}$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$ but not both. Here, $A^c = \mathbb{N} \backslash A$ denotes the complement of A in $\mathbb{N}$.

4. No finite subset of $\mathbb{N}$ is a member of $\mathcal{F}$.

Consequently, the following properties hold:

a. $\emptyset \notin \mathcal{F}$ and $\mathbb{N} \in \mathcal{F}$, where $\emptyset$ denotes the empty set.

b. If $\{A_1, A_2, \ldots, A_k\}$ is a finite collection of mutually disjoint subsets of $\mathcal{F}$, then no more than one of them is a member of $\mathcal{F}$. If in addition $\cup_{j=1}^{k} A_j = \mathbb{N}$, then exactly one of the A_j is a member of $\mathcal{F}$.

c. Every cofinite set (i.e., the complement of a finite set) in $\mathbb{N}$ is a member of $\mathcal{F}$.

d. $\mathcal{F}$ is a maximal filter in the following sense: A *proper filter* on $\mathbb{N}$ is by definition a collection $\mathcal{G}$ of subsets of $\mathbb{N}$ with $\emptyset \notin \mathcal{G}$ and satisfying Conditions 1 and 2 above. There is no proper filter that is larger than $\mathcal{F}$ in the sense that $\mathcal{F}$ is a proper subset of $\mathcal{G}$.

There are many nonprincipal ultrafilters on $\mathbb{N}$. Let us choose and fix our attention on one of them, say, $\mathcal{F}$. Also, let $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be two sequences of real numbers. We call these sequences *equivalent modulo* $\mathcal{F}$—or simply *equivalent* when a chosen and fixed $\mathcal{F}$ is understood—if $\{n \in \mathbb{N} : x_n = y_n\} \in \mathcal{F}$. This is truly an equivalence relation. As a result, the set of all sequences of reals is partitioned into equivalence classes, each of which is defined to be a *hyperreal*.[3] Each member of an equivalence class is a *representative* of that hyperreal, and that hyperreal is denoted by $X = \langle x_n \rangle$ or $X = \langle x_1, x_2, x_3, \ldots \rangle$, where $\{x_n\}_{n=0}^{\infty}$ is any such representative. Let $\mathbb{R}$ denote the real line. Every $x \in \mathbb{R}$ has a hyperreal version $\langle x, x, x, \ldots \rangle$. In this way, we view the real line $\mathbb{R}$ as being a subset of the set ${}^*\mathbb{R}$ of all hyperreals, in which case it is convenient to use the same symbol for the real and

[3]It is conventional to refer to hyperreal numbers as "hyperreals" and to real numbers as "reals." We will in general use capital letters to denote hyperreals and lower-case letters to denote reals.

the hyperreal. For example, 2 is a real, and 2 also denotes the corresponding hyperreal $\langle 2, 2, 2, \ldots \rangle$.

If a condition depending upon n holds for all n in some set $F \in \mathcal{F}$, we will simply say that it holds "for almost all n" or simply "a.e.".[4] Thus, as above, the hyperreals $X = \langle x_n \rangle$ and $Y = \langle y_n \rangle$ are defined to be *equal* (i.e., $X = Y$) if $\{n \in \mathbb{N} : x_n = y_n\} = F \in \mathcal{F}$, and we say in this case that $x_n = y_n$ a.e. Furthermore, addition, multiplication, inequality, and absolute value are defined componentwise on the representatives of hyperreals. That is, if $X = \langle x_n \rangle$ and $Y = \langle y_n \rangle$, then $X + Y = \langle x_n + y_n \rangle$ and $XY = \langle x_n y_n \rangle$. Also, $X < Y$ means $x_n < y_n$ a.e., and $X \leq Y$ is defined similarly. Furthermore, $|X| = \langle |x_n| \rangle$. Finally, ${}^*\mathbb{R}_+$ will denote the set of all nonnegative hyperreals: $X = \langle x_n \rangle \in {}^*\mathbb{R}_+$ if and only if $x_n \geq 0$ a.e.

The hyperreal $X = \langle x_n \rangle$ is called *infinitesimal* if, for every positive real ϵ, we have $\{n \in \mathbb{N} : |x_n| < \epsilon\} \in \mathcal{F}$, that is, if $|x_n| < \epsilon$ a.e. Also, $X = \langle x_n \rangle$ is called *unlimited* if $|x_n| > \epsilon$ a.e. for every positive real ϵ. Thus, the reciprocal $X^{-1} = \langle x_n^{-1} \rangle$ of an infinitesimal $\langle x_n \rangle$ is unlimited, and conversely. A *limited* hyperreal is one that is not unlimited. Thus, $X = \langle x_n \rangle$ is limited if and only if there is a positive real number γ such that $|x_n| < \gamma$ a.e. A hyperreal that is neither infinitesimal nor unlimited is called *appreciable*. Thus, $X = \langle x_n \rangle$ is appreciable if there exist ϵ and γ with $0 < \epsilon < \gamma < \infty$ such that $\epsilon < x_n < \gamma$ a.e. Around each real $X = \langle x, x, x, \ldots \rangle$ in ${}^*\mathbb{R}$, there is a set of hyperreals $Y = \langle y_1, y_2, y_3, \ldots \rangle$ that are infinitesimally close to $\mathbf{x}$ (i.e., $|X - Y|$ is infinitesimal for each such $\mathbf{y}$). The set of such hyperreals is called the *halo* of $\mathbf{x}$, and $\mathbf{x}$ is called the *shadow* or *standard part* of every $\mathbf{y}$ in that halo.

Since every cofinite set is a member of every nonprincipal ultrafilter, any of the adjectives: infinitesimal, appreciable, limited, and unlimited holds for $X = \langle x_n \rangle$ whenever the corresponding inequality on x_n holds for all n in a cofinite subset of $\mathbb{N}$. Moreover, we are free to change the values of x_n in $X = \langle x_n \rangle$ for all n in any subset of $\mathbb{N}$ not in $\mathcal{F}$; this will not alter $\mathbf{x}$.

[4] The abbreviation "a.e." stands for "almost everywhere". Although brief and convenient, "a.e." is rather a misnomer, for the set of those n for which the condition holds can be a very small subset of $\mathbb{N}$.

4 The General Idea

Let us sketch out how we will build a nonstandard version of a given transfinite graph $\mathcal{G}^\nu$, postponing some definitions to later sections.

In analogy with the set of real numbers as the initial set of "individuals" from which the hyperreal numbers are derived through an ultrapower construction, we now choose the set $(\mathcal{G}_f)$ of all finite graphs as the initial set of "individuals" from which our nonstandard graphs will be derived. Let $\mathcal{S}$ be the set of all sequences of finite graphs, and let the nonprincipal ultrafilter $\mathcal{F}$ be chosen. Two sequences $\{\mathcal{G}_n^0\}_{n\in N}$ and $\{\mathcal{H}_n^0\}_{n\in N}$ with $\mathcal{G}_n^0, \mathcal{H}_n^0 \in (\mathcal{G}_f)$ for all n will be considered *equivalent (with respect to)* $\mathcal{F}$ if $\{n\colon \mathcal{G}_n^0 = \mathcal{H}_n^0\} \in \mathcal{F}$. Here, $\mathcal{G}_n^0 = \mathcal{H}_n^0$ denotes a graph isomorphism. This is truly an equivalence relationship and it partitions $\mathcal{S}$ into equivalence classes, which we refer to as *nonstandard graphs*. Such a nonstandard graph is denoted by $\langle \mathcal{G}_n^0 \rangle$, where the $\mathcal{G}_n^0$ are members of any one of the sequences in the equivalence class. Just what these nonstandard graphs are appears to be nebulous, but we can ascertain a significance for some of them as follows.

But first, let us digress for a moment. Let $\mathcal{G}^\nu$ be a given, transfinite graph having countably many branches. We will reduce $\mathcal{G}^\nu$ to a finite graph $\mathcal{G}_0^0$ by "opening" or "shorting" and then "removing" all but finitely many branches of $\mathcal{G}^\nu$. Then, we will "restore" those branches finitely many at a time to obtain an expanding[5] sequence $\{\mathcal{G}_n^0\}_{n=0}^\infty$ of finite graphs that fills out $\mathcal{G}^\nu$. However, after all the branches are restored, this restoration process may generate more shortings between tips than exist in the original graph $\mathcal{G}^\nu$, as we shall see in the examples of Sec. 7. If additional shorting between tips are not (resp. are) generated, we shall say that $\mathcal{G}^\nu$ is *restorable* (resp. is *not restorable*).

So, with $\mathcal{G}^\nu$ being a given, restorable, transfinite graph having countably many branches, we choose a sequence $\{\mathcal{G}_n^0\}_{n\in N}$ of finite networks $\mathcal{G}_n^0$ that restores $\mathcal{G}^\nu$ without creating any additional shortings of tips, as was just explained. Then, any class $\langle \mathcal{G}_n^0 \rangle$ of equivalent sequences of finite graphs (modulo $\mathcal{F}$) is taken to be a *nonstandard version of* $\mathcal{G}^\nu$ and is denoted by $\mathcal{G}_{\rm ns}^\nu$. Since the choices of the openings and shortings of branches and the sequence of restorations can be varied, $\mathcal{G}^\nu$ will have many nonstandard versions $\mathcal{G}_{\rm ns}^\nu$ with respect to

[5] By "expanding," we mean that $\mathcal{G}_n^0$ is a subgraph of $\mathcal{G}_{n+1}^0$ for every n.

the chosen $\mathcal{F}$.

More generally, the sequence $\{\mathcal{G}_n^0\}_{n=0}^{\infty}$ need not be expanding, nor need the $\mathcal{G}_n^0$ be finite graphs or even graphs of rank 0. However, in this paper we will not explore these possibilities except for a few remarks in Sec. 11.

Later on in Sec. 9, we will construct a nonstandard graph by inverting the operations of opening-removing and shorting-removing branches and using these inverse operations to build the sequence $\{\mathcal{G}_n^0\}_{n=0}^{\infty}$ without referring to a given transfinite graph. The graph that may result after the sequence is completed will be called a *buildable graph.* As a special case, every restorable transfinite graph will be buildable, as will be proven in Sec. 10. Also, if the structure obtained through such a building process is a transfinite graph as defined in Sec. 2, then that transfinite graph will automatically be restorable. It may be that structures more general than the transfinite graphs defined in Sec. 2 can be achieved through a building process.

5 Opening and Shorting Branches: Restorable Graphs

We define "opening" and "shorting" branches in $\mathcal{G}^\nu$ by first assigning to each branch b a positive real number g_b, not necessarily the same for different branches. Later on, when discussing random walks, g_b will be the "branch conductance" and $r_b = 1/g_b$ the "branch resistance," but presently g_b is used merely for defining shorting and opening b and keeping track of where b is located.

We *open* b by setting $g_b = 0$, and we *short* b by setting $r_b = 0$ (equivalently, $g_b = \infty$). An opened or shorted branch is *restored* by reassigning to it its original conductance g_b. A branch for which $0 < g_b < \infty$ will be called a *conductive branch* or simply a *conductance.*

We wish to generate an expanding sequence of finite graphs that fills out $\mathcal{G}^\nu$ and at the same time maintain the connections that the transfinite nodes provide in $\mathcal{G}^\nu$. To this end, we proceed as follows. Remember that the branch set $\mathcal{B}$ of $\mathcal{G}^\nu$ is countably infinite.

Procedure 5.1.

1. Number all the branches using the natural numbers $j = 0, 1, 2, \ldots$.

2. For each nonopen nonelementary tip, choose one of its representative paths and short every branch in that path. Open all other branches.

3. Restore branches finitely many at a time, starting with the lowest numbered branches and restoring others in accordance with increasing branch numbers. Number these steps of restoring finitely many branches at a time using the natural numbers $n = 0, 1, 2, \ldots$.

There are many ways of following Procedure 5.1 because of the different ways of numbering branches, choosing shorted representative paths, and restoring branches finitely many at a time. This will lead to many nonstandard versions of $\mathcal{G}^\nu$ even when $\mathcal{F}$ is fixed.

At the nth step of the restoration process of part 3 of Procedure 5.1, let $\mathcal{M}_n^\nu$ be the same graph as $\mathcal{G}^\nu$ but with the conductances existing right after that step and with all the other branches being shorted or opened. A maximal set of maximal nodes in $\mathcal{M}_n^\nu$ that are pairwise connected through paths of shorted branches will be called a *proximity* in $\mathcal{M}_n^\nu$. Furthermore, each maximal node in $\mathcal{M}_n^\nu$ that is not connected to any other maximal node through a path of shorted branches will also comprise a *proximity* in $\mathcal{M}_n^\nu$, or more particularly a *singleton proximity*. Each maximal node belongs to some proximity in $\mathcal{M}_n^\nu$, which we will call the nth *proximity* of that node. For fixed n the proximities partition the set of maximal nodes in $\mathcal{M}_n^\nu$. We also say that the embraced tips of a maximal node *lies in* the proximity of that node.

As n increases, the proximities of a given maximal node shrink, in general, and never increase because shorted branches are being restored. That is, upon denoting the nth proximity of a given maximal node x^α by $\mathcal{P}_n(x^\alpha)$, we have $\mathcal{P}_n(x^\alpha) \supset \mathcal{P}_{n+1}(x^\alpha)$ for every n. As $n \to \infty$, every maximal node in $\mathcal{P}_n(x^\alpha)$ other than x^α may eventually disappear from the $\mathcal{P}_n(x^\alpha)$. (See Examples 6.1 and 6.3 below.) If this happens for every maximal node x^α in $\mathcal{G}^\nu$, we say that $\mathcal{G}^\nu$ is *restorable*, and also say that $\{\mathcal{M}_n^\nu\}_{n=0}^\infty$ is a sequence that *restores* $\mathcal{G}^\nu$. However, this may not happen; there may be one or more other nodes that remain in $\mathcal{P}_n(x^\alpha)$ for all n. (See Examples 6.2 and 6.3 below.) As a result, there will be tips that are not shorted together (i.e., are not embraced by the same node) in $\mathcal{G}^\nu$ but remain shorted together in $\mathcal{M}_n^\nu$ through paths of shorted branches for every n. In this latter case, we say that $\mathcal{G}^\nu$ is *not restorable*. Given two maximal nodes x^α and y^β in $\mathcal{G}^\nu$, we will say that their sequences of proximities, namely, $\{\mathcal{P}_n(x^\alpha)\}_{n=0}^\infty$ and $\{\mathcal{P}_n(y^\beta)\}_{n=0}^\infty$, are *eventually disjoint* if there exists a natural number m such that $\mathcal{P}_n(x^\alpha) \cap \mathcal{P}_n(y^\beta) = \emptyset$ for every $n \geq m$.

Two tips of any ranks in $\mathcal{G}^\nu$ are called *nondisconnectable* if every representative path of one tip meets every representative path of the other. Equivalently, those tips are nondisconnectable if and only if every two representative paths, one for each tip, meet at infinitely many maximal nodes [20, page 58].

Theorem 5.2. *A necessary condition for $\mathcal{G}^\nu$ to be restorable is the following: If any two tips of any ranks in $\mathcal{G}^\nu$ are nondisconnectable, then either they are shorted (i.e., are embraced by the same node) or at least one of them is open (i.e., is the sole member of a maximal singleton node).*

Proof. Indeed, if this condition is not satisfied by some pair of tips, then those tips will not be open and will be embraced by different maximal nodes. Moreover, the two shorted representative paths chosen for them in part 2 of Procedure 5.1 will meet for every n because only finitely many shorted branches are restored at every step of the restoration. Hence, two maximal nodes containing those two tips will be in the same proximity for every n. Thus, $\mathcal{G}^\nu$ is not restorable. □

The following necessary and sufficient condition for the restorability of $\mathcal{G}^\nu$ is readily derived from our definitions.

Theorem 5.3. *$\mathcal{G}^\nu$ is restorable if and only if it is possible to choose shorted representative paths for every nonopen nonelementary tip in $\mathcal{G}^\nu$ such that, for every pair of maximal nodes, the corresponding sequences of proximities in which those two nodes lie are eventually disjoint.*

Every countable, conventionally infinite graph having at least one 0-tip can be represented by a 1-graph all of whose 0-tips are open (i.e., all of whose 1-nodes are singletons). Such a graph is restorable because there is no shorting of 0-tips. We might say that such a graph is "open at infinity."

In contrast, for an example of a conventionally infinite graph that is "shorted at infinity," consider any countable, conventionally infinite, locally finite graph having at least one 0-tip and consider also the associated 1-graph consisting of a single 1-node containing all the

0-tips. This too is restorable. Indeed, the sequence of proximities containing the 1-node keeps shrinking so that the sequence of proximities of each 0-node is eventually disjoint from the former sequence and also from the sequence of proximities of every other 0-node.

6 Nonstandard Graphs

From each $\mathcal{M}_n^\nu$ we obtain a unique finite graph $\mathcal{G}_n^0$ as follows. A conductive branch having one or both of its elementary tips in a proximity in $\mathcal{M}_n^\nu$ is said to be *incident* to that proximity. Every proximity in $\mathcal{M}_n^\nu$ having at least one incident conductive branch is replaced by a 0-node whose incident branches are conductive branches that are incident to that proximity. Thus, all the shorted and opened branches are discarded, and $\mathcal{G}_n^0$ consists of all the conductive branches with the incidences as stated. Any proximity in $\mathcal{M}_n^\nu$ having no incident conductive branch is not used in the definition of $\mathcal{G}_n^0$.

Thus, $\{\mathcal{G}_n^0\}_{n=0}^\infty$ is an expanding sequence of finite graphs that "fills out" $\mathcal{G}^\nu$ in the sense that, for every n, $\mathcal{G}_{n+1}^0$ is obtained from $\mathcal{G}_n^0$ by a finite number of restorations of opened and shorted branches and every branch of $\mathcal{G}^\nu$ eventually appears as a conductive branch in $\mathcal{G}_n^0$ for all n sufficiently large.

As was defined in Sec. 4, $\langle\mathcal{G}_n^0\rangle$ is taken to be a *nonstandard version* $\mathcal{G}_{\rm ns}^\nu$ of $\mathcal{G}^\nu$ modulo $\mathcal{F}$ so long as $\mathcal{G}^\nu$ is restorable and an appropriate set of shorted representative paths is chosen to satisfy the condition of Theorem 5.3. In this case, $\{\mathcal{M}_n^\nu\}_{n=0}^\infty$ restores $\mathcal{G}^\nu$, and we will also say that $\{\mathcal{G}_n^0\}_{n=0}^\infty$ *restores* $\mathcal{G}^\nu$.

It should be noted that at each step of the restoration process, $\mathcal{G}_n^0$ remains a finite graph. Only when the restoration process is completed (so that all branches are restored) can 0-nodes of infinite degree and transfinite nodes appear. It is only then that a transfinite graph can spring into view.

As branches are restored in part 3 of Procedure 5.1, the following will occur. Given any maximal node x^α of $\mathcal{G}^\nu$, there will be some natural number m depending on x^α for which the mth proximity $\mathcal{P}_m(x^\alpha)$ will have at least one incident conductive branch. Then, the same will be true for every $n \geq m$. Thus, for each such n, $\mathcal{P}_n(x^\alpha)$ becomes a 0-node x_n^0 in $\mathcal{G}_n^0$. So, we have a mapping $x^\alpha \mapsto x_n^0$ defined for all but finitely many values of n. We will call x_n^0 the nth *image* of x^α. For a fixed n, more than one maximal node in $\mathcal{G}^\nu$ may have the same image. However, if $\mathcal{G}^\nu$ is restorable and if shorted representative paths are chosen appropriately according to Theorem 5.3, every two maximal nodes will eventually have different images. In this case, we take $x_{\rm ns}^\alpha = \langle x_n^0 \rangle$ to be the nonstandard image of x^α in $\mathcal{G}_{\rm ns}^\nu = \langle \mathcal{G}_n^0 \rangle$.

When discussing nonstandard random walks in Sec. 12, we will be assigning[6] node voltages $u(x_n^0)$ to x_n^0 to get the hyperreal node voltage $U(x_{\rm ns}^\alpha) = \langle u(x_n^0) \rangle$ at $x_{\rm ns}^\alpha$. With all the hyperreal node voltages so determined, all the hyperreal branch voltage and branch currents will be determined as well.

7 Three examples

Example 7.1. Consider again the 1-graph $\mathcal{G}^1$ shown in Fig. 1(a). It has two kinds of "infinities," a 0-node z^0 of countably infinite degree and a 1-node w^1 that connects the infinite extremity of a ladder graph to a branch β_1. The 0-node z^0 is represented by the lower horizontal line. The resistor symbols denote branches having positive conductances. The upper horizontal path of the infinitely many branches b_k is a one-ended 0-path that reaches the 1-node w^1, which shorts the 0-tip of that 0-path to an elementary tip of the branch β_1; β_1 is connected in series with the branch β_0, which in turn is incident to z^0. With branches numbered as shown, one way of getting a sequence $\{\mathcal{G}_n^0\}_{n=0}^\infty$ is to short $b_1, b_3, b_5, \ldots$, open $b_2, b_4, b_6, \ldots$, and then restore branches two at a time according to b_{2n-1} and b_{2n}, $n = 1, 2, 3, \ldots$. Thus, $\mathcal{G}_0^0$ consists only of the branches b_0, β_0, and β_1 connected in a loop. Fig. 1(b) shows $\mathcal{M}_n^1$, which determines $\mathcal{G}_n^0$ as consisting of the branches $b_0, b_1, \ldots, b_{2n-1}, b_{2n}, \beta_1, \beta_0$ connected together as a finite ladder. (Here, $x_n^0 = x^0$ for $n > 0$, and $y_n^0 = y^0$ for all n.) $\mathcal{G}^1$ is restorable in this way. Of course, there are many other ways of restoring $\mathcal{G}^1$.

Example 7.2,. The 1-graph of Fig. 2(a) is not restorable in any way. It consists of a one-way-infinite series connection of pairs of parallel branches. The 0-tip of the upper (resp. lower) one-ended 0-path of even (resp. odd) numbered branches is connected through the

[6] For the finitely many values of n for which x_n^0 does not exist, we can assign any value to $u(x_n^0)$ without altering the hyperreal $U(x_{\rm ns}^\alpha)$.

1-node x_0^1 (resp. x_1^1) to an elementary tip of the branch β_0 (resp. β_1).[7] These two 0-tips are nondisconnectable. So, upon applying Procedure 5.1, we will find that the two 1-nodes will always reside in the same proximity of $\mathcal{M}_n^1$, whatever be n. Consequently, they will coalesce when all branches are restored no matter what choices are made when applying Procedure 5.1. The restored 1-graph will be that of Fig. 2(b). So truly, the 1-graph of Fig. 2(a) is not restorable.

Example 7.3. Fig. 3 shows a 1-graph that is restorable, but, to restore it, the shorted representative paths of its 0-tips have to be chosen appropriately. The lower (resp. upper) horizontal line denotes a single 0-node x^0 (resp. 1-node x^1). A countable infinity of one-ended 0-paths start at x^0 and reach x^1 through 0-tips. If the shorted representatives of those 0-tips are all chosen short enough so that they do not meet x^0, then any sequence of branch restorations will restore that 1-graph. However, if infinitely many of them are chosen long enough so that they meet x^0, then x^0 and x^1 will coalesce into a single 1-node (embracing x^0), and the 0-paths will form loops, yielding a different 1-graph. Thus, the 1-graph of Fig. 3 can be restored when the shorted representatives of the 0-tips have been so chosen that no more than finitely may of them meet x^0.

8 Eventual Connectedness

Given any restorable transfinite graph $\mathcal{G}^\nu$, let us choose any restoration sequence in accordance with Theorem 5.3, giving thereby $\{\mathcal{M}_n^\nu\}_{n=0}^\infty$ and $\{\mathcal{G}_n^0\}_{n=0}^\infty$. As before, each node x^0 in $\mathcal{M}_n^\nu$ has an image x_n^0 in $\mathcal{G}_n^0$.

Lemma 8.1 *For any fixed n, every one-ended γ-path P^γ in $\mathcal{M}_n^\nu$ $(0 \leq \gamma < \nu)$ has at most finitely many branches that are not shorted.*

Proof. First, consider the case where γ is a natural number. Let Q^γ be the shorted representative γ-path for the γ-tip of P^γ occurring in $\mathcal{M}_n^\nu$. If Q^γ is larger than P^γ, then our conclusion holds. So, assume that $P^\gamma \backslash Q^\gamma$ exists as a path.[8] $P^\gamma \backslash Q^\gamma$ has at most finitely many γ-nodes (perhaps none at all) because the one and only γ-tip of P^γ has been removed. Assume $P^\gamma \backslash Q^\gamma$ does have γ-nodes. Thus, it has only finitely many $(\gamma-1)$-tips. Let $R_k^{\gamma-1}$

[7] There are uncountably many other 0-tips, but these are all open and not shown.

[8] $P^\gamma \backslash Q^\gamma$ denotes the path induced by the branches in P^γ that are not in Q^γ.

$(k = 1, \ldots, K)$ be the shorted representative paths for them occurring in $\mathcal{M}_n^\nu$. Then, either $(P^\gamma \backslash Q^\gamma) \backslash \cup_{k=1}^K R_k^{\gamma-1}$ does not exist as one or more paths, in which case our conclusion follows, or it consists of only finitely many paths of ranks no larger than $\gamma - 1$. In the latter case, each of those paths can have no more than finitely many $(\gamma - 1)$-nodes. (Otherwise, it would have a $(\gamma - 1)$-tip, but such have been removed.) Assume that at least some of those paths have $(\gamma - 1)$-nodes. They will have only finitely many $(\gamma - 2)$-tips. Upon removing the shorted representative paths for those $(\gamma - 2)$-tips, we will have at most finitely many $(\gamma - 2)$-paths remaining, all of which will have only finitely many $(\gamma - 2)$-nodes. Continuing this reasoning through decreasing ranks, we will either eliminate all paths, in which case all the branches of P^γ are shorted, or we will have remaining only finitely many 0-paths. The branches of those 0-paths are finite in number and are in P^γ.[9] This yields our conclusion.

Now, consider the case where the rank ν of $\mathcal{G}^\nu$ is ω. By what has already been proven for the finite ranks, we need only consider now a one-ended $\vec{\omega}$-path $P^{\vec{\omega}}$ (i.e., $\gamma = \vec{\omega}$).[10] The shorted representative $\vec{\omega}$-path $Q^{\vec{\omega}}$ for the $\vec{\omega}$-tip of $P^{\vec{\omega}}$ embraces all the nodes in $P^{\vec{\omega}}$ of all ranks larger than some natural number μ. So, $P^{\vec{\omega}} \backslash Q^{\vec{\omega}}$ either does not exist or is a path of rank μ or less. Thus, we can now apply the argument in the preceding paragraph to obtain our conclusion again. □

Lemma 8.2. *Given any two nodes x^α and y^β in $\mathcal{G}^\nu$ $(0 \leq \alpha, \beta \leq \nu)$, there is a natural number k for which x_n^0 and y_n^0 are connected in $\mathcal{G}_n^0$ for every $n \geq k$ (equivalently, there is a path in $\mathcal{M}_n^\nu$ terminating at x^α and y^β all of whose branches are either shorted or restored).*

Proof. By the connectedness of $\mathcal{G}^\nu$, there is a two-ended γ-path P^γ in $\mathcal{G}^\nu$ $(\gamma \geq \max(\alpha, \beta))$ terminating at x^α and y^β. Since P^γ is two-ended, it has at most finitely many $(\gamma - 1)$-tips, say, m of them.[11] We can partition P^γ into m one-ended $(\gamma - 1)$-paths whose union is P^γ. By Lemma 8.1, each of these paths have only finitely many branches that are restored or open in $\mathcal{M}_n^\nu$. The finitely many open branches in all of those paths taken together are also finite in number, and they will all be restored in $\mathcal{M}_n^\nu$ for some sufficiently large n. □

[9] Those last branches may be conductive or opened—or shorted because of other shorted representatives of other tips.

[10] $\vec{\omega}$ is the arrow rank immediately preceding ω. [19, Sec 5.2], [20, Sec. 2.3], [22, Sec. 2.3].

[11] If $\gamma = \omega$, $\gamma - 1$ denotes $\vec{\omega}$.

9 Appending and Inserting Branches: Buildable Graphs

Rather than starting with a transfinite graph and following Procedure 5.1 in order to restore it (obtaining perhaps a different graph), we may build a transfinite graph one branch at a time through two simple operations applied infinitely often. We will now define those operations as applied to any transfinite graph $\mathcal{G}^\nu$, but later on we will build a transfinite graph through an infinite sequence of applications of those two operations to finite 0-graphs. So, let $\mathcal{G}^\nu$ be given.

The first operation will be called *appending* a branch b to $\mathcal{G}^\nu$. The appending is done by first creating b as a new branch; then, each tip of b is either appended to a node of $\mathcal{G}^\nu$ as an additional embraced elementary tip or is made an elementary tip of a newly created 0-node. If b is a self-loop, both of its tips are appended to one node of $\mathcal{G}^\nu$, or the newly created 0-node is a doublet. If b is not a self-loop, then either a tip of b is appended to a node of $\mathcal{G}^\nu$ or a new singleton node is created that contains that tip, and similarly for the other tip of b. Thus, we may have b's tips appended to existing nodes of $\mathcal{G}^\nu$, or one tip might be appended to an existing node and the other contained in a newly created singleton node, or both might be in two newly created singleton nodes—one tip to each. When b's tips are appended to two nodes in the same component of $\mathcal{G}^\nu$, the connectivity of that component increases.

The second operation will be called *inserting* a branch b to $\mathcal{G}^\nu$. To *insert* a newly created branch b into $\mathcal{G}^\nu$, choose a nonsingleton node x^γ, partition its embraced tips into two nonempty sets to obtain two nodes x^α and x^β, and then append one elementary tip of b to x^α and the other to x^β. Inserting a branch will not increase connectivity.

A transfinite graph can be built by starting from a finite 0-graph $\mathcal{H}_0^0$ and appending or inserting branches, one branch at a time, to get an increasing sequence $\{\mathcal{H}_j^0\}_{j=0}^\infty$ of finite 0-graphs. Here, j also numbers the branches as they are introduced. This construction need not lead to a transfinite graph because no nonsingleton node of rank greater than 0 need arise. On the other hand. if might produce a transfinite graph.

Example 9.1. We can build the 1-graph of Fig. 1(a), without referring to it and some associated graphs $\mathcal{M}_n^\nu$, as follows. Let $\mathcal{H}_0^0$ be the three-branch loop consisting of b_0, β_0, and β_1. Then, split the 0-node between b_0 and β_1 and insert b_1 so that a tip of b_1 and a tip of b_0 are in the doublet 0-node x^0 and the other tip of b_1 and a tip of β_1 are in another doublet 0-node. This yields $\mathcal{H}_1^0$ as a four-branch loop. Next, append b_2 to the node between b_1 and β_1 and to the node between b_0 and β_0 to get $\mathcal{H}_2^0$. Continue this process of inserting b_3, appending b_4, inserting b_5, appending b_6, and so on as indicated in Fig 1(a). After an infinity of branches are so introduced, we will obtain the 1-graph of Fig. 1(a). Note that this transfinite graph springs into view only after the infinite process is completed. Before that, only finite graphs occur. □

Any graph that is obtained after an infinite sequence of appending and/or inserting of branches is applied will be called *buildable*. Were we to apply appendings only, the graph obtained would be a conventionally infinite one, which, if it has 0-tips, would be open at infinity. An infinity of insertions must be used if transfinite nodes (other than singleton 1-nodes) are to be obtained.

10 Restorability Implies Buildability

Theorem 10.1. *A restorable graph is buildable.*

Proof. Let $\mathcal{G}^\nu$ be a restorable countable graph of rank ν with a given branch numbering b_j ($j = 0, 1, 2, \ldots$). Let $\mathcal{M}_n^\nu$ and $\mathcal{G}_n^0$ ($n = 0, 1, 2, \ldots$) be two corresponding sequences of graphs chosen for a restoration of $\mathcal{G}^\nu$, as defined in Secs. 5 and 6. We get from $\mathcal{M}_n^\nu$ to $\mathcal{M}_{n+1}^\nu$ by restoring finitely many branches. It does not matter in which order those branches are restored; the result $\mathcal{M}_{n+1}^\nu$ will be the same. So, choose any order of restoration for those finitely many branches.

We build $\mathcal{G}^\nu$ by starting with the finite graph $\mathcal{G}_0^0$. We need to specify how to obtain $\mathcal{G}_{n+1}^0$ from $\mathcal{G}_n^0$ by appending and/or inserting branches, whatever be n. Moreover, we will do this one branch at a time. So, let $j = j_n + 1, j_n + 2, \ldots, j_{n+1}$ be the numbers for the branches that are restored in sequence when proceeding from $\mathcal{M}_n^\nu$ to $\mathcal{M}_{n+1}^\nu$. Also, set $\mathcal{L}_{j_n}^\nu = \mathcal{M}_n^\nu$ and $\mathcal{L}_{j_{n+1}}^\nu = \mathcal{M}_{n+1}^\nu$, and let $\mathcal{L}_j^\nu$ ($j = j_n, j_n + 1, j_n + 2, \ldots, j_{n+1}$) be the graph $\mathcal{G}^\nu$ with restored values g_b for branches b with numbers up to and including j but with $g_b = 0$ (an opened branch) or $g_b = \infty$ (a shorted branch) for the branches b with numbers above j. We use the same openings and shortings of branches as those used in Procedure 5.1. With n traversing the natural numbers, we thus obtain a sequence $\{\mathcal{L}_j^\nu\}_{j=0}^\infty$ of restorations, one branch at a time, culminating in the restorable graph $\mathcal{G}^\nu$.

Next, let $\mathcal{H}_j^0$ be the finite graph obtained from $\mathcal{L}_j^\nu$ in the same way as $\mathcal{G}_n^0$ was obtained from $\mathcal{M}_n^\nu$ (see the beginning of Sec. 6). Thus, $\{\mathcal{H}_j^0\}_{j=0}^\infty$ is a sequence of finite graphs that fills out $\mathcal{G}^\nu$ such that exactly one branch is restored when going from $\mathcal{H}_j^0$ to $\mathcal{H}_{j+1}^0$ and moreover $\mathcal{H}_0^0 = \mathcal{G}_0^0$.

The proximities of $\mathcal{L}_j^\nu$ can be partitioned into two classes: first, those that have conductive branches incident to them and, second, those that do not. A proximity of the first kind appears as a node of $\mathcal{H}_j^0$, whereas a proximity of the second kind does not appear as a node of $\mathcal{H}_j^0$. Let b be the branch that is restored when proceeding from $\mathcal{L}_j^\nu$ to $\mathcal{L}_{j+1}^\nu$.

Let us now enumerate the different ways a branch b may be restored in the transition from $\mathcal{L}_j^\nu$ to $\mathcal{L}_{j+1}^\nu$ and the corresponding operation that takes $\mathcal{H}_j^0$ to $\mathcal{H}_{j+1}^0$. Each of the first eight ways can be accomplished by appending a branch. The ninth way requires the insertion of a branch.

A branch that is not a self-loop will be called a *regular branch.*

1. First, assume that b is an opened self-loop in $\mathcal{L}_j^\nu$. Restore g_b to make b conductive in $\mathcal{L}_{j+1}^\nu$. Thus, b appears as a conductive self-loop in $\mathcal{L}_{j+1}^\nu$ and in $\mathcal{H}_{j+1}^0$ as well. If b is incident to a proximity of the first kind (resp. second kind) in $\mathcal{L}_j^\nu$, its tips are appended to the 0-node of $\mathcal{H}_j^0$ corresponding to that proximity (resp. its tips are made members—and the only members—of a newly created 0-node in $\mathcal{H}_{j+1}^0$). In either case, b has been appended to $\mathcal{H}_j^0$ to get $\mathcal{H}_{j+1}^0$.

2. Assume that, in $\mathcal{L}_j^\nu$, b is an opened regular branch such that each of its tips is in a proximity of the first kind, the same proximity (resp. two different proximities) for the two tips. The restoration of b results in each tip being appended to the corresponding node in $\mathcal{H}_j^0$, and b appears as a self-loop in $\mathcal{H}_{j+1}^0$ (resp. b appears as a regular branch in $\mathcal{H}_{j+1}^0$). In either case, b has been appended to $\mathcal{H}_j^0$ to get $\mathcal{H}_{j+1}^0$.

3. Assume that, in $\mathcal{L}_j^\nu$, b is an opened regular branch such that each of its tips is in a

proximity of the second kind, the same proximity (resp. different proximities) for the two tips. Now, the restoration of b produces b as an isolated self-loop (resp. isolated regular branch) in $\mathcal{H}_{j+1}^0$. Here again, b has been appended to $\mathcal{H}_j^0$ to get $\mathcal{H}_{j+1}^0$.

4. Assume that, in $\mathcal{L}_j^\nu$, b is an opened regular branch such that one of its tips is in a proximity of the first kind and the other is in a proximity of the second kind. Upon restoration, b appears as an end-branch of a component of $\mathcal{H}_{j+1}^0$. Here too, b has been appended to $\mathcal{H}_j^0$ to get $\mathcal{H}_{j+1}^0$.

5. Assume that b is a shorted self-loop in $\mathcal{L}_j^\nu$. Upon restoration, b appears as a self-loop in $\mathcal{H}_{j+1}^0$. If, in $\mathcal{L}_j^\nu$, b is incident to a proximity of the first (resp. second) kind, then b is appended to a component of $\mathcal{H}_j^0$ (resp. is appended as an isolated self-loop) to get $\mathcal{H}_{j+1}^0$.

6. Assume that, in $\mathcal{L}_j^\nu$, b is a shorted regular branch such that both of its tips are in a proximity of the second kind. Because b is a shorted branch, there can be only one proximity in $\mathcal{L}_j^\nu$ in which b's tips occur. If there is (resp. is not) a path of shorted branches connecting b's two tips other than the path provided by b itself, b appears in $\mathcal{H}_{j+1}^0$ as an isolated self-loop (resp. as an isolated regular branch). In either case, this result can be achieved by appending b to $\mathcal{H}_j^0$ as an isolated self-loop or isolated regular branch to get $\mathcal{H}_{j+1}^0$.

7. Assume that, in $\mathcal{L}_j^\nu$, b is a shorted regular branch such that both of its tips are in a proximity of the first kind, and assume furthermore that there is a path of shorted branches connecting b's tips other than the path provided by b itself. Upon restoration, b appears in $\mathcal{H}_{j+1}^0$ as a self-loop having adjacent branches. This result, too, can be achieved by appending b to $\mathcal{H}_j^0$ to get $\mathcal{H}_{j+1}^0$.

8. Assume that, in $\mathcal{L}_j^\nu$, b is a shorted regular branch such that both of its tips are in a proximity of the first kind, assume that there is no path of shorted branches connecting b's tips other than the path provided by b itself, and assume that conductive branches are incident to that proximity on only one side of b (i.e., only one of b's nodes is such

that there is a path of shorted branches—a trivial path of just a single node allowed—starting at that node, reaching a conductive branch, and not passing through b). Upon restoration, b appears as an end-branch—but not an isolated branch—of $\mathcal{H}^0_{j+1}$ (i.e., one but not both of its nodes is of degree 1). Here too, we can get from $\mathcal{H}^0_j$ to $\mathcal{H}^0_{j+1}$ by appending b to $\mathcal{H}^0_j$.

9. Finally, assume that, in $\mathcal{L}^\nu_j$, b is a shorted regular branch such that both of its tips are in a proximity of the first kind, assume that there is no path of shorted branches connecting b's tips other than the path provided by b itself, and assume that conductive branches appear on both sides of b (i.e., from each node of b there is a path of shorted branches—again a trivial path is allowed—starting at that node and reaching a conductive branch without passing through b). Here at last is the one and only case where an insertion must be used in order to get from $\mathcal{H}^0_j$ to $\mathcal{H}^0_{j+1}$ when b is restored. In particular, b partitions the nodes of the said proximity into two sets. The elementary tips of the conductive branches incident to that proximity comprise a 0-node in $\mathcal{H}^0_j$. To insert b, partition those tips into the two sets induced by the partitioning provided by b. Those two sets provide two 0-nodes and then b's two tips are appended to them, one tip to each. In this way, b is inserted in $\mathcal{H}^0_j$ to get $\mathcal{H}^0_{j+1}$.

We have now listed all possible ways of getting from $\mathcal{H}^0_j$ to $\mathcal{H}^0_{j+1}$ when b is restored. Appendings work in the first eight cases, and an insertion must be used in the ninth case.

Now, upon restoring branches one-by-one, we get from $\mathcal{H}^0_{j_n} = \mathcal{G}^0_n$ to $\mathcal{H}^0_{j_{n+1}} = \mathcal{G}^0_{n+1}$. Since n is arbitrary, we have in fact the sequence $\{\mathcal{G}^0_n\}^\infty_{n=0}$ which by hypothesis restores $\mathcal{G}^\nu$. Moreover, these one-by-one appendings and/or insertings of branches show that $\mathcal{G}^\nu$ is buildable through the sequence $\{H^0_j\}^\infty_{j=0}$, where $\mathcal{H}^0_0 = \mathcal{G}^0_0$. This proves the theorem. □

A conclusion that can now be drawn is that any *restorable transfinite graph can be built by repeatedly applying infinitely often two simple operations (appending and inserting branches) starting from a finite graph.* In this way, transfinite graphs appear as natural extensions of conventional graphs, extensions that have been overlooked until recently.

11 Some Comments, Questions, and Speculations

Rather than using a given restorable transfinite graph to guide the building of that graph, we can apply an arbitrarily chosen infinite sequence of appendings and insertings of branches. That sequence must contain an infinity of insertions if one-ended paths yielding nonopen transfinite tips and thereby nonsingleton nodes are to be built. But even then, such transfinite nodes need not arise through a particular building process. On the other hand, other sequences with infinitely many insertions do generate transfinite nodes. Moreover, graphs of ranks higher than ω can certainly be built, but how much higher has not been determined.

Building a graph without any guidance provided by the restoration of a given transfinite graph leaves the result indeterminate. We are faced with the ancient Aristotelian dichotomy of a potential infinity and an actual infinity [13, page 3]. No matter how finitely many steps we take in building a graph, we still have on hand only a finite graph. The infinite process must be completed before a transfinite graph is achieved, as will certainly be the case when restoring a restorable one. What may arise from an arbitrary but completed infinite building process? Can we get something more than what is presently available from our theory of transfinite graphs? Let us consider this more closely.

There does not seem to be any restriction on the countable ranks that can be achieved through an appropriate building process—or is there? Can we obtain any countable rank, such as a rank that is a nameless and unidentified countable ordinal larger than any specific countable ordinal that has heretofore been examined [13]? We may be able to reach higher ranks by appending and inserting uncountably many branches to transfinite graphs (rather than finite ones) at each step of the building process. After all, the appending and inserting of branches were defined in Sec. 9 on any transfinite graph. For an example of the appending of uncountably many branches all at once, consider the infinite binary tree; it has uncountably many 0-tips, each of which can be made the sole member of a singleton 1-node; then, a branch can be appended to each of those 1-nodes simultaneously. Furthermore, transfinite paths, rather than single branches, can be inserted all at once into a transfinite node by first partitioning the node. Thus, greater generality than that provided by Procedure 5.1 is available.

So, can we generate a sequence of buildable graphs whose countable ranks increase unboundedly? Can we then "complete" that process to get a transfinite graph of rank $\aleph_1$? How can a transfinite graph of uncountable rank be defined? We are groping through uncharted territory with these speculations.

12 Random Walks on Restorable Transfinite Graphs

We continue to assume that $\mathcal{G}^\nu$ is a connected transfinite graph with countably many branches. Henceforth, we assume in addition that $\mathcal{G}^\nu$ is restorable (and therefore buildable) through an expanding sequence $\{\mathcal{G}_n^0\}_{n=0}^0$ of finite graphs $\mathcal{G}_n^0$. Thus, $\mathcal{G}^\nu$ has nonstandard versions modulo a chosen $\mathcal{F}$. Even when $\mathcal{F}$ is fixed, $\mathcal{G}^\nu$ has many nonstandard versions, as explained above. We fix our attention on one of them $\mathcal{G}_{\rm ns}^\nu = \langle \mathcal{G}_n^0 \rangle$ henceforth. As before, we assume that each branch b is assigned a conductance g_b $(0 < g_b < \infty)$.

Because each $\mathcal{G}_n^0$ is is a finite graph, we can readily lift many standard theorems concerning random walks on finite graphs into a nonstandard setting. We shall list some of them in this section and illustrate them with the transfinite graph of Fig. 1(a). These theorems can be derived from analyses of finite graphs treated as electrical networks.

Our prior development of random walks on transfinite graphs [20, Chap. 7], [22, Chap. 8] used standard analysis and required the imposition of many restrictions on the graphs. Moreover, the standard probability of a random walk leaving a transfinite node is 0. To obtain nontrivial results, only the exceptional cases where the random walker does leave transfinite nodes was admitted; this is the "roving" assumption [20, page 197], [22, page 156], which conditions probabilities on those exceptional cases. No such restrictions on $\mathcal{G}^\nu$ are now needed other than the restorability of $\mathcal{G}^\nu$, but now probabilities are hyperreals, possibly infinitesimals.

In the following, we assume that a random walker wanders on a finite graph with conductances g_b in accordance with the nearest-neighbor rule: The probabilities of proceeding from a node x to the nodes adjacent to x are proportional to the conductances between x and the adjacent nodes. A slight complication arises for us. When constructing the finite graphs $\mathcal{G}_n^0$ in accordance with Procedure 5.1, some $\mathcal{G}_n^0$ may be found to have parallel branches and/or self-loops. The nearest-neighbor rule as stated does not encompass these. To overcome this problem, we combine parallel branches into a single branch by summing conductances. Also, we delete self-loops; in effect, we will be ignoring the steps a random walker may take around self-loops and are only analyzing the random walks that are reduced in this way. This conforms with the fact that, when an electrical source is appended to two nodes, the currents in self-loops are 0, and thus node voltages are unaffected by the deletion of self-loops.

The first theorem we lift is the Nash-Williams rule [14]. Let x^0, y^0 and z^0 be three nodes in a finite graph $\mathcal{G}^0$. Let prob(sx^0,ry^0,bz^0) denote the probability that a random walker, after starting from the node x^0 and following the nearest-neighbor rule, will reach y^0 before reaching z^0.[12] The Nash-Williams rule asserts that prob(sx^0,ry^0,bz^0) is equal to the voltage at x^0 when y^0 is held at 1 volt and z^0 is held at 0 volt. That voltage at x^0 is determined by Kirchhoff's laws and Ohm's law applied to the electrical network whose graph is $\mathcal{G}^0$ with an additional branch for the voltage source appended to y^0 and z^0 and whose electrical branch parameters are the branch conductances g_b except for the voltage source.

Now, let $\mathcal{G}^\nu_{\rm ns} = \langle \mathcal{G}^0_n \rangle$ be a nonstandard version of $\mathcal{G}^\nu$ having branch conductances, and let x^α, y^β, and z^γ be three maximal nodes in $\mathcal{G}^\nu$.[13] As a consequence of restorability, the three nodes x^α, y^β, and z^γ will eventually[14] appear as different 0-nodes x^0_n, y^0_n, and z^0_n in $\mathcal{G}^0_n$. We let $x^\alpha_{\rm ns} = \langle x^0_n \rangle$, $y^\beta_{\rm ns} = \langle y^0_n \rangle$, and $z^\gamma_{\rm ns} = \langle z^0_n \rangle$ be the corresponding nonstandard nodes in $\mathcal{G}^\nu_{\rm ns}$. Moreover, by Lemma 8.2, x^0_n, y^0_n, and z^0_n will eventually be connected in $\mathcal{G}^0_n$. Consequently, the Nash-Williams rule can eventually be applied to $\mathcal{G}^0_n$ to get the following nonstandard result.[15]

Theorem 12.1. *The hyperreal probability that a random walker on $\mathcal{G}^\nu_{\rm ns}$, after starting from $x^\alpha_{\rm ns}$, will reach $y^\beta_{\rm ns}$ before reaching $z^\gamma_{\rm ns}$ is*

$$\mathrm{PROB}(\mathrm{s}x^\alpha_{\rm ns}, \mathrm{r}y^\beta_{\rm ns}, \mathrm{b}z^\gamma_{\rm ns}) \;=\; \langle \mathrm{prob}(\mathrm{s}x^0_n, \mathrm{r}y^0_n, \mathrm{b}z^0_n \rangle \qquad (2)$$

where, for each n, the term on the right-hand side is given by the Nash-Williams rule applied

[12] More generally, we can replace y^0 and z^0 by two disjoint sets of nodes and then apply the following analysis by shorting together the nodes of each set.

[13] Here too, we can get a more general result by replacing y^β and z^γ by disjoint sets of nodes.

[14] By "eventually" we mean "for n all sufficiently large."

[15] As was mentioned before, we will use capital letters to denote hyperreals and lower case letters for reals.

to $\mathcal{G}_n^0$.

In the following examples, we will be working with ladder graphs. That structure along with the 1 siemen conductance of every branch leads to the Fibonacci numbers $F(k)$, $(k = 0, 1, 2, \ldots)$. These are defined recursively by $F(0) = F(1) = 1$ and $F(k+2) = F(k+1) + F(k)$. We have

$$F(k) \ = \ \frac{\lambda_1^{k+1} - \lambda_2^{k+1}}{\sqrt{5}} \tag{3}$$

where $\lambda_1 = (1+\sqrt{5})/2 = 1.618\ldots$ and $\lambda_2 = (1-\sqrt{5})/2 = -0.618\ldots$.

Example 12.2 Consider again the 1-graph $\mathcal{G}^1$ of Fig. 1(a). Every branch is assigned a conductance of 1 siemen. A random walker on a nonstandard version of $\mathcal{G}^1$ starts at $x_{\rm ns}^0$ and wanders, possible through the 1-node $w_{\rm ns}^1$, to reach $y_{\rm ns}^0$. What is the probability PROB($\mathrm{s}x_{\rm ns}^0$,$\mathrm{r}y_{\rm ns}^0$,$\mathrm{b}z_{\rm ns}^0$) that the random walker reaches $y_{\rm ns}^0$ before reaching $z_{\rm ns}^0$? To answer this, we have to specify the nonstandard version $\mathcal{G}_{\rm ns}^\nu = \langle \mathcal{G}_n^0 \rangle$ we are using.

Short and open branches as in Example 8.1 (see Fig. 1(b)). Let $\mathcal{G}_0^0$ be the loop consisting of branches b_0, β_0, and β_1. Then, restore branches two at a time according to b_{2n-1} and b_{2n} $(n = 1, 2, 3, \ldots)$. Thus, $\mathcal{G}_n^0$ is the ladder graph of Fig. 1(b) restored up to the branch b_{2n}. Then, according to the Nash-Williams rule for the finite graph $\mathcal{G}_n^0$, the probability of the random walker, starting at x^0 and reaching the node y^0 before reaching the node z_n^0 is the voltage $u_n(x^0)$ when y^0 is held at 1 volt and z_n^0 is held at 0 volt. A straightforward computation gives $u_n(y^0) = 1/F(2n+2)$. Thus,

$$\mathrm{PROB}(\mathrm{s}x_{\rm ns}^0, \mathrm{r}y_{\rm ns}^0, \mathrm{b}z_{\rm ns}^0) \ = \ \left\langle \frac{1}{F(2n+2)} \right\rangle . \tag{4}$$

If instead we use the nonstandard graph obtained by restoring b_1, b_2, b_3, then b_4, b_5, b_7, then b_6, b_9, b_{11}, and so one (that is, we restore one vertical and two horizontal branches at each step $n = 1, 2, 3, \ldots$) we obtain

$$\mathrm{PROB}(\mathrm{s}x_{\rm ns}^0, \mathrm{r}y_{\rm ns}^0, \mathrm{b}z_{\rm ns}^0) \ = \ \left\langle \frac{1}{F(2n+2) + nF(2n+1)} \right\rangle . \tag{5}$$

Both of these results are infinitesimals, but (5) is smaller than (4), as is to be expected since the random walker must pass through more resistance in $\mathcal{G}_n^0$ in order to reach y^0 for the second case as compared to the first case. □

Another standard result concerns the escape probability $p_{\rm esc}(x^0 \to y^0)$ in a finite graph $\mathcal{G}^0$. This is the probability of a random walker, after starting from x^0, reaches y^0 before returning to x^0. The result [2, page 304] asserts that $p_{\rm esc}(x^0 \to y^0) = c_{\rm eff}(x^0, y^0)/c_{x^0}$, where $c_{\rm eff}(x^0, y^0)$ is the input conductance between x^0 and y^0 and c_{x^0} is the total conductance incident to x^0. We have $c_{\rm eff}(x^0, y^0) = 1/u(x^0)$, where $u(x^0)$ is the node voltage at x^0 with respect to y^0 chosen as ground (i.e., choose $u(y^0) = 0$) when a current source injects 1 ampere at x^0 and extracts 1 ampere at y^0.

This, too, can be immediately lifted into the following for a nonstandard version $\mathcal{G}^\nu_{\rm ns} = \langle \mathcal{G}^0_n \rangle$ of a restorable ν-graph $\mathcal{G}^\nu$. As always, $x^\alpha_{\rm ns} = \langle x^0_n \rangle$ and $y^\beta_{\rm ns} = \langle y^0_n \rangle$ are nonstandard nodes in $\mathcal{G}^\nu_{\rm ns}$ corresponding to the nodes x^α and y^β of $\mathcal{G}^\nu$.

Theorem 12.3. *The hyperreal escape probability that a random walker on $\mathcal{G}^\nu_{\rm ns}$, after starting from a node $x^\alpha_{\rm ns}$, reaches another node $y^\beta_{\rm ns}$ before returning to $x^\alpha_{\rm ns}$ is*

$$P_{\rm ESC}(x^\alpha_{\rm ns} \to y^\beta_{\rm ns}) \;=\; \langle p_{\rm esc}(x^0_n \to y^0_n) \rangle \;=\; \left\langle \frac{c_{\rm eff}(x^0_n, y^0_n)}{c_{x^0_n}} \right\rangle. \qquad (6)$$

If the right-hand side is an infinitesimal, $\mathcal{G}^\nu_{\rm ns}$ is *recurrent from $x^\alpha_{\rm ns}$ to $y^\beta_{\rm ns}$*, and, if it is appreciable, $\mathcal{G}^\nu_{\rm ns}$ is *transient from $x^\alpha_{\rm ns}$ to $y^\beta_{\rm ns}$*.

The corresponding standard concept relates to a conventionally infinite graph whose 0-tips are all shorted together into a single 1-node y^1; then, that graph is called recurrent if $p_{\rm esc}(x^0 \to y^1) = 0$ and is called transient if $p_{\rm esc}(x^0 \to y^1) > 0$. Since we are only dealing with real numbers in the standard case, recurrence or transience does not depend upon the choice of x^0. However, in the nonstandard case, the hyperreal $P_{\rm ESC}(x^\alpha_{\rm ns} \to y^\beta_{\rm ns})$ does depend upon the choice of $x^\alpha_{\rm ns}$. Furthermore, unlike the standard case, we can now quantify and compare recurrence and transience for different nonstandard versions $\mathcal{G}^\nu_{\rm ns}$ of $\mathcal{G}^\nu$, even when x^α amd y^β are fixed in $\mathcal{G}^\nu$. How $\mathcal{G}^\nu$ is restored will affect these hyperreal probabilities.

Similarly,, in the recurrent case, when dealing with a given nonstandard version of a conventionally infinite graph with a 1-node y^1 that contains all 0-tips, the hyperreal $P_{\rm ESC}(x^0_{\rm ns} \to y^1_{\rm ns})$ will, in general, take on different infinitesimal values depending of the choice of $x^0_{\rm ns}$, and thus we can compare the sizes of these infinitesimal escape probabilities for different $x^0_{\rm ns}$.

Example 12.4. For the standard 1-graph $\mathcal{G}^1$ of Fig. 1(a) wherein all branch conductances are 1 siemen, consider again the nonstandard version obtained by shorting and opening branches and then restoring them two at a time as in the first part of Example 12.2. We determine the escape probability from $x^0_{\rm ns}$ to $y^0_{\rm ns}$. In order to get $c_{\rm eff}(x^0_n, y^0_n)$, we need to determine the node voltage $u(x^0_n)$ when $u(y^0_n) = 0$ and a current source injects 1 ampere into x^0_n and extracts 1 ampere from y^0_n. This is easily computed by using superposition; first apply the current source from z^0_n to x^0_n, then apply the current source from y^0_n to z^0_n; finally, add the results. We get

$$c_{\rm eff}(x^0_n, y^0_n) \;=\; \frac{1}{u(x^0_n)} \;=\; \frac{F(2n+3)}{2(F(2n+2)-1)}.$$

Since $c_{x^0_n} = 2$, we obtain

$$P_{\rm ESC}(x^0_{\rm ns} \to y^0_{\rm ns}) \;=\; \left\langle \frac{F(2n+3)}{4(F(2n+2)-1)} \right\rangle.$$

The right-hand side is an appreciable hyperreal less than $\langle 1 \rangle$. Its shadow is 0.4045..., as can be seen by using (3). It is not infinitesimal because the random walker can go from $x^0_{\rm ns}$ to $y^0_{\rm ns}$ in two steps by passing through $z^0_{\rm ns}$. Thus, $\mathcal{G}^\nu_{\rm ns}$ is transient from $x^0_{\rm ns}$ to $y^0_{\rm ns}$.

Similarly, to get the escape probability from $w^1_{\rm ns}$ to $y^0_{\rm ns}$ with respect to the same restoration sequence, we determine $c_{\rm eff}(w^0_n, y^0_n) = F(2n+1)/F(2n)$. Now, $c_{w^0_n} = 3$. Thus,

$$P_{\rm ESC}(w^1_{\rm ns} \to y^0_{\rm ns}) \;=\; \left\langle \frac{F(2n+3)}{3F(2n+2)} \right\rangle.$$

This has the shadow 0.5393.... This is a larger escape probability because w^1 is closer to y^0 than is x^0. □

Some other standard results concern "times" and "transversals" [2, Secs. IX.2 and IX.3]. Let $s_{z^0}(x^0 \to y^0)$ be the sojourn time (that is, the expected number of occurrences) that the random walker is at node z^0 before it reaches node y^0, given that it starts at x^0; x^0 and z^0 may be the same node, counting the start as one occurrence. Then, $s_{z^0}(x^0 \to y^0) = c_{z^0} u(z^0)$ where c_{z^0} is the total conductance incident to z^0 and $u(z^0)$ is the node voltage at z^0 when a current of 1 ampere is injected into x^0 and extracted at y^0 and with y^0 taken as the ground node (i.e., $u(y^0) = 0$). In particular, $s_{x^0}(x^0 \to y^0) = c_{x^0}/c_{\rm eff}(x^0, y^0)$. Furthermore, let $\vec{b}$ be a branch with an orientation, and let $e_{\vec{b}}(x^0 \to y^0)$ be the expected difference between the number of occurrences that the random walker traverses b in the direction of b's orientation minus the number of occurrences it traverses b in the reverse direction, given that it starts at x^0 and stops when it first reaches y^0. Then, $e_{\vec{b}}(x^0 \to y^0)$ is equal to the current in b measured in the direction of b's orientation, when again 1 ampere is injected at x^0 and extracted at y^0.

These results, too, can be immediately lifted for a nonstandard version $\mathcal{G}^\nu_{\rm ns} = \langle \mathcal{G}^0_n \rangle$ of a restorable transfinite graph $\mathcal{G}^\nu$. By virtue of Lemma 8.2, given any two nodes x^α and y^β in $\mathcal{G}^\nu$ and their corresponding nodes x^0_n and y^0_n in $\mathcal{G}^0_n$, we can inject 1 ampere into x^0_n and extract 1 ampere at y^0_n for all sufficiently large n. Similarly, any branch b of $\mathcal{G}^\nu$ will be restored for all sufficiently large n. Also, let z^γ be another node of $\mathcal{G}^\nu$ (possibly, $z^\gamma = x^\alpha$), and let z^0_n be its corresponding node in $\mathcal{G}^0_n$. Then, a node voltage $u(z^0_n)$ and a branch current $i_{\vec{b},n}$ will exist for all but finitely many values of n. So, we can state the following nonstandard versions for sojourn times and branch transversals.

Theorem 12.5. *Let $x^\alpha_{\rm ns} = \langle x^0_n \rangle$, $y^\beta_{\rm ns} = \langle y^0_n \rangle$, and $z^\gamma_{\rm ns} = \langle z^0_n \rangle$ be three nodes in $\mathcal{G}^\nu_{\rm ns}$ (possibly, $x^\alpha_{\rm ns} = z^\gamma_{\rm ns}$), and let $\vec{b}$ be an oriented branch in $\mathcal{G}^\nu_{\rm ns}$. Inject 1 ampere into $x^\alpha_{\rm ns}$ and extract 1 ampere from $y^\beta_{\rm ns}$. Then, for the random walker starting at $x^\alpha_{\rm ns}$ and stopping when it first reaches $y^\beta_{\rm ns}$, the hyperreal sojourn time at $z^\gamma_{\rm ns}$ is*

$$S_{z^\gamma_{ns}}(x^\alpha_{\rm ns} \to y^\beta_{\rm ns}) \;=\; \langle c_{z^0_n} u(z^0_n) \rangle \tag{7}$$

and the hyperreal expected difference in b's transversal numbers with respect to b's orientation is

$$E_{\vec{b}}(x^\alpha_{\rm ns} \to y^\beta_{\rm ns}) \;=\; \langle i_{\vec{b},n} \rangle \tag{8}$$

where, with respect to $\mathcal{G}^0_n$ and all n sufficiently large, $u(z^0_n)$ is the node voltage at z^0_n with respect to y^0_n taken as ground, $c_{z^0_n}$ is the total conductance incident at z^0_n, and $i_{\vec{b},n}$ is the current in $\vec{b}$ measured with respect to b's orientation. If $z^\gamma_{\rm ns} = x^\alpha_{\rm ns}$, then

$$S_{x^\alpha_{ns}}(x^\alpha_{\rm ns} \to y^\beta_{\rm ns}) \;=\; \left\langle \frac{c_{x^0_n}}{c_{\rm eff}(x^0_n, y^0_n)} \right\rangle. \tag{9}$$

Example 12.6. Consider again the 1-graph of Fig. 1(a) with all branch conductances being 1 sieman, and also let $\mathcal{G}^\nu_{\rm ns} = \langle \mathcal{G}^0_n \rangle$ be the nonstandard version of it for the restoration process specified in Example 12.4 (see the first part of Example 12.2). Orient b_1 and β_1 from left to right. Then, straightforward calculations yield the following.

$$S_{w^1_{ns}}(x^0_{ns} \to z^0_{ns}) = \left\langle \frac{2}{F(2n+3)} \right\rangle \qquad (10)$$

$$S_{x^0_{ns}}(x^0_{ns} \to z^0_{ns}) = \left\langle \frac{F(2n+2)}{F(2n+3)} \right\rangle \qquad (11)$$

$$E_{\vec{b}_1}(x^0_{ns} \to z^0_{ns}) = \left\langle \frac{F(2n+1)}{F(2n+3)} \right\rangle \qquad (12)$$

$$E_{\vec{\beta}_1}(x^0_{ns} \to z^0_{ns}) = \left\langle \frac{1}{F(2n+3)} \right\rangle \qquad (13)$$

As is to be expected from the locations of the nodes and branches, (10) and (13) are infinitesimal and (11)and (12) are appreciable. The shadows of (11) and (12) are $1/\lambda_1 = 0.618\ldots$ and $1/\lambda_1^2 = 0.381\ldots$ respectively. □

As for mean return time $h(x^0, x^0)$ and commute time $c(x^0, y^0)$, we have the following for a finite graph all whose branch conductances are 1 siemen. $h(x^0, x^0)$ is the expected number of branch transversals that a random walker makes after starting from x^0 and then returning to x^0 for the first time. For this we have $h(x^0, x^0) = 2|\mathcal{B}|/d(x^0)$, where $|\mathcal{B}|$ is the number of branches in the graph and $d(x^0)$ is the degree of x^0. Also, $c(x^0, y^0)$ is the expected number of branch transversals the random walker makes in going from x^0 to y^0 and returning to x^0 for the first time. In this case, $c(x^0, y^0) = 2|\mathcal{B}|/c_{\text{eff}}(x^0, y^0)$. Thus, we have the following, where $|\mathcal{B}_n|$ denotes the number of branches in $\mathcal{G}_n$.

Theorem 12.7 *Assume that every branch of $\mathcal{G}^\nu_{ns}$ has a conductance of 1 siemen. Then the hyperreal mean return time $H(x^\alpha_{ns}, x^\alpha_{ns})$ and mean commute time $C(x^\alpha_{ns}, y^\beta_{ns})$ are given by*

$$H(x^\alpha_{ns}, x^\alpha_{ns}) = \left\langle \frac{2|\mathcal{B}_n|}{d(x^0_n)} \right\rangle \qquad (14)$$

and

$$C(x^\alpha_{ns}, y^\beta_{ns}) = \left\langle \frac{2|\mathcal{B}_n|}{c_{\text{eff}}(x^0_n, y^0_n)} \right\rangle \qquad (15)$$

where $|\mathcal{B}_n|$ is the number of branches in $\mathcal{G}^0_n$.

Example 12.8. Continuing Example 12.6, we have the hyperreals

$$H(x^0_{ns}, x^0_{ns}) = H(y^0_{ns}, y^0_{ns}) = \langle 2n+3 \rangle,$$

$$H(w^1_{\rm ns}, w^1_{\rm ns}) = \left\langle \frac{4n+6}{3} \right\rangle,$$
$$H(z^0_{\rm ns}, z^0_{\rm ns}) = \left\langle \frac{4n+6}{n+2} \right\rangle,$$
$$C(x^0_{\rm ns}, y^0_{\rm ns}) = \left\langle 4(2n+3)\frac{F(2n+2)-1}{F(2n+3)} \right\rangle.$$

□

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Figure Captions

(There are only figure numbers, but no figure texts.)

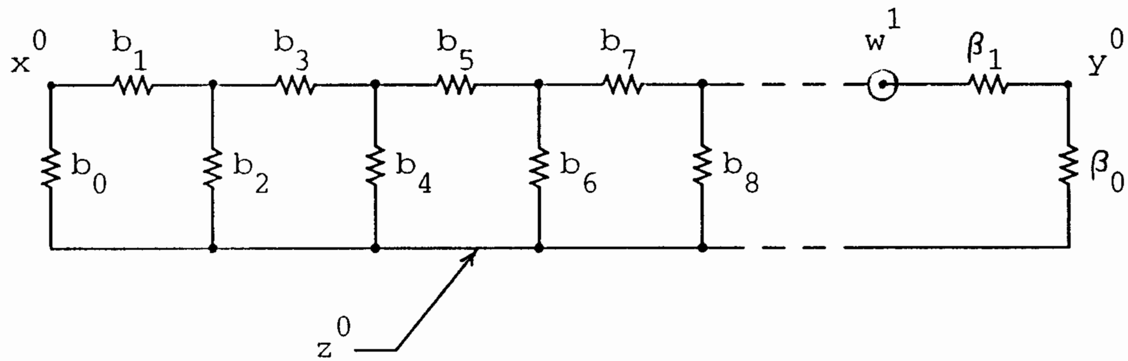
Fig. 1(a)

Fig. 1(b)

Fig. 2(a)

Fig. 2(b)

Fig. 3



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Size	Document Number
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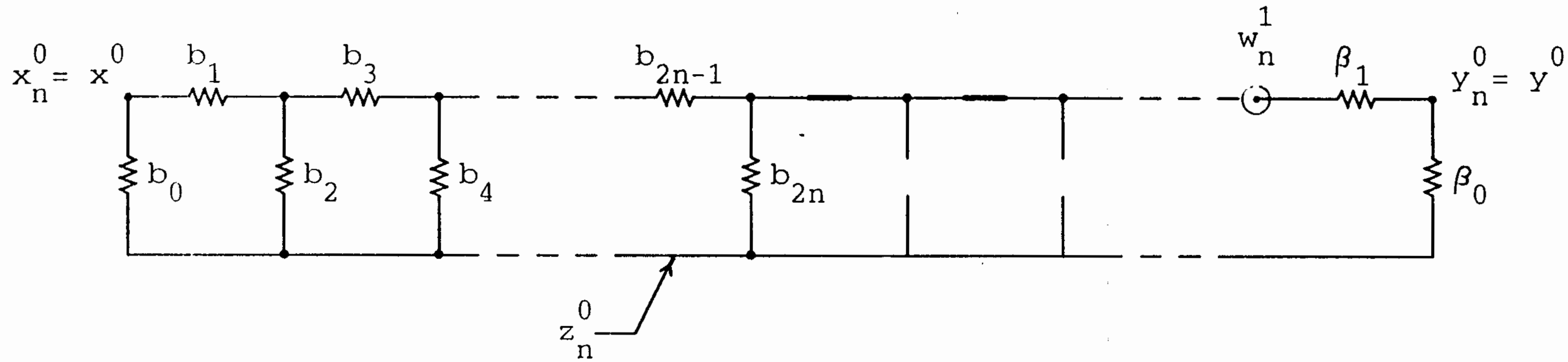
Date:	Saturday, October 14, 2000
Sheet	1 of 1



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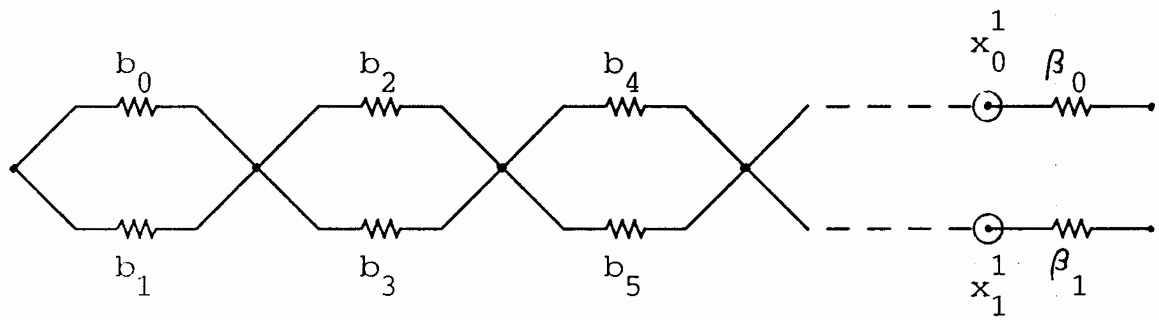

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Fig. 2(a)

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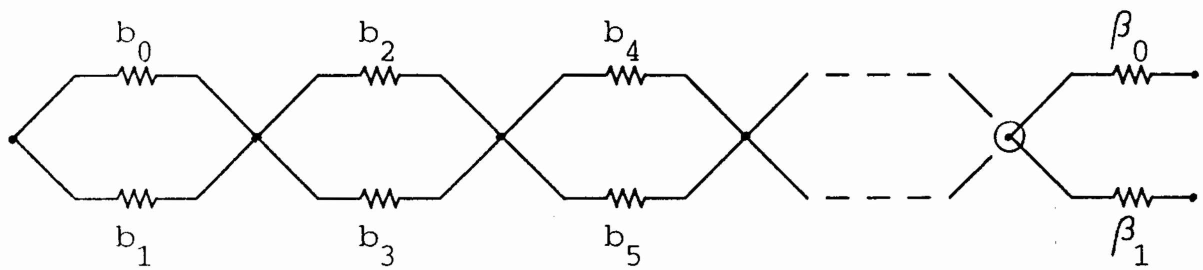
Date: Saturday, October 14, 2000 Sheet 1 of 1



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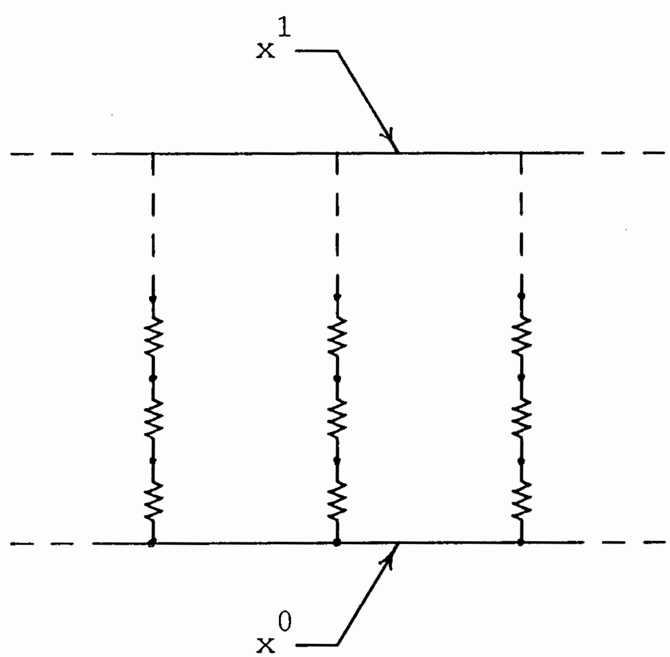
Date:	Saturday, October 14, 2000	Sheet 1 of 1	



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Date:	Saturday, October 14, 2000	Sheet 1	of 1