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NONSTANDARD TRANSFINITE GRAPHS

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Abstract — For any given sequence of transfinite graphs all of the same rank, a nonstandard transfinite graph of that same rank is constructed. The procedure is similar to an ultrapower construction of an internal set from a sequence of subsets of the real line, but now the primitive entities are the nodes of various ranks instead of real numbers.

1 Introduction

In several prior works [1], [2], [3], [4], [5], the idea of nonstandard transfinite graphs and networks was introduced and investigated. The basic idea in those works was to start with a given transfinite graph, to reduce it to a finite graph by shorting and opening branches, and then to obtain an expanding sequence of finite graphs by restoring branches sequentially. If all this is done in an appropriate fashion, it may happen that the sequence of finite graphs fills out and restores the original transfinite graph once the restoration process is completed. If in addition there is an assignment of electrical parameters to the branches, we finally obtain sequences of branch currents and branch voltages, from which hyperreal currents and voltages can be derived. The latter will then automatically satisfy Kirchhoff's laws even though Kirchhoff's laws may on occasion be violated in the original transfinite network—an important advantage of this nonstandard approach.

However, this is only a partial construction of a nonstandard graph in the sense that the completion of the restoration process—if successful—results in the original standard transfinite graph. The sequence of restorations only provides a means of constructing hyperreal currents and voltages satisfying Kirchhoff's laws. A more general approach might start with an arbitrary sequence of transfinite graphs and construct from that a nonstandard graph in much the same way as an internal set in the hyperreal line ${}^*\mathbb{R}$ is constructed from a given sequence of subsets of the real line $\boldsymbol{R}$, that is, by means of an ultrapower construction [6, page 125]. In this case, the resulting nonstandard transfinite graph has nonstandard branches and nonstandard nodes of various ranks. Thus, that nonstandard transfinite graph is much different from those of the prior works cited above.

Our objective in this work is to develop this latter approach to nonstandard graphs. The standard 0-graphs are the (possibly infinite) conventional graphs, and the standard graphs of higher ranks are nontrivially transfinite. The nonstandard versions of these graphs contain, in addition to analogues of standard branches and nodes, nonstandard branches and nodes that have no standard analogues—in much the same way as unlimited hyperreals are different from real numbers. Our procedure is presented in detail for the ranks 0 and 1 and merely summarized for higher ranks of transfiniteness because it extends to higher ranks in much the same way.

Various definitions and results concerning transfinite graphs are invoked throughout this work. Whenever this occurs, references to [7] are given wherein any needed information can be found.

2 Nonstandard 0-Graphs

A standard 0-graph $\mathcal{G}^0$ is a conventional (finite or infinite) graph $\mathcal{G}^0 = \{\mathcal{B}, \mathcal{X}^0\}$, where $\mathcal{X}^0$ is the set of its nodes (called 0-nodes henceforth) and $\mathcal{B}$ is the set of its branches. Each branch $b \in \mathcal{B}$ designates a two-element set $b = \{x^0, y^0\}$ with $x^0, y^0 \in \mathcal{X}^0$ and $x^0 \neq y^0$; b and x^0 are said to be incident and so, too, are b and y^0. Also, x^0 and y^0 are said to be adjacent through b. We allow neither parallel branches (i.e., multigraphs) nor self-loops (i.e., branches that are each incident to just one 0-node).

Next, let $\langle \mathcal{G}_n^0 \rangle_{n \in \boldsymbol{N}}$ be a given sequence of 0-graphs, where $\boldsymbol{N}$ is the set of all natural numbers $n = 0, 1, 2, \ldots$. For each n, we have $\mathcal{G}_n^0 = \{\mathcal{B}_n, \mathcal{X}_n^0\}$, where $\mathcal{B}_n$ is the set of branches and $\mathcal{X}_n^0$ is the set of 0-nodes. Also, for each n, we label all the 0-nodes in order to distinguish them from each other; however, in the following we suppress this labeling—it being understood. Furthermore, let $\mathcal{F}$ be a chosen nonprincipal ultrafilter on $\boldsymbol{N}$ [6, pages 18-19].

In the following, $\langle x_n^0 \rangle$ will denote a sequence of 0-nodes with $x_n^0 \in \mathcal{X}_n^0$ for all n. A nonstandard 0-node ${}^*x^0$ is an equivalence class of such sequences of 0-nodes, where two such sequences $\langle x_n^0 \rangle$ and $\langle y_n^0 \rangle$ are taken to be equivalent if $\{n : x_n^0 = y_n^0\} \in \mathcal{F}$, in which case we write "$\langle x_n^0 \rangle = \langle y_n^0 \rangle$ a.e." or say that $x_n^0 = y_n^0$ "for almost all n." We also write ${}^*x^0 = [x_n^0]$, where it is understood that the x_n^0 are the members of any one sequence in the equivalence class.

That this truly defines an equivalence class can be shown as follows. Reflexivity and symmetry being obvious, consider transitivity: Given that $\langle x_n^0 \rangle = \langle y_n^0 \rangle$ a.e. and that $\langle y_n^0 \rangle = \langle z_n^0 \rangle$ a.e., we have $N_{xy} = \{n : x_n^0 = y_n^0\} \in \mathcal{F}$ and $N_{yz} = \{n : y_n^0 = z_n^0\} \in \mathcal{F}$. By the properties of a filter, $N_{xy} \cap N_{yz} \in \mathcal{F}$. Moreover, $N_{xz} = \{n : x_n^0 = z_n^0\} \supset (N_{xy} \cap N_{yz})$. Therefore, $N_{xz} \in \mathcal{F}$. Hence, $\langle x_n^0 \rangle = \langle z_n^0 \rangle$ a.e.; transitivity holds. We let ${}^*\mathcal{X}^0$ denote the set of nonstandard 0-nodes.

Next, we define the nonstandard branches: Let ${}^*x^0 = [x_n^0]$ and ${}^*y^0 = [y_n^0]$ be two nonstandard 0-nodes. This time, let $N_{xy} = \{n : \{x_n^0, y_n^0\} \in \mathcal{B}_n\}$ and $\overline{N}_{xy} = \{n : \{x_n^0, y_n^0\} \notin \mathcal{B}_n\}$. Since $\mathcal{F}$ is an ultrafilter, exactly one of N_{xy} and $\overline{N}_{xy}$ is a member of $\mathcal{F}$. If it is N_{xy}, then ${}^*b = [\{x_n^0, y_n^0\}] = \{{}^*x^0, {}^*y^0\}$ is defined to be a nonstandard branch; that is, ${}^*b$ is an equivalence class of sequences $\langle b_n \rangle$ of branches $b_n = \{x_n^0, y_n^0\} \in \mathcal{B}_n$, $n = 0, 1, 2, \ldots$.[1] We let ${}^*\mathcal{B}$ denote the set of nonstandard branches. On the other hand, if $\overline{N}_{xy} \in \mathcal{F}$, then $[\{x_n^0, y_n^0\}]$ is not a nonstandard branch.

We shall now show that this definition is independent of the representatives chosen for the 0-nodes. Let $[\{x_n^0, y_n^0\}]$ and $[\{v_n^0, w_n^0\}]$ represent the same nonstandard branch. We want to show that, if $\langle x_n^0 \rangle = \langle v_n^0 \rangle$ a.e., then $\langle y_n^0 \rangle = \langle w_n^0 \rangle$ a.e. Suppose $\langle y_n^0 \rangle \neq \langle w_n^0 \rangle$ a.e. Then, $\{n : x_n^0 = v_n^0\} \cap \{n : y_n^0 \neq w_n^0\} \in \mathcal{F}$. Thus, there is at least one n for which the three nodes $x_n^0 = v_n^0$, y_n^0, and w_n^0 are all incident to the same standard branch—in violation of the definition of a branch.

Next, we show that we truly have an equivalence relationship for the set of all sequences of standard branches. Reflexivity and symmetry being obvious again, consider transitivity: Let ${}^*b = [\{x_n^0, y_n^0\}]$, ${}^*\bar{b} = [\{\bar{x}_n^0, \bar{y}_n^0\}]$, ${}^*\tilde{b} = [\{\tilde{x}_n^0, \tilde{y}_n^0\}]$, and assume that ${}^*b = {}^*\bar{b}$ and ${}^*\bar{b} = {}^*\tilde{b}$. We

[1] Incidence between a nonstandard branch and a nonstandard node and adjacency between two nonstandard nodes are defined in much the same way.

want to show that ${}^*b = {}^*\tilde{b}$. We have $N_{b\bar{b}} = \{n\colon \{x_n^0, y_n^0\} = \{\bar{x}_n^0, \bar{y}_n^0\}\} \in \mathcal{F}$ and $N_{\bar{b}\tilde{b}} = \{n\colon \{\bar{x}_n^0, \bar{y}_n^0\} = \{\tilde{x}_n^0, \tilde{y}_n^0\}\} \in \mathcal{F}$. Moreover, $N_{b\tilde{b}} = \{n\colon \{x_n^0, y_n^0\} = \{\tilde{x}_n^0, \tilde{y}_n^0\}\} \supset (N_{b\bar{b}} \cap N_{\bar{b}\tilde{b}}) \in \mathcal{F}$. Therefore, $N_{b\tilde{b}} \in \mathcal{F}$. Thus, ${}^*b = {}^*\tilde{b}$, as desired.

Finally, we define a nonstandard 0-graph ${}^*\mathcal{G}^0$ to be the pair ${}^*\mathcal{G}^0 = \{{}^*\mathcal{B}, {}^*\mathcal{X}^0\}$.

3 Nonstandard 1-Graphs

A standard 1-graph $\mathcal{G}^1$ is a triplet $\mathcal{G}^1 = \{\mathcal{B}, \mathcal{X}^0, \mathcal{X}^1\}$, where $\mathcal{B}$ and $\mathcal{X}^0$ are sets of standard branches and standard 0-nodes as before and $\mathcal{X}^1$ is a set of standard 1-nodes [7, Sec. 2.1]. Each 1-node x^1 contains a set of 0-tips [7, page 20] of the 0-graph $\mathcal{G}_n^0 = \{\mathcal{B}, \mathcal{X}^0\}$ and at most one 0-node unique to that 1-node. That is, x^1 either contains no 0-node or exactly one 0-node x^0 and that 0-node is not a member of any other 1-node. In the latter case, we write $x^0 \in x^1$.

Now, let $\langle \mathcal{G}_n^1 \rangle_{n \in \boldsymbol{N}}$ be a given sequence of standard 1-graphs. So, for each $n = 0, 1, 2, \ldots$, we have $\mathcal{G}_n^1 = \{\mathcal{B}_n, \mathcal{X}_n^0, \mathcal{X}_n^1\}$. Thus, for almost all n, $\{\mathcal{B}_n, \mathcal{X}_n^0\}$ is a conventionally infinite 0-graph with at least one 0-tip. In addition to the labeling of all the 0-nodes, we label all the 1-nodes in order to distinguish them as well; again we suppress this labeling—it being understood. As before, we let $\mathcal{F}$ be a chosen and fixed nonprincipal ultrafilter. With these items in hand, we construct the sets ${}^*\mathcal{B}$ and ${}^*\mathcal{X}^0$ exactly as before.

In the following, $\langle x_n^1 \rangle$ will denote a sequence of 1-nodes where $x_n^1 \in \mathcal{X}_n^1$ for each n. The definition of a nonstandard 1-node ${}^*x^1$ is virtually the same as that for a nonstandard 0-node. Specifically, ${}^*x^1$ is an equivalence class of such sequences of 1-nodes, where two such sequences $\langle x_n^1 \rangle$ and $\langle y_n^1 \rangle$ are defined to be equivalent if $\{n\colon x_n^1 = y_n^1\} \in \mathcal{F}$. That this truly defines an equivalence class is shown exactly as in the case of 0-nodes. When equivalence holds, we write "$\langle x_n^1 \rangle = \langle y_n^1 \rangle$ a.e." We also write ${}^*x^1 = [x_n^1]$, where again it is understood that the x_n^1 are members of one of the sequences in the equivalence class.

By definition [7, page 22], each standard 1-node x_n^1 of $\mathcal{G}_n^1$ is a set $\mathcal{T}_{n\tau}^0 \cup \mathcal{N}_{n\tau}^0$, where $\cup_\tau \mathcal{T}_{n\tau}^0$ is a partition of the set of 0-tips in the 0-graph $\mathcal{G}_n^0 = \{\mathcal{B}_n, \mathcal{X}_n^0\}$ and $\mathcal{N}_{n\tau}^0$ is either the empty set or a singleton whose sole member is a 0-node in $\mathcal{X}_n^0$, that is not a member of any other 1-node in $\mathcal{G}_n^1$. If $\{n\colon x_n^0 \in x_n^1\} \in \mathcal{F}$, we write ${}^*x^0 = [x_n^0] \in {}^*x^1 = [x_n^1]$.

From the given sequence $\langle \mathcal{G}_n^1 \rangle_{n \in \mathbb{N}}$, we can define the nonstandard 0-tips as follows: Let $\langle t_n^0 \rangle$ be a sequence of 0-tips, one from each $\mathcal{G}_n^0$. Two such sequences $\langle t_n^0 \rangle$ and $\langle s_n^0 \rangle$ will be taken to be equivalent if $\{n : t_n^0 = s_n^0\} \in \mathcal{F}$. This is truly an equivalence relationship; its transitivity can be proven exactly as transitivity was proven for equivalent sequences of 0-nodes. We write ${}^*t^0 = [t_n^0]$, where the t_n^0 are the members of any one sequence in the equivalence class. Now, given any sequence $\langle x_n^1 \rangle$ of standard 1-nodes, we can choose a sequence $\langle t_n^0 \rangle$ of standard 0-tips with $t_n^0 \in x_n^1$ for every n. Thus, we can write ${}^*t^0 = [t_n^0] \in {}^*x^1 = [x_n^1]$. On the other hand, since for each n every t_n^0 is a member of some x_n^1, we have that ${}^*t^0 = [t_n^0]$ is a member of some ${}^*x^1 = [x_n^1]$.

We should now show that, if ${}^*x^1 = {}^*y^1$, where ${}^*x^1 = [x_n^1]$ and ${}^*y^1 = [y_n^1]$, and if a 0-tip t_n^0 of $\mathcal{G}_n^0$ is a member of x_n^1 for almost all n, then t_n^0 is a member of y_n^1 for almost all n. Suppose this is not true. Thus, for almost all n, $t_n^0 \in x_n^1$ and $t_n^0 \notin y_n^1$. Hence, it is not true that ${}^*x^1 = {}^*y^1$. We can conclude that, if ${}^*x^1 = {}^*y^1$, then $t_n^0 \in x_n^1$ if and only if $t_n^0 \in y_n^1$, again for almost all n.

We should also show that, if ${}^*x^0 \in {}^*x^1$, if ${}^*y^0 \in {}^*y^1$, and if ${}^*x^1 = {}^*y^1$, then ${}^*x^0 = {}^*y^0$. Set $N_x = \{n : x_n^0 \in x_n^1\}$, $N_y = \{n : y_n^0 \in y_n^1\}$, and $N_{xy1} = \{n : x_n^1 = y_n^1\}$. We have $N_x, N_y, N_{xy1} \in \mathcal{F}$. Therefore, $N_x \cap N_y \cap N_{xy1} \in \mathcal{F}$. We want to show that $N_{xy0} = \{n : x_n^0 = y_n^0\} \in \mathcal{F}$. Suppose $N_{xy0} \notin \mathcal{F}$. Then, $\overline{N}_{xy0} = \{n : x_n^0 \neq y_n^0\} \in \mathcal{F}$. Hence, $\overline{N}_{xy0} \cap N_x \cap N_y \cap N_{xy1} \in \mathcal{F}$. This means that there exists at least one n for which $x_n^0 \in x_n^1$, $y_n^0 \in y_n^1$, $x_n^1 = y_n^1$, but $x_n^0 \neq y_n^0$. This a a contradiction because each 1-node embraces no more than one 0-node by definition of a 1-node. Similarly, the assumption that ${}^*x^0 \in {}^*x^1$, ${}^*x^1 = {}^*y^1$, and ${}^*y^1$ does not embrace a nonstandard 0-node leads to the contradiction that, for some n, $x_n^1 = y_n^1$ contains a 0-node and does not contain a 0-node. In this way, our definition of a nonstandard 1-node conforms with that of a standard 1-node.

Altogether then, we let ${}^*\mathcal{X}^1$ denote the set of nonstandard 1-nodes and define the nonstandard 1-graph ${}^*\mathcal{G}^1$ to be the triplet:

$$ {}^*\mathcal{G}^1 = \{{}^*\mathcal{B}, {}^*\mathcal{X}^0, {}^*\mathcal{X}^1\}. $$

4 Nonstandard Graphs of Higher Ranks

Virtually the same arguments establish nonstandard graphs of higher ranks. For instance, given a sequence $\langle \mathcal{G}_n^2 \rangle_{n \in N}$ of standard 2-graphs $\mathcal{G}_n^2 = \{\mathcal{B}_n, \mathcal{X}_n^0, \mathcal{X}_n^1, \mathcal{X}_n^2\}$, where $\mathcal{X}_n^2$ is a set of standard 2-nodes, we construct the sets $^*\mathcal{B}$, $^*\mathcal{X}^0$, $^*\mathcal{X}^1$ as before. Then, the nonstandard 2-nodes $^*x^2$ are equivalence classes of sequences $\langle x_n^2 \rangle$ of standard 2-nodes, one 2-node from each $\mathcal{G}_n^2$, with $\langle x_n^2 \rangle$ being equivalent to $\langle y_n^2 \rangle$ if $\{n : x_n^2 = y_n^2\} \in \mathcal{F}$. Nonstandard 1-tips $^*t^1 = [t_n^1]$ are defined much as are nonstandard 0-tips, and it can be shown that, if $^*x^2 = {}^*y^2$ and if $^*t^1 \in {}^*x^2$, then $^*t^1 \in {}^*y^2$. It can also be shown that, if $^*x^\alpha \in {}^*x^2$ and $^*y^\beta \in {}^*y^2$, where $0 \leq \alpha, \beta \leq 1$, and if $^*x^2 = {}^*y^2$, then $^*x^\alpha = {}^*y^\beta$ (which implies of course that $\alpha = \beta$). Then, with $^*\mathcal{X}^2$ denoting the set of all nonstandard 2-nodes, we define the nonstandard 2-graph $^*\mathcal{G}^2$ arising from $\langle \mathcal{G}_n^2 \rangle$ as the quadruplet

$$^*\mathcal{G}^2 = \{^*\mathcal{B}, {}^*\mathcal{X}^0, {}^*\mathcal{X}^1, {}^*\mathcal{X}^2\}.$$

It is easy to set up a recursive argument through which nonstandard graphs of all natural ranks can be defined. Then, applying the same arguments as above with all those nonstandard graphs of all natural-number ranks in hand, we can define the nonstandard graphs of ranks $\vec{\omega}$ and then ω. Then, proceeding recursively, we can define nonstandard graphs of ranks $\omega + k$ ($k = 1, 2, 3, \ldots$), then to the ranks $\omega + \vec{\omega} = \omega \vec{\cdot} 2$, then $\omega \cdot 2$, and so on to still higher ranks.

5 Comparison of Two Different Constructions of Nonstandard Transfinite Graphs

As was mentioned in the Introduction, our prior papers on nonstandard transfinite graphs were based on expanding sequences of finite graphs that filled out transfinite graphs. This approach is subsumed by the following general procedure: As the basic set, we start with the set $\mathbb{G}$ of all graphs (both finite and transfinite). Then, we partition the set $\mathbb{G}^N$ of all sequences of graphs into equivalence classes by considering two sequences $\langle \mathcal{G}_n \rangle$ and $\langle \mathcal{H}_n \rangle$ to be equivalent if $\{n : \mathcal{G}_n = \mathcal{H}_n\} \in \mathcal{F}$, where $\mathcal{G}_n = \mathcal{H}_n$ denotes a graphical isomorphism. Any such equivalence class is denoted by $[\mathcal{G}_n]$, where The $\mathcal{G}_n$ are the members of any one sequence in the class. We then define $[\mathcal{G}_n]$ to be a nonstandard graph. This ultrapower construction is the same as that for the hyperreal numbers except that the real line $\mathbb{R}$ is replaced by the set $\mathbb{G}$ of all graphs. In our prior works, we used a special case of this construction by restricting the sequences $\langle \mathcal{G}_n \rangle$ as stated above. The advantage of that prior procedure is that it allowed us to replace the real currents and voltages in a standard transfinite electrical network by hyperreal currents and voltages which would always satisfy Kirchhoff's laws. This was a consequence of the fact that Kirchhoff's laws are satisfied in finite networks. Kirchhoff's laws are not always satisfied by real currents and voltages in transfinite networks.

In contrast, this paper starts with a sequence of 0-graphs and uses its 0-nodes as the basic elements from which nonstandard 0-nodes and nonstandard branches are constructed in a way analogous to the construction of an internal set from a sequence of sets in $\mathbb{R}$. This yields a nonstandard 0-graph. Then, the 1-nodes of a sequence of 1-graphs are used to obtain nonstandard 1-nodes in a similar way to get a nonstandard 1-graph—and so on to the nonstandard graphs of higher ranks.

Thus, we have two different procedures yielding two different kinds of nonstandard graphs. Further research on both of these seems inviting. For instance, instead of the ultrapower construction used in both cases, a transfer principle should yield our results more efficiently. This would require a symbolic language that encompasses transfinite graphs.

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