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TRANSMISSION NETWORKS

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Abstract — Proposed herein is a new method for solving arbitrary networks of nonuniform transmission ladders and lines. Two cases are considered, the first occurring when all circuit elements are purely resistive and in general nonlinear and the second occurring when they are linear but in general have both resistive and reactive components. The first case arises for example when the DC operating point of a nonlinear interconnection network is to be determined, and the second arises for example when the transient behavior of a network of linear, possibly nonuniform, RLCG transmission lines and ladders is desired. Moreover, our method eliminates the voltages at internal nodes of a ladder and only treats nodes at which three or more ladders or lines meet or just one ladder or line terminates. The internal node voltages are obtained at the end of our procedure with very little additional computation. This provides a computational advantage, which may save orders of magnitude in computation time. A similar facility accrues to cascades of distributed lines and ladders.

1 Introduction

We propose a novel way of analyzing networks of lumped and/or distributed transmission ladders and lines, which we feel has some intrinsic theoretical value and in certain circumstances provides some computational advantages. The method works when the ladders and lines possess certain properties that are not overly restrictive. Two cases are considered. The first concerns the determination of the DC operating point for a transmission network whose elements may be nonlinear functions of voltage or current and whose ladders and lines may be nonuniform. Thus, we will be dealing with purely resistive structures. In the

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second case, the parameters are required to be linear, but now reactances are allowed. For example, we now encounter linear but possibly nonuniform RLCG lines. Our procedure yields a means of efficiently computing the transient behavior of the network.

The first model, the nonlinear purely resistive one has not we claim been previously analyzed with the power of our proposed method. This is primarily a theoretical contribution to the study of electrical circuits. Nonetheless, there is at least one practical configuration that is encompassed by the first model. Consider a fan-out circuit from the collector terminal of a BJT feeding the base terminals of several other BJTs. The output of the first BJT may be represented by a voltage source in series with a nonlinear resistor connected from ground to the input of the fan-out circuit. The outputs of the fan-out circuit are connected to the nonlinear base-emitter resistances of the other BJTs. Linear resistances for the interconnection wires might be adequate, but some interconnects use a reverse-biased p-n junction to get isolation from ground, in which case there is a nonlinear distributed conductance to ground along the length of the interconnect under DC conditions. The question is how much of the output voltage from the first BJT appears as input voltages on the other BJTs — a DC operating point problem. Our method provides a solution. Moreover, our method applies just as well to far more complicated interconnection configurations such as those having transmission loops.

The second case, a network of linear resistance-reactance transmission ladders and lines, is of greater practical interest. Our method provides a new approach to the computation of transient behavior for arbitrary networks of interconnects modeled by nonuniform RLCG transmission lines. We do not allow coupling between lines except when these too are modeled by network elements, the simplest case being lumped capacitors connected to nodes along the lines.

We can treat this second case as a special case of the the first one by restricting the complex variable s for the Laplace transform to a real positive number. Then, all reactances appear as resistances. The transient behavior can then be obtained numerically by applying the Gaver-Stehfest algorithm [11], which needs only finitely many real positive values of s, ten such values being typical and adequate. However, this algorithm works well only when the transients are monotonic or close to monotonic. For more oscillatory transients other algorithms using complex values of s, such as the Singhal-Vlach algorithm [10], [13, Chapter 10], are available. The linear version of our method can be directly extended to this complex case.

How does our method compare to prior works? First of all, it applies to arbitrary networks of transmission ladders and lines, not just to simple configurations of them. Furthermore, it allows the replacement of any cascade of such ladders and lines by a single grounded two-port, thereby eliminating many unknowns before the network equations are to be solved. For example, consider a nonuniform RLCG transmission line. For only certain simple distributed-parameter variations along the line can the line's two-port parameters (e.g., its A,B,C,D parameters) be determined in closed form. For other nonuniform variations, one might replace the line by a lumped ladder network, but this then introduces many additional nodes. Rather than treating the node voltages within each ladder as unknowns — as SPICE would do, our method treats the entire ladder by input-output operators and only their input and output voltages are taken as unknowns. After these fewer unknowns are determined, the internal-node voltages can be obtained with very little additional computation. This same advantage accrues to cascades of nonlinear nonuniform resistive ladders and lines.

A number of methods have been proposed for finding the voltage-current regime in a finite nonlinear resistive network (not to mention the many existence and uniqueness theorems that have appeared). A probably incomplete list of them is the following: [1], [2], [3], [5], [8], [12], [14]. None of these take advantage of the special structure of a transmission network to reduce the number of simultaneous nonlinear equations that must be iteratively solved.

Our method has another advantage over SPICE when DC operating-point analyses are made. This advantage arises when some internal node of a ladder network has incident resistors that are all large. An almost open circuit appears at that node, and the corresponding Jacobian matrix appears close to being singular (i.e., that matrix has a row and a column of near-zero entries). In this case, SPICE can collapse. However, such a situation presents no problem at all for our technique. Our recursive procedure for eliminating such internal node voltages passes through such a node as easily as any other node, and the almost singularity is then masked out by other nodes of the ladder. After the reduced number of unknowns are determined by our procedure, the eliminated voltages are then easily determined.

2 Some Preliminary Ideas

In the following, R^n denotes n-dimensional real Euclidean space with its conventional norm $\|x\| = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}$ for $x = (x_1, \ldots, x_n) \in R^n$. $[a, b]$ denotes a compact interval in R^1 with endpoints a and b and $a < b$. We shall take the adjectives "nonlinear" and "nonuniform" as subsuming "linear" and "uniform"; that is, "linear" is treated as a special case of "nonlinear," and "uniform" is treated as a special case of "nonuniform." We say "operating point" in place of "DC operating point."

By an *increasing locus* we will mean the graph in the (v, i)-plane of a function $v \mapsto i$ that is continuous and strictly monotonically increasing and is such that $i \to +\infty$ as $v \to +\infty$ and $i \to -\infty$ as $v \to -\infty$. By a *decreasing locus* we will mean the graph in the (v, i)-plane of a function $v \to i$ that is continuous and strictly monotonically decreasing and is such that $i \to -\infty$ as $v \to +\infty$ and $i \to +\infty$ as $v \to -\infty$. (The same definitions result when we reverse the roles of v and i.) A *horizontal locus* (resp. *vertical locus*) is a straight line that is horizontal, i.e., i remains constant (resp. vertical, i.e., v remains constant).

Duffin established the following fundamental about 50 years ago [4].

Duffin's Theorem: A finite network of nonlinear resistors excited by independent voltage sources has a unique operating point if the voltage-current characteristic curve of every resistor is an increasing locus.

Henceforth, we always assume the following for the branches (i.e., one-ports) in our network.

Conditions 2.1. Every branch has a voltage-current characteristic curve that is an increasing locus.

We let $G : v \mapsto i$ and $R : i \mapsto v$ denote the corresponding bijective mappings. In general, $G(0) = h$ need not be 0. When this is so, $R(0) = e$ will not be 0 either, and moreover there will be an independent source implicitly encompassed within the branch. This is illustrated in Figure 1.

In addition to branches, we may have distributed transmission lines. More generally, our networks will consist of connections of grounded two-ports. A typical two-port is shown in Figure 2 along with the polarity conventions for its input voltage-current pair $x_0 = (v_0, i_0) \in R^2$ and for its output voltage-current pair $x_1 = (v_1, i_1) \in R^2$. The *forward mapping* of the two-port is the operator $f : x_0 \mapsto x_1$ and its backward mapping is $b : x_1 \mapsto x_0$. The two-ports we allow are specified in the next section, and in every case their forward and backward mappings will always exist, as determined by Kirchhoff's laws and the nonlinear version of Ohm's law; moreover, f and b will be inverse mappings : $b = f^{-1}$. (In the special case where the two-port contains only linear positive resistors but no sources, the backward mapping is represented by the chain matrix of A, B, C, D parameters and the forward mapping by the inverse of that chain matrix.)

3 Building Blocks

Let us now point out several kinds of grounded two-ports that will serve as the basic building blocks for our resistive transmission networks. The networks will be obtained by connecting such two-ports at their port terminals without violating the ground configuration.

3a. The single grounding branch. We will say that a branch is *grounding* if exactly one of its two nodes is the ground node. A two-port having just one branch, a grounding branch, is shown in Figure 3(a); the branch is shown in the Thevenin form. The forward mapping of the two-port is given by

$$v_1 = v_0, \tag{1}$$

$$i_1 = i_0 - g(v_0 + e). \tag{2}$$

3b. The single floating resistor. We will call a branch *floating* if neither of its nodes is the ground node. A two-port having just one branch, a floating branch, is shown in Figure 3(b); it is convenient now to represent the branch in the Norton form. The forward mapping of the two-port is given by

$$v_1 = v_0 - r(i_0 + h), \tag{3}$$

$$i_1 = i_0. \tag{4}$$

3c. The lumped ladder network. Consider now a ladder network obtained by alternately cascading the two-ports of Figures 3(a) and 3(b). The result is a cascade of el-sections as shown in Figure 3(c). For the first el-section the forward mapping $f_0 : (v_0, i_0) \mapsto (v_1, i_1)$ is given by

$$v_1 = v_0 - r_0(i_1 + h_0), \tag{5}$$

$$i_1 = i_0 - g_0(v_0 + e_0), \tag{6}$$

and similarly for the subsequent el-sections. The forward mapping of the ladder is then the composition

$$f = f_{n-1} f_{n-2} \cdots f_0 : (v_0, i_0) \mapsto (v_n, i_n),$$

where f_k is the forward mapping of the kth el-section. Moreover, the first branch of the ladder may be a floating branch, and the last branch may be a grounding branch. In both cases, f may be obtained by altering f_0 and f_{n-1} in accordance with (1) through (4). When all the sources in the cascade are absent, the ladder of Figure(c) becomes the basic lumped transmission line, which in general may be nonuniform and nonlinear.

3d. The distributed resistive transmission line. The last of our building blocks is the distributed version of the lumped transmission line. It is illustrated in Figure 3(d). It too has no internal sources and is in general nonuniform and nonlinear. In particular, the voltage $v(l)$ and current $i(l)$ depend upon the distance l along the line from the input at $l = 0$ to the output at $l = L$. We shall abuse notation by letting v and i be the values $v(l)$ and $i(l)$ of the line voltage and line current at an unspecified point l along the line; thus, v and i denote the range values $v(l)$ and $i(l)$ in volts and amperes respectively, instead of the corresponding mappings $l \mapsto v(l)$ and $l \mapsto i(l)$. (A more precise notation would encumber our equations unnecessarily.) Furthermore, $r(l, i)$ will denote the rate of voltage decrease at the point l and current i, and $g(l, v)$ will denote the rate of current decrease at the point l and voltage v. Thus, r and g represent the effects of distributed, nonuniform, and nonlinear series resistance and conductance-to-ground for the line. With the prime denoting differentiation with respect to l, we have the line equations:

$$v'(l) = -r(l, i(l)), \tag{7}$$

$$i'(l) = -g(l, v(l)). \tag{8}$$

They govern the variations of v and i in the forward direction from input to output. Moreover, they can be written more concisely by using a matrix-like notation. Set $x(l) = (v(l), i(l)) \in R^2$ for each $l \in [a, b]$. Then,

$$x'(l) = \begin{bmatrix} v'(l) \\ i'(l) \end{bmatrix} = \begin{bmatrix} 0 & -r(l,\cdot) \\ -g(l,\cdot) & 0 \end{bmatrix} \begin{bmatrix} v(l) \\ i(l) \end{bmatrix} = F(l, x(l)). \tag{9}$$

Thus, F maps $[0, L] \times R^2$ into R^2. Here too, we will abuse notation by letting x and $x(l)$ denote the same value in R^2.

A solution to these equations for $0 \leq l \leq L$ and for the initial conditions $v(0) = v_0$ and $i(0) = i_0$ yields the forward mapping f of the line. Upon setting $x_0 = (v_0, i_0) \in R^2$ and $x_1 = (v_1, i_1) \in R^2$, where $v_1 = v(L)$ and $i_1 = i(L)$, we have $f : x_0 \mapsto x_1$. The backward mapping is $b : x_1 \mapsto x_0$. These mappings exist under certain conditions which we now specify.

Let W represent some nonvoid open interval in R^1. We shall say that *$r(l, i)$ is Lipschitz in i on W and that $g(l, v)$ is Lipschitz in v on W both uniformly with respect to all $l \in [0, L]$* if there exists a constant K_W not depending on l such that

$$|r(l, i) - r(l, \hat{i})| = K_W |i - \hat{i}| \tag{10}$$

and

$$|g(l, v) - g(l, \hat{v})| = K_W |v - \hat{v}|. \tag{11}$$

for all $l \in [0, L]$ and for all $i, \hat{i}, v, \hat{v} \in R^1$. These conditions do not restrict the growth of $r(l, i)$ or of $g(l, v)$ as $|i|$ or $|v|$ tends to ∞; we need merely increase the constant K_W as W increases in size.

Henceforth, the following conditions are assumed.

Conditions 3.1.

(a) $r(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are continuous functions from $[0,L] \times R^1$ into R^1. Moreover, for each l, we have that $r(l,0) = g(l,0) = 0$ and that $r(l,\cdot)$ and $g(l,\cdot)$, as functions of their second arguments, are strictly monotonically increasing functions and are bijections from R^1 onto R^1.

(b) For each nonvoid open interval W in R^1, $r(l,i)$ is Lipschitz in i on W and $g(l,v)$ is Lipschitz in v on W both uniformly with respect to all $l \in [0,L]$.

Condition 3.1(1) implies that $F(\cdot,\cdot)$ is continuous from $[0,L] \times R^2$ into R^2. By Condition 3.1(2), $F(l,x)$ is Lipschitz with respect to x on $W \times W$ uniformly for all $l \in [0,L]$; that is,

$$\|F(l,x) - F(l,\hat{x}\| \leq K_W \|x - \hat{x}\| \tag{12}$$

for all $l \in [0,L]$ and all $x, \hat{x} \in W \times W$. These properties allow us to invoke a standard theorem (see, for instance, Theorem 1 on page 297 of [6]) to assert the following.

Theorem 3.2. *Let $l_a \in [0,L]$ and $x_a \in R^2$. Assume Conditions 3.1. Then, there exists a unique solution $x(l)$ to (9) defined for all $l \in [0,L]$ such that $x(l_a) = x_a$. Moreover, the mapping $l \mapsto x(l)$ is differentiable from $[0,L]$ into R^2.*

Proof. The cited theorem in [6] only asserts the unique existence of a differentiable x on an open interval $J \subset [0,L]$ with $l_a \in J$. However, since the conclusion holds for every $l_a \in [0,L]$, we can get the unique existence of the trajectory for all $l \in [0,L]$ by piecing together the open intervals (open with respect to $[0,L]$) around each point of $[0,L]$. Indeed, if J is one such interval and if c is one of its limit end points, then there will be another open interval $\hat{J}$ containing c, and the two solutions x on J and $\hat{x}$ on $\hat{J}$ must coincide on $J \cap \hat{J}$ if that cited theorem is to hold at any point of $J \cap \hat{J}$. ♣

4 Mappings of Loci

In this section we shall note that the two-port building blocks forward map decreasing loci into decreasing loci and similarly for horizontal and vertical loci.

Lemma 4.1.

(a) The two-port for a grounding branch (Figure 3(a)) forward maps a decreasing locus or horizontal locus into a decreasing locus, and forward maps a vertical locus into a vertical locus.

(b) The two-port for a floating branch (Figure 3(b)) forward maps a decreasing locus or a vertical locus into a decreasing locus and forward maps a horizontal locus into a horizontal locus.

(c) Any ladder network of grounding branches and floating branches having at least one grounding branch and one floating branch (such as in Figure 3(c)) forward maps each decreasing locus, each vertical locus, and each horizontal locus into a decreasing locus.

(d) Every distributed resistive transmission line (Figure 3(d)) forward maps each decreasing locus, each vertical locus, and each horizontal locus into a decreasing locus.

Note. Similar properties hold for backward mappings when dealing with increasing loci.

Proof. Assertions (a) and (b) follow directly from Conditions 2.1 and equations (1) through (4). Compositions of the mappings corresponding to assertions (a) and (b) then leads to assertion (c). Finally, to verify assertion (d), let

$$\begin{aligned} Q_4 &= \{(v,i) : v > 0, i < 0\}, \\ V_+ &= \{(v,i) : v > 0, i = 0\}, \\ I_- &= \{(v,i) : v = 0, i < 0\}. \end{aligned}$$

Thus, Q_4 is the open fourth quadrant of the (v,i)-plane, V_+ is its positive voltage axis, and I_- is its negative current axis. Also, let $y = (v_y, i_y)$ and $x = (v_x, i_x)$ be two different points in the voltage-current plane for a point l on the transmission line, and consider the difference $y' - x'$ between the tangent vectors of the locus. By (1) and (2),

$$y'(l) - x'(l) = (r(l, i_x(l)) - r(l, i_y(l)), g(l, v_x(l)) - g(l, v_y(l))).$$

According to the monotonicity properties asserted in Condition 3.1(a), we have the following results at each value of l. If $y - x \in Q_4$, then $y' - x' \in Q_4$. If $y - x \in V_+$, then $y' - x' \in I_-$. If $y - x \in I_-$, then $y' - x' \in V_+$. Since these hold whatever be l and since trajectories cannot stay fixed except at the origin, it follows immediately once more that the forward mapping of the line sends a decreasing, horizontal, or vertical locus into a decreasing locus. □

5 Transmission Networks and Schematics

As was indicated above, we will throughout this paper take a *transmission network* to be a network obtained by connecting the two-ports of Section 3 at their port terminals while maintaining the common ground node throughout. The *schematic* is the graph obtained by representing each maximal cascade in the transmission network by a line segment, which we shall refer to as a *leg*. Thus, each node of the schematic corresponds to a non-ground node of the transmission network incident to three or more two-ports or incident to just one two-port (the non-ground nodes of Figure 3(a) being treated as different nodes). Non-ground nodes incident to exactly two two-ports are interior nodes of a cascade and do not appear in the schematic. For example, Figure 4(a) shows a transmission network consisting of nine maximal ladders. Two of them close on themselves to form transmission self-loops. The corresponding schematic is shown in Figure 4(b). It has nine legs, whereas the network of Figure 4(a) has 63 branches.

Our procedure for solving a transmission network is based upon the schematic, and in particular upon the choice of a spanning tree in the schematic. It is advantageous but not necessary to choose a spanning tree with the minimum number of ends, for this will lead to the minimum number of unknowns and thereby to the minimum number of independent equations that need to be solved. That minimum number is equal to the number of end nodes of the chosen spanning tree plus the number of chords in the schematic. Each self-loop counts as a chord. Thus, for Figure 4(b) that number is $3 + 4 = 7$. SPICE on the other hand will require 30 unknowns, one for each non-ground node. Later on, we will argue that the equations our procedure generates are no overly complicated and that the entire procedure does provide a computational advantage over SPICE.

6 Transmission Trees

Before we present our general method for solving a transmission network, let us consider a *transmission tree*, that is, the special case where the schematic is a tree. The unknowns are now the voltages at the end nodes of the tree. To be specific, we shall explain our method in terms of the transmission tree whose schematic is shown in Figure 5. v_p will denote the node voltage at node n_p. Choose any end node and label it n_0. Then place a number k alongside every node in the schematic, where k is the number of legs in the schematic tree between that node and node n_0; n_0 gets the number 0. We shall refer to k as the *level* of the node. Thus, there may be several nodes at the same level. The forward mapping of each leg will be oriented from its node of higher level to its node of lower level, as is indicated by the arrowheads in Figure 5. Next, number all the legs and denote the mth leg by l_m. The input voltage and input current for l_m will be denoted by v_m and i_m, and the output voltage and output current by u_m and j_m, as is indicated in Figure 6. Furthermore, let us treat any grounding branch at a node of the schematic as part of one of the ladders incident to that node. Thus, the input current to any end leg is 0.

To solve the transmission tree of Figure 5, we start at level 5 and forward map $(v_1, 0)$, $(v_2, 0)$, and $(v_3, 0)$ through l_1, l_2, and l_3 respectively to get (u_1, j_1), (u_2, j_2), and (u_3, j_3) as functions of the unknowns v_1, v_2, and v_3 respectively. Equating node voltages at n_9, we get $u_1 = u_2 = u_3$. This represents two independent equations for Kirchhoff's voltage law around two loops, The first one corresponding to $u_1 = u_2$ proceeds from ground, then along l_1 and back along l_2, and then to ground. The one corresponding to $u_2 = u_3$ proceeds from ground, then along l_2 and back along l_3, and then to ground. The equation for $u_1 = u_3$ is a combination of the first two. Then, upon summing the output currents from l_1, l_2, and l_3, we get the input current i_9 to l_9: $i_9 = j_1 + j_2 + j_3$. The input voltage v_9 to l_9 is $v_9 = u_1 = u_2 = u_3$.

(Later on, when we use an iterative method to solve the nonlinear equations, the output voltages u_1, u_2, and u_3 will not in general be the same at each step of the iteration. In this case, we designate the input voltage v_9 as equal to any one of the output voltages, say, u_1 — or alternatively to the average or some other combination of u_1, u_2, and u_3. The same thing is done at the other internal nodes of the schematic tree. In this way, the only unknown voltages being sought in the iterative stage of the solution procedure are the end-node voltages $v_0, \ldots, v_8$. Once these are obtained, all the node voltages of the transmission tree can be obtained through forward or backward mappings without any further iterations.)

We now repeat this procedure with forward mappings from level 4 to level 3 to get two more independent equations: $v_{10} = u_4 = u_9$ and $v_{11} = u_5 = u_6$. Summations of currents yield $i_{10} = j_4 + j_9$ and $i_{11} = j_5 = j_9$. These u's and j's are also functions of the unknown end-node voltages. Forward mappings from level 3 to level 2 gives two more independent equations: $v_{12} = u_7 = u_{10} = u_{11}$. It also gives $i_{12} = j_7 + j_{10} + j_{11}$. From level 2 to level 1, we have one more independent equation, $v_{13} = u_8 = u_{12}$ and in addition $i_{13} = j_8 + j_{12}$. Finally, at the initially chosen end node n_0, we have two more independent equations involving the unknown end-node voltages, namely, $v_0 = u_{13}$ and $j_{13} = 0$. Altogether, we have nine independent equations in the nine unknown end-node voltages. Our equations insure that Kirchhoff's laws and the nonlinear form of Ohm's law are satisfied everywhere, and therefore they will yield the unique operating point for the network. Duffin's theorem insures the last conclusion when all elements are lumped. If there are distributed lines, Duffin's theorem can still be invoked by taking each distributed line as the limit of a sequence of discretizations of that line.

That we will always have the correct number of independent equations for any transmission tree follows from the following theorem.

Lemma 6.1. For any schematic that is a tree, the number of independent equations generated by voltage equalities at all the non-end nodes is two less than the number of end nodes.

Proof. Let E be the number of end-nodes (counting n_0 as well), let I be the number of non-end nodes, and let d_i be the degree of the ith non-end node. The total number of nodes is $E + I$. Since we are dealing with a tree, the total number of legs is $E + I - 1$; that number is also $(E + \sum_{i=1}^{I} d_i)/2$. Thus,

$$I = -\frac{E}{2} + 1 + \frac{1}{2}\sum_{i=1}^{I} d_i.$$

Now, the number of independent voltage equations generated by voltage equalities is $\sum_{i=1}^{I}(d_i - 2)$. Hence,

$$\sum_{i=1}^{I}(d_i - 2) = \sum_{i=1}^{I} d_i - 2I = E - 2.$$

as asserted. □

It may be informative to note how the monotonicities of the resistances come into play in our method. Consider again node n_1 of Figure 5 and assume that every leg has at least one grounding branch. (When this is not so, the leg will simply be the two-port of Figure 3(b), and our argument can be easily adjusted to accommodate this case.) If v_1 is adjusted upward, then the input current of leg l_1 remains at 0, but, by virtue of Lemma 6.1, the output voltage u_1 of l_1 adjusts upward, and the output current j_1 of l_1 adjusts downward. A similar assertion holds for every other end-node voltage. Consequently, if v_1 is adjusted upward, then v_2 and v_3 must also be adjusted upward in order to satisfy Kirchhoff's voltage law. In this case, j_1, j_2, and j_3 adjust downward, and therefore i_9 adjusts downward too.

We can now argue in the same way for legs l_9 and l_4 to conclude that v_4 must also be adjusted upward if the equation $u_9 = u_4 = v_{10}$ is to hold. In fact, these latter voltages adjust upward too. Furthermore, the currents at node n_{10} satisfy $i_{10} = j_9 + j_4$. It follows as before that i_{10} adjusts downward.

These monotonic relationships can be traced throughout our procedure and lead to the following conclusion. If any end-node voltage is adjusted upward, all other end-node voltages (including v_0) must also adjust upward if node voltage equations are to be satisfied. Moreover, output currents will adjust downward. In particular, for leg l_{13} and node n_0, we have that j_0 adjusts downward. But, at node n_0 we must have $j_0 = 0$. Since an increasing v_0 corresponds to a decreasing j_0, there is a unique value of v_0 for which $j_0 = 0$. All the other voltages and currents in the transmission tree can now be obtained from the found value of v_0 and from $j_0 = 0$ through backward mappings of the legs or parts of the legs.

Let us also observe that v_0 and then all voltages and currents can be obtained through a very simple graphical procedure. We start with a horizontal line at zero current at each of the end nodes other than n_0. These horizontal loci are forward mapped through the legs of the tree in accordance with our procedure to obtain a single decreasing locus at n_0. Where that locus crosses the zero-current axis is the value of v_0. This can be done quite efficiently through the secant method, and convergence is assured because that single locus is decreasing. During the procedure, the output locus of each leg should be stored in memory. Then, the pair $(v_0, 0)$ can be successively backward mapped, with the output voltage-current pair of any leg being determined by the input voltage of its succeeding leg, to obtain all voltages and currents. This reduction of the problem to finding where a single decreasing locus crosses the abscissa works for every transmission tree.

On the other hand, instead of this graphical procedure, we might work with analytical expressions for the forward and backward mappings obtained from formulas for the elements of the transmission tree. Then, any standard procedure for solving a system of nonlinear equations can be employed, as for instance the Newton-Raphson method [9] if characteristic curves are given by continuously differentiable functions, or the secant method otherwise. Although the forward and backward mappings will now be combinations of the element formulas of possibly very high order if there are many elements in a cascade, the number of unknowns will remain equal to the number of end nodes. Thus, for example, the Jacobian matrix — in the case of the Newton-Raphson method — will remain of low order even though the determination of its terms may require extended differentiations. The latter is straightforward; it does not involve root solving. Thus, we feel that our method provides a computational advantage over SPICE. We witnessed this when trying several examples.

7 Transmission Networks with Transmission Loops

The most general kind of transmission network is one whose schematic has loops. Each of the latter reflect a *transmission loop* within the original network, that is, a sequence of two-port cascades connected terminal-to-terminal with the last terminal connected to the first one and possibly with other two-ports connected at intermediate terminals.

Again it is easier to explain our method by referring to an example. Let it be the transmission network of Figure 4(a); its schematic is shown in Figure 4(b). We have chosen a tree with the minimal number of ends and have designated one of them as the node n_0 at level 0. Passing along the tree from n_0 we label the levels at all the other nodes as shown and orient each tree leg from higher level to lower level. As for the chord legs that are not self-loops, we orient them in the same way, that is, from higher level to lower level. The self-loop legs (and, for more general networks, the legs connecting nodes at the same level) are given any orientation. As always, we will treat any grounding branch at a node of the schematic as part of one of the legs incident to that node.

This time, the unknown independent variables will be all the end node voltages, namely, v_0, v_1 and v_2 along with the input currents in each chord, namely, i_6, i_7, i_8, and i_9 (see Figures 4 and 5). We now analyze in much the same way as for a transmission tree by following the chosen spanning tree from higher levels to lower levels in order to write the needed independent equations.

At the highest level, level 4, we have the inputs $(v_1, 0)$ and $(v_2, 0)$. Forward mapping them, we get u_1, j_1, u_2, and j_3 as functions of v_1 and v_2. Equating voltages at n_3, we have $u_1 = u_2$; this provides one independent equation in v_1 and v_2. Also, $v_3 = u_1 = u_2$ gives the input voltage v_3 to legs l_3, l_6, and l_7. Then, the input current for leg l_3 is $i_3 = j_1 + j_2 - i_6 - i_7$. We can now forward map (v_3, i_3) along leg l_3 to get (u_3, j_3) as a function of v_1, v_2, i_6, and i_7. Here, $u_3 = v_4$. Similarly, the forward mappings of (v_4, i_8) and (v_4, i_9) along l_8 and l_9 respectively give (u_8, j_8) and (u_9, j_9) as functions of v_1, v_2, i_6, i_7, i_8, and i_9. Equating voltages, we have $v_4 = u_8$ and $v_4 = u_9$. These two independent equations represent Kirchhoff's voltage law around the self-loops l_8 and l_9. Moreover, $i_4 = j_3 - i_8 + j_8 - i_9 + j_9$. Proceeding further along the tree, we forward map (v_4, i_4) along l_4 and also (v_3, i_6) and (v_3, i_7) along l_6 and l_7. This allows us to write $v_5 = u_4 = u_6 = u_7$, which represents Kirchhoff's voltage law around the loops l_3-l_4-l_6 and l_3-l_4-l_7, two more independent equations. Furthermore, $i_5 = j_4 + j_6 + j_7$, and a forward mapping of (v_5, i_5) gives (u_5, j_5). Two more independent equations are obtained from $u_5 = v_0$ and $j_5 = 0$. Altogether we have seven independent equations in the seven unknowns, and Kirchhoff's laws and Ohm's nonlinear law are satisfied everywhere. After solving for the unknowns, we can use forward and backward mappings to get all the voltages and currents in the transmission network.

(At the iterative stages of solving nonlinear equations, u_1 and u_2 will in general be different; so we may set v_3 equal to one of them, say u_1, thereby eliminating v_3 as an unknown for the itertions. Similarly, we may set $v_5 = u_3$ and $v_5 = u_4$.)

The secant method can be used to solve these equations when element characteristics are given as numerical data, and the Newton-Raphson method is applicable when those characteristics are given as continuously differentiable functions. There are of course other methods for solving nonlinear equations such as homotopy methods.

Example 7.1. To check our method against SPICE, we took it that each maximal ladder in Figure 4(a) was uniform with the number of el-sections shown therein. The shunting conductances g_k and the series resistances r_k for every one of the el-sections in the kth leg ($k = 1, \ldots, 9$) were chosen to be

$$\begin{aligned} g_1(v) = g_2(v) = g_3(v) = g_5(v) &= 0.0005\,(v^3 + 3v), \\ r_1(i) = r_2(i) = r_3(i) = r_5(i) &= 0.05\,(i^3 + 2i), \\ g_4(v) = g_6(v) = g_7(v) = g_8(v) = g_9(v) &= 0.0005\,(2v^3 + v), \\ r_4(i) = r_6(i) = r_7(i) = r_8(i) = r_9(i) &= 0.05\,(2i^3 + i), \end{aligned}$$

with $e = h = 0$. Also, we assumed the existence of grounding branches at nodes n_0, n_1, and n_2 and set their conductances g_{gk} and series voltages sources e_k ($k = 0, 1, 2$) equal to the following values.

$$\begin{aligned} g_{g1}(v) &= 0.1\,v, & e_1 &= 4, \\ g_{g2}(v) &= 0.1\,v, & e_2 &= 3, \\ g_{g0}(v) &= 0.1\,v, & e_0 &= 1. \end{aligned}$$

We implemented our method with a C program to solve for our seven unknowns and also ran a SPICE program, which required 30 unknowns. The results for the end-node voltages v_0, v_1, and v_2 for both programs are shown in Table 1. They match to the fifth decimal. The computation time for SPICE was .54 seconds, whereas that for our method was .02 seconds.

8 The General Procedure

Let us now state explicitly our general procedure for solving resistive nonlinear transmission networks having perhaps transmission loops.

Draw its schematic, and choose a spanning tree in it. (It is preferable but not necessary to choose a tree with the minimum number of ends.) Any grounding branch at a node of the schematic should be treated as part of an incident leg. Choose any end node n_0 of the tree and let its level be 0. The level of every other node is the number of legs in the tree between that other node and n_0. Next, orient every leg from its node of higher level to its node of lower level, but, if there is no difference in those levels, orient the leg arbitrarily.

TABLE 1
The End-Node Voltages
From Example 7.1

	Our Method	SPICE
v_1	1.92666	1.92666
v_2	1.90643	1.90643
v_3	1.85257	1.85257

Similarly, arbitrarily orient any leg that forms a self-loop. Consequently, at each node other than n_0, there will be exactly one outgoing tree leg, but any number of incoming tree legs, that number being zero when and only when that node is an end node of the tree. Also, at every node other than n_0, there can be any number of outgoing chord legs and any number of incoming chord legs. At n_0 there can be any number of incoming chord and tree legs but no outgoing legs unless there are self-loop legs incident to n_0. The unknown variables are the node voltages at the end nodes of the tree and also the input currents to each chord leg.

We will be using the forward mappings of the legs. These are directly available if each leg is a ladder whose elements are specified either numerically or as explicit functions. If there is a distributed line anywhere, its forward mapping must be determined or, failing that, approximated by a ladder with appropriately chosen element characteristics.

Assume for now that all element characteristics are given as explicit functions. Consider all the nodes at the highest level. They will be some of the end nodes of the chosen spanning tree of the schematic. Their voltages and the input currents of all their outgoing chords are unknowns. Forward map these quantities along their respective legs to get functions of those unknowns. This yields the output voltages and currents of all the incoming chords at those nodes of highest level. Let n_k be one of those nodes. Equate the output voltages of the incoming chords at n_k to that node's voltage v_k. This will yield a number of independent equations in the unknowns, that number being equal to the number of incoming chords at n_k. Next, sum algebraically all the currents entering and leaving n_k to get the input current i_m of the unique tree leg l_m incident to n_k as a function of the unknowns. Forward map (v_k, i_m) along that tree leg l_m. Do this at all nodes of highest level.

All the forward mappings performed so far yield some or all of the incoming currents at each node n_q at the next-to-highest level (and possibly other such currents for nodes at still lower levels) as functions of the unknowns at the highest level. Moreover, the voltage v_q at n_q is equal to every one of the output voltages of the incoming legs at n_q. (During the iteration stages of the nonlinear numerical procedure, we may set v_q equal to just one of those output voltages — or alternatively to their average value.) The input currents of the outgoing chords at n_q are also unknowns. Forward map them and v_q too along their respective legs. Do this for all nodes at the next-to-highest level. Thus, we now have all the output voltages and output currents of the incoming chords and incoming tree legs at n_q again as functions of unknowns. Upon equating all the output voltages at n_q to each other and to v_q, we will have still more independent equations. The number of these will be one less than the number of incoming tree legs at n_q plus the number of incoming chord legs at n_q. Next, at each node at the next-to-highest level, say, n_q again, sum algebraically all the currents entering and leaving n_q to get the input current i_n of the unique outgoing tree leg l_n incident to n_q. Forward map (v_q, i_n) along l_n.

This process can be continued, working successively from nodes at higher levels to nodes at lower levels. Upon equating output voltages for all the incoming legs at each node — and doing so finally at the initially chosen 0-level node n_0, we will have a number of independent equations that is two less than the number of unknowns. Indeed, this follows from Lemma 6.1 (which is stated for a transmission tree) and the fact that each chord provides one more equation. We then set the unknown v_0 equal to the common output voltages for the incoming legs at n_0 and set the algebraic sum of all the output and input currents for the legs incident to n_0 equal to 0 to get two more equations. (If there are no self-loop legs incident to n_0, there are no outgoing chords at n_0.)

Now, solve these independent nonlinear equations by whatever numerical procedure that works. This presents the same difficulties that adhere to all such nonlinear problems, but we now have the advantage in general of dealing with far fewer unknown variables than conventional circuit analyses require. Of course, our procedure requires the determination of the forward mappings of all the legs, but these are simply compositions of element and distributed-line functions and do not require root solving; that is, no equations need to be solved to get those compositions. To be sure, those compositions will have many terms when there are many elements in each leg, but a computer can implement these easily.

In fact, our entire procedure can be automated. Despite the several steps inherent in our procedure, it is quite straightforward with only a few computational rules.

Finally, let us take up the case where some or all of the element and distributed-line characteristics are given as numerical data. This means that any such distributed line will in fact be approximated by a lumped ladder. A numerical technique that can be used in this case is the secant method [9]. To use it, one makes an initial guess of numerical values for all the unknowns. These values can be forward mapped throughout the network using the numerically given element characteristics. (This can be done rapidly; no iterative solving is involved.) This yields a measure of the error produced by the initial guess. A second guess is made and treated similarly. The secant method then uses those two error measures to make a better numerical guess of all the unknowns. The process can be continued by using at each step the last two error measures to get a sequence of sets of values for the unknowns, which (hopefully) will converge to the operating-point values.

9 Transient Behavior of Resistive-Reactive Transmission Networks

Henceforth, we restrict our attention to linear transmission networks that have reactances in addition to resistances. As before, the distributed transmission lines may be nonuniform, but linearity is now needed because the Laplace transformation will be used to allow an application of our procedure in the transformed domain. The basic idea is that an inductive impedance Ls and a capacitive impedance $1/Cs$ act like resistances when the complex variable s is restricted to the real positive axis R_+. Moreover, there are numerical algorithms, such as the Gaver-Stehfest algorithm [11], by which a transient response can be computed from a knowledge of a Laplace transform at a finite number of points on R_+. The Gaver-Stehfest algorithm is very fast and works well when the transient is continuous, bounded, and either monotonic or close to monotonic (as for example when the inductances are small as compared to the capacitances — a not uncommon situation).

Thus, given the finitely many points on R_+, we repeat our procedure for each value of s at each of those points and then employ the Gaver-Stehfest algorithm to compute transient behavior. This requires the determination of the forward mappings for each such s, and in the case of an RLCG nonuniform transmission line, whose forward mapping cannot be determined explicitly, a lumped ladder network may be used to get an approximation for the needed forward mapping. On the other hand, because of linearity our procedure now involves nothing more than matrix manipulations, and the independent equations can be solved by inverting a matrix. As a result, our method can be extended to complex values of s when the transmission network is linear. This allows us to use other algorithms for inverting the Laplace transformation such as the Singhal-Vlach algorithm [10], [13, Chapter 10], which employs complex quantities.

Let us be more specific about the forward and backward mappings of a nonuniform linear distributed RLCG transmission line. The line voltage $v(l,t)$ and line current $i(l,t)$ are now time-varying and depend upon the spatial distance l along the line and the time t. With the same polarities as those shown in Figure 3(d), the equations governing the line are now

$$\frac{\partial v(l,t)}{\partial l} = -L(l)\frac{\partial i(l,t)}{\partial t} - R(l)i(l,t), \tag{13}$$

$$\frac{\partial i(l,t)}{\partial l} = -C(l)\frac{\partial v(l,t)}{\partial t} - G(l)v(l,t). \tag{14}$$

The distributed parameters $L(l)$, $R(l)$, $C(l)$, and $G(l)$ are respectively the series inductance in henries per meter, the series resistance in ohms per meter, the capacitance-to-ground in farads per meter, and the conductance-to-ground in mhos per meter. (Henceforth, $L(l)$ denotes the distributed inductance, whereas L — without the argument symbol (l) — denotes the length of the line.) Upon applying the distributional Laplace transformation [15], we convert these equations into the following.

$$\frac{dV(l,s)}{dl} = -(L(l)s + R(l))I(l,s) \tag{15}$$

$$\frac{dI(l,s)}{dl} = -(C(l)s + G(l))V(l,s) \tag{16}$$

(The use of the distributional Laplace transformation allows us to incorporate initial values into the transforms of derivatives [15].) In order to invoke the results of Subsection 3d, we need merely assume the following.

Conditions 9.1. $L(l)$, $R(l)$, $C(l)$, and $G(l)$ represent continuous positive functions of l on the compact interval $[0, L]$.

Hence, there are two positive constants K and M such that $L(l)$, $R(l)$, $C(l)$, and $G(l)$ are all bounded above by K and bounded below by M. Again we abuse notation by letting

V represent $V(l,s)$ and I represent $I(l,s)$, rather than the corresponding functions. Then, for fixed $s > 0$ again,

$$(l, I) \mapsto (L(l)s + R(l))I$$

and

$$(l, V) \mapsto (C(l)s + G(l))V.$$

are continuous functions from $[0,L] \times R^1$ into R^1, which obviously satisfy all of Conditions 3.1 (with K_W replaced by $Ks + K$). Upon setting $X = X(l,s) = (V(l,s), I(l,s))$, we can therefore restate Theorem 3.2 for this case as follows.

Theorem 9.2. Fix $s > 0$, and choose $l_a \in [0,L]$ and $X_a(s) \in R^2$. Assume Conditions 9.1. Then, there exists a unique solution $X(l,s)$ to (15) and (16) defined for all $l \in [0,L]$ such that $X(l_a,s) = X_a(s)$. Moreover, the mapping $l \mapsto X(l,s)$ is differentiable from $[0,L]$ into R^2.

Needless to say, all the other results of Sections 3 and 4 also carry over to the present case. In particular, in the Laplace-transform domain the line has an input-output representation with a forward mapping $F(s) : X(0,s) \mapsto X(L,s)$ and a backward mapping $B(s) = F(s)^{-1}$: $X(L,s) \mapsto X(0,s)$. Thus, we have the following version of Lemma 4.1(d).

Lemma 9.3. A nonuniform linear distributed RLCG line whose parameters satisfy Condition 9.1 has a Laplace-transformed input-output representation whose forward mapping $F(s)$ sends each decreasing locus, each vertical locus, and each horizontal locus into a decreasing locus (when f is replaced by $F(s)$ and where s is a fixed positive number.)

Note. The loci we now deal with are straight lines — again because of linearity.

We now have all we need in order to empower our version of a transient analysis.

Example 9.4. Consider the transmission network shown in schematic form in Figure 7. Each line represents an exponentially tapered RC line, all identical. A linear Thevenin branch with a (time-varying) voltage e and a 1 Ω resistor appears at node n_1. This will be the one and only source driving the network. All other nodes of the schematic have 1 Ω grounding resistors. We have chosen a spanning tree with just two ends, and n_0 is chosen as the node furthest to the right. The other node indices coincide with the node levels. Each leg is oriented as indicated in Figure 7 by arrowheads on the leg, the direction of the forward mapping for the leg. Thus, the distributed linear series resistance $R(l)$ and distributed linear shunting capacitance $C(l)$ are given by $R(l) = R_0 e^{\alpha l}$ and $C(l) = C_0 e^{-\alpha l}$, where R_0, C_0, and α are all positive constants and l varies from 0 to L in accordance with the arrowheads. We call this a *widening taper* and use the dotted arrows in Figure 7 to indicate the direction of widening. A negative α would indicate a *narrowing taper.* Thus, in the direction of the leg orientations, the legs l_1, l_2, l_4, and l_5 have widening tapers, and legs l_3 and l_6 have narrowing tapers.

In the widening direction, each line has the matrix description

$$\begin{bmatrix} V(L,s) \\ I(L,s) \end{bmatrix} = W \begin{bmatrix} V(0,s) \\ I(0,s) \end{bmatrix} = \begin{bmatrix} w_{00} & w_{01} \\ w_{10} & w_{11} \end{bmatrix} \begin{bmatrix} V(0,s) \\ I(0,s) \end{bmatrix}$$

where, in accordance with [7],

$$\begin{aligned} w_{00} &= e^{\alpha L/2}(\cosh \psi L - \frac{\alpha}{2\psi} \sinh \psi L), \\ w_{01} &= -\frac{R_0}{\psi} e^{\alpha L/2} \sinh \psi L, \\ w_{10} &= \left(\frac{\alpha^2}{4\psi R_0} - \frac{\psi}{R_0} \right) e^{-\alpha L/2} \sinh \psi L, \\ w_{11} &= e^{-\alpha L/2} \left(\cosh \psi L + \frac{\alpha}{2\psi} \sinh \psi L \right), \\ \psi &= \left(\frac{\alpha^2}{4} + sR_0 C_0 \right)^{1/2}. \end{aligned}$$

Thus, W is the forward mapping for legs l_1, l_2, l_4, and l_5. The backward mapping for these legs is

$$W^{-1} = \begin{bmatrix} w_{11} & -w_{01} \\ -w_{10} & w_{00} \end{bmatrix}$$

because $\det W = 1$. The forward mapping N for legs l_3 and l_6, which is the mapping in the narrowing directions, are obtained by replacing α by $-\alpha$ within W, and their backward mapping is obtained in the same way from W_{α}^{-1}.

We now have five unknowns: v_0, v_3, i_4, i_5, and i_6. An application of our procedure, starting at node n_3 and taking into account the currents in the grounding branches leads to the following matrix equations. From node n_3

$$\begin{bmatrix} u_3 \\ j_3 \end{bmatrix} = N \begin{bmatrix} v_3 \\ e - v_3 - i_4 - i_6 \end{bmatrix},$$

$$\begin{bmatrix} u_4 \\ j_4 \end{bmatrix} = W \begin{bmatrix} v_3 \\ i_4 \end{bmatrix},$$

$$\begin{bmatrix} u_6 \\ j_6 \end{bmatrix} = N \begin{bmatrix} v_3 \\ i_6 \end{bmatrix}.$$

From node n_2,

$$\begin{bmatrix} u_2 \\ j_2 \end{bmatrix} = W \begin{bmatrix} u_3 \\ -u_3 + j_3 - i_5 \end{bmatrix},$$

$$\begin{bmatrix} u_5 \\ j_5 \end{bmatrix} = W \begin{bmatrix} u_3 \\ i_5 \end{bmatrix}.$$

From node n_1,

$$u_4 = u_2, \tag{17}$$

$$\begin{bmatrix} u_1 \\ j_1 \end{bmatrix} = W \begin{bmatrix} u_2 \\ j_2 + j_4 - u_2 \end{bmatrix}.$$

At node n_0,

$$v_0 = u_1 = u_5 = u_6, \tag{18}$$

$$v_0 = j_1 + j_5 + j_6. \tag{19}$$

Equations (18) provide three independent equations, and (17) and (19) provide the other two needed to solve for the five unknowns.

We chose $e(t)$ to be the unit-step function and solved for the Laplace transform $V_0(s)$ of $v_0(t)$ at ten values of s in R_+ as required by a particular case of the Gaver-Stehfest algorithm, which then yielded $v_0(t)$. We then did the same thing for $v_3(t)$. The results are shown in Figure 8. We also computed $v_1(t)$ and $v_2(t)$; their curves are virtually the same as that for $v_0(t)$. We were unable to find a version of SPICE that encompasses nonuniform distributed lines. However, the Singhal-Vlach algorithm is available for checking our inversion of the Laplace transformation. We used the version of that algorithm employing twenty poles and seven zeros in their Padé approximation. Plots of the results for $v_0(t)$ and $v_3(t)$ are indistinguishable from those obtained from the Gaver-Stehfest algorithm. Indeed, their numerical values for $t = .2, .4, \ldots, 6.0$ agree to three places and differ in the fourth place by just one or two digits except at $t = .2$ where our choices of the numbers of poles and zeros effects the very beginning of the transient.

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FIGURE LEGENDS

Figure 1. **(a)** The form of the locus of the voltage-current characteristic for any branch. Here, $h = G(0)$ and $e = R(0) = G^{-1}(0)$.

(b) The Thevenin form of that branch. $g : v + e \mapsto i$ is the function obtained by horizontally shifting the characteristic of (a) to make it pass through the origin. Thus, e is an implicit independent voltage source encompassed by that characteristic.

(c) The Norton form of that branch. $r : i+h \mapsto v$ is the function obtained by vertically shifting the characteristic of (a) to make it pass through the origin. Thus, h is an implicit independent current source encompassed by that characteristic.

Figure 2. A two-port.

Figure 3. **(a)** A grounding branch as a two-port.

(b) A floating branch as a two-port.

(c) A lumped ladder network as a cascade of el-sections.

(d) A distributed transmission line.

Figure 4. **(a)** A transmission network consisting of nine lumped ladders.

(b) The corresponding schematic. The heavy lines denote a chosen spanning tree in the schematic. That tree has the minimum possible number of ends. The number near each node indicates the level of that node.

Figure 5. The schematic of a transmission tree. The number near each node indicates the level of that node.

Figure 6. The input voltage v_m, input current i_m, output voltage u_m, and output current j_m for the leg l_m.

Figure 7. The schematic of a transmission network consisting of six identical exponentially tapered RC distributed lines. The heavily drawn lines denote the chosen spanning tree. The dotted arrows indicate the directions of the widening tapers. The arrowheads indicate the directions of the forward mappings. The node-voltage indices and the node levels are the same in this example. All grounding resistors are 1 Ω.

Figure 8. The node voltages $v_0(t)$ and $v_3(t)$ of Example 9.4.

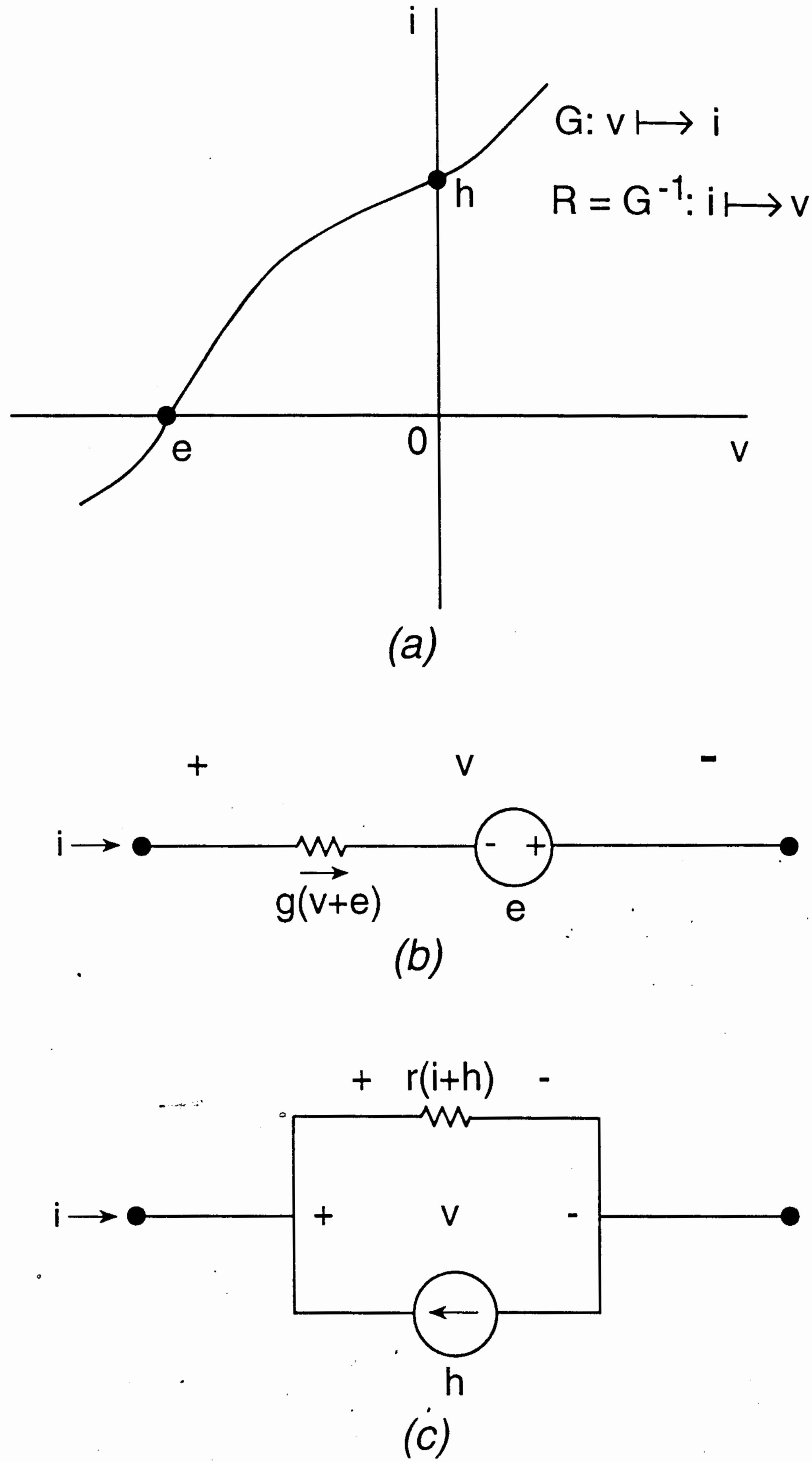


Fig. 1 a/b/c

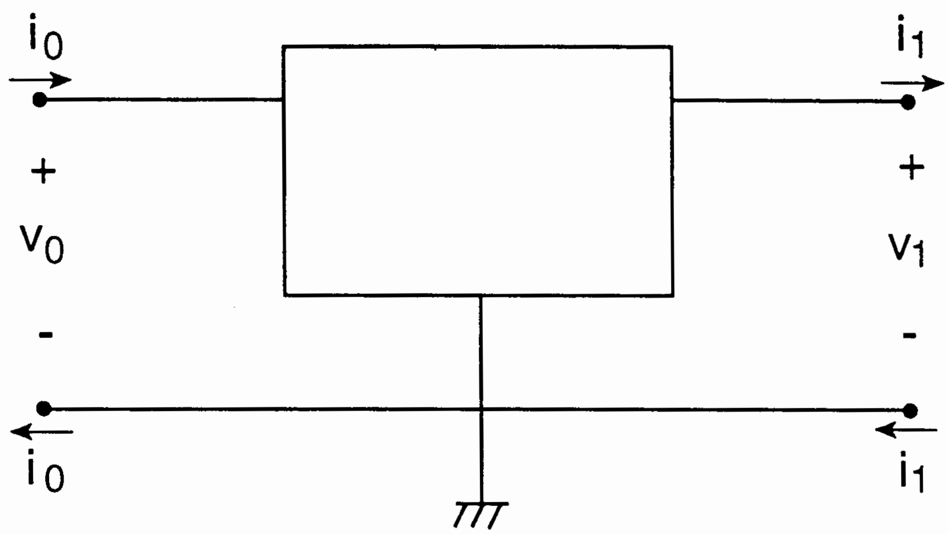


Fig. 2

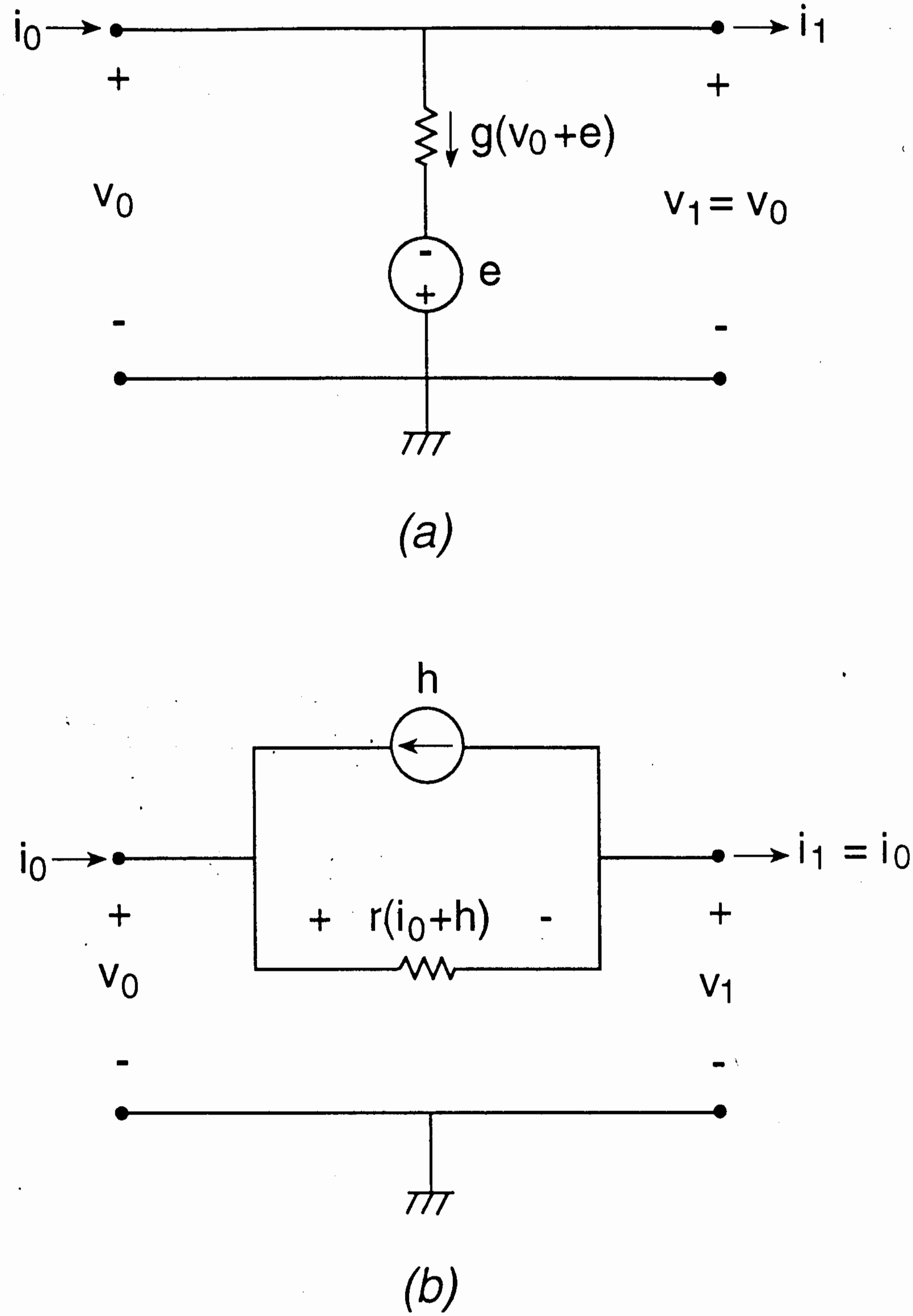
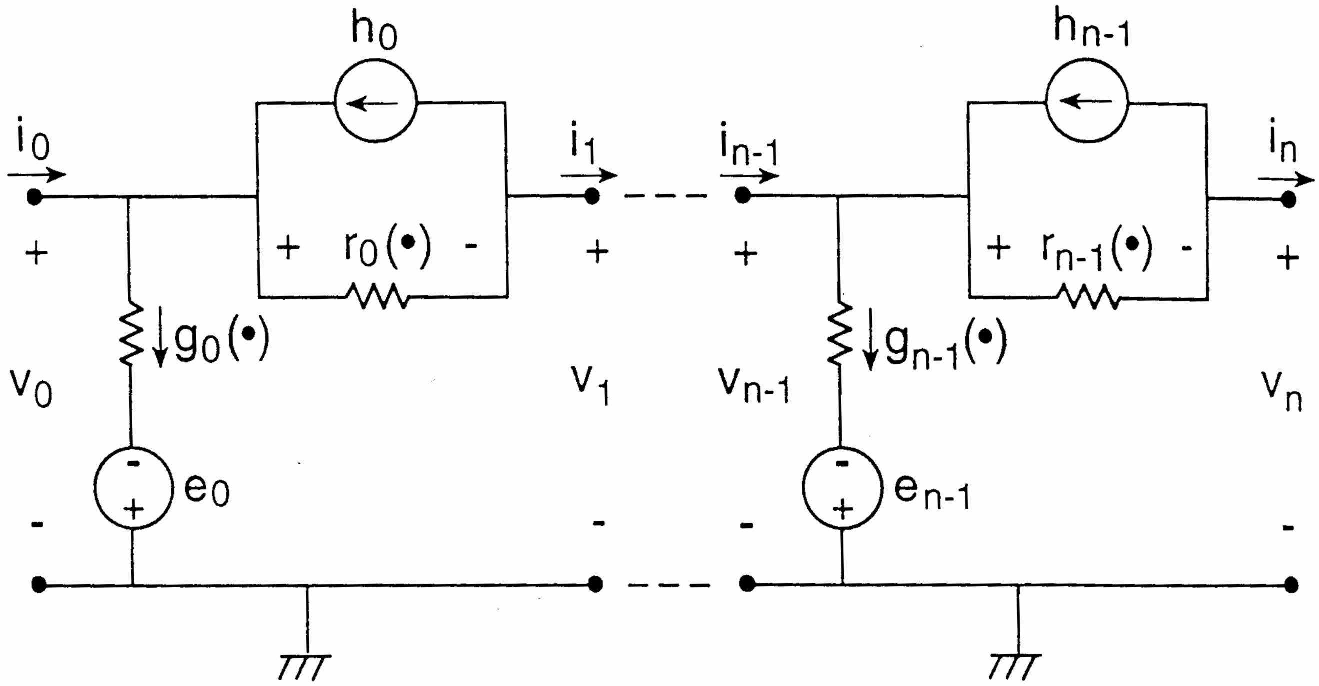
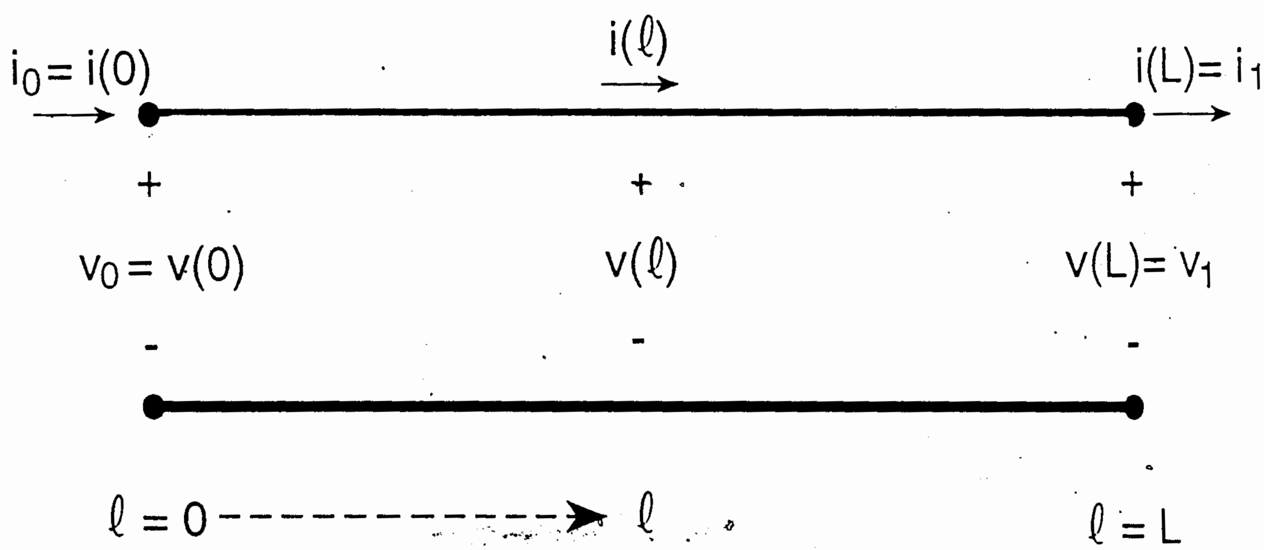


Fig. 3 a/b



h_0
h_{n-1}
i_0
i_1
i_{n-1}
i_n
+ $r_0(\bullet)$ -
+ $r_{n-1}(\bullet)$ -
$g_0(\bullet)$
$g_{n-1}(\bullet)$
v_0
v_1
v_{n-1}
v_n
e_0
e_{n-1}

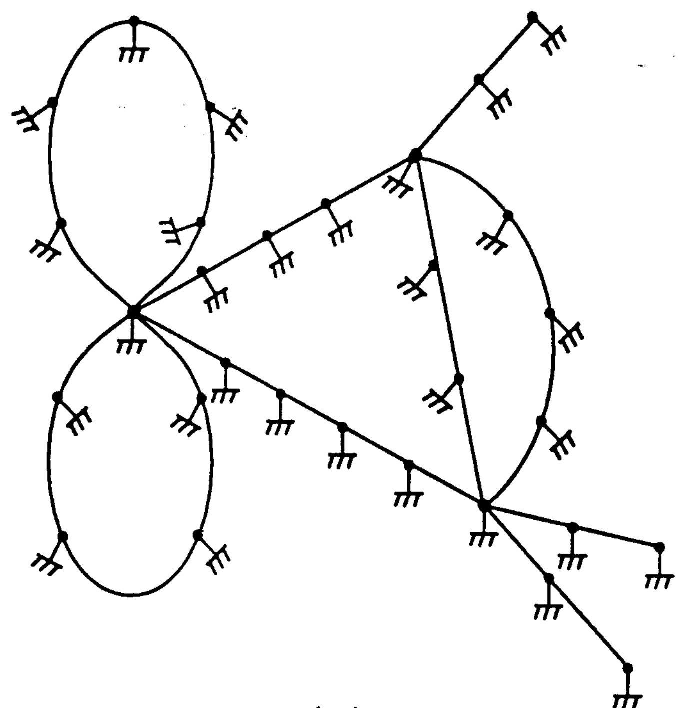
(c)



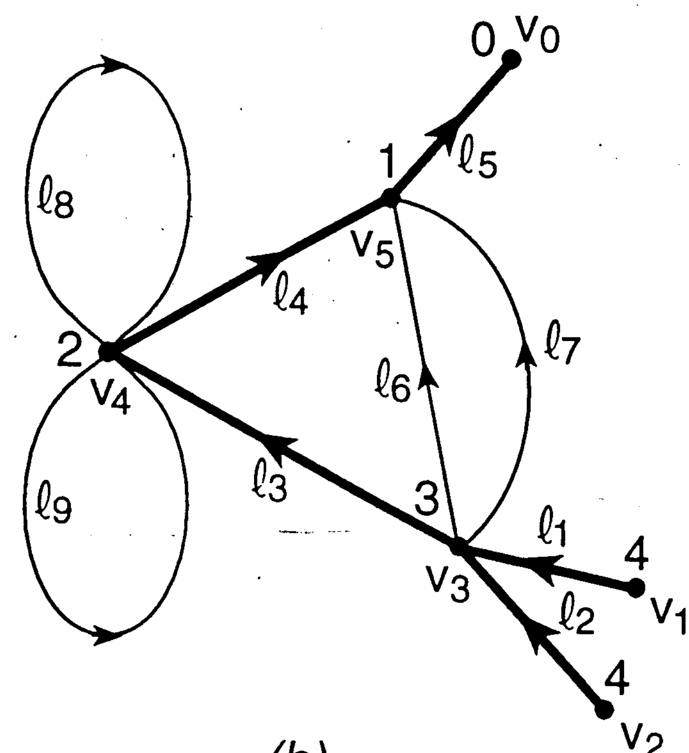
$i_0 = i(0)$
$i(\ell)$
$i(L) = i_1$
$v_0 = v(0)$
$v(\ell)$
$v(L) = v_1$
$\ell = 0$
ℓ
$\ell = L$

(d)

Fig. 3 c/d

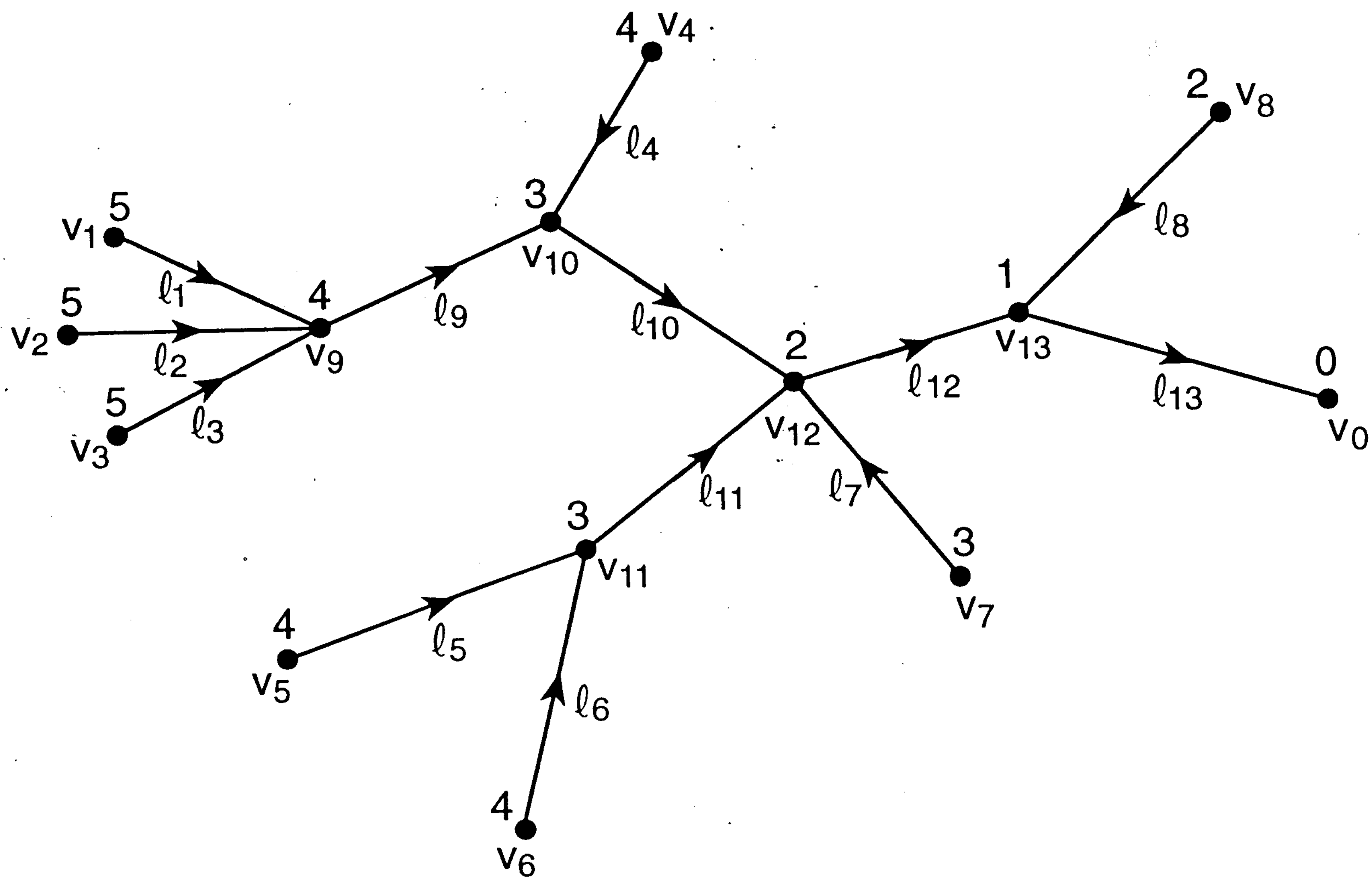


(a)



(b)

Fig. 4 a/b



5
v1
5
v2
5
v3
4
v9
3
v10
4 v4
3
v11
4
v5
4
v6
2
v12
3
v7
1
v13
2 v8
0
v0
ℓ1
ℓ2
ℓ3
ℓ4
ℓ5
ℓ6
ℓ7
ℓ8
ℓ9
ℓ10
ℓ11
ℓ12
ℓ13

Fig. 5

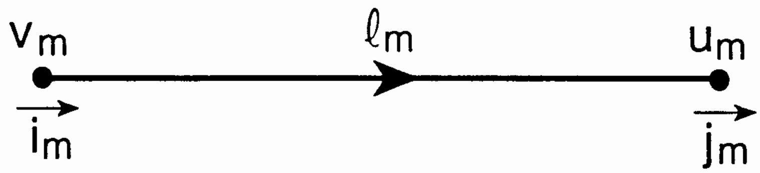


Fig. 6

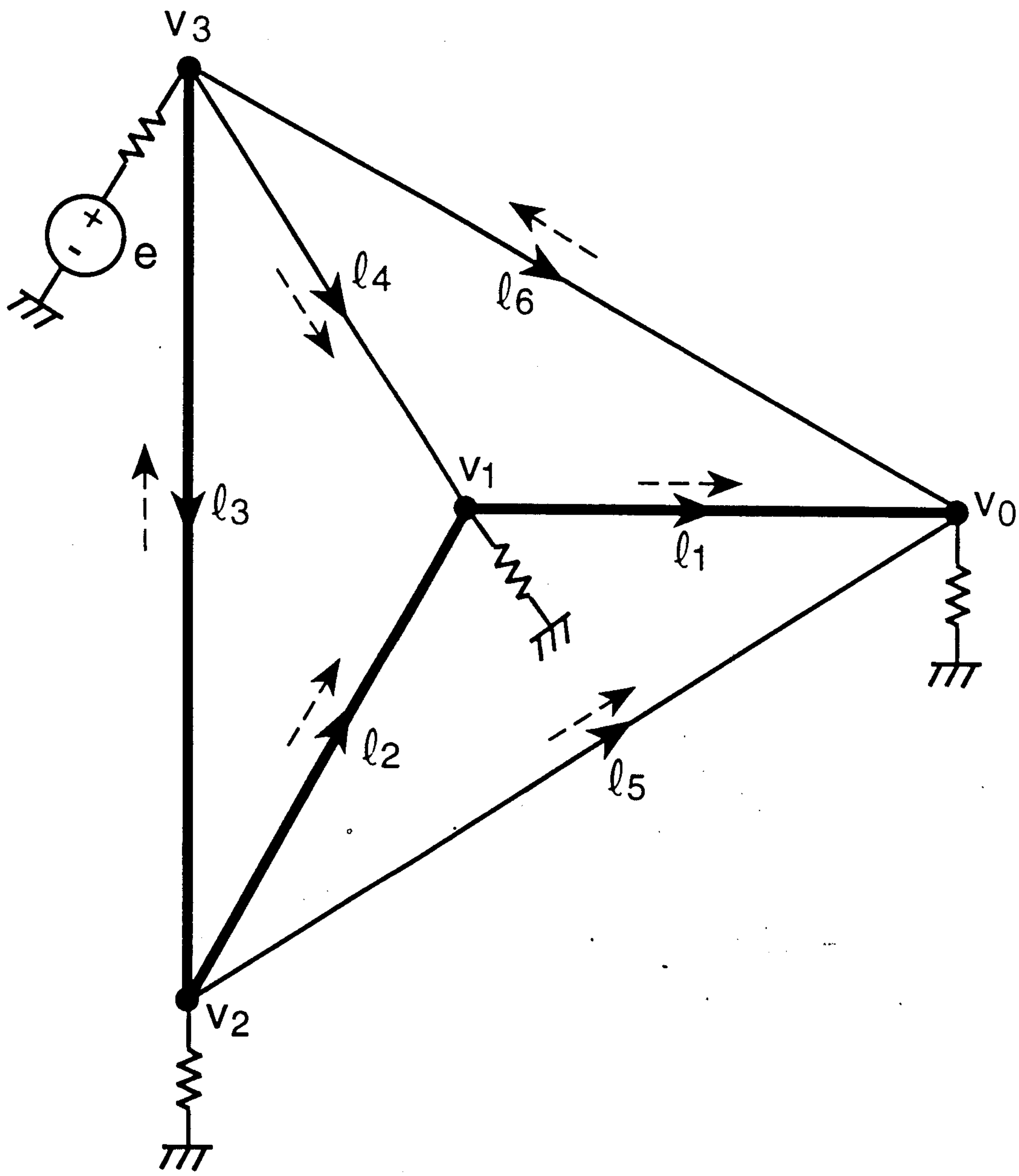
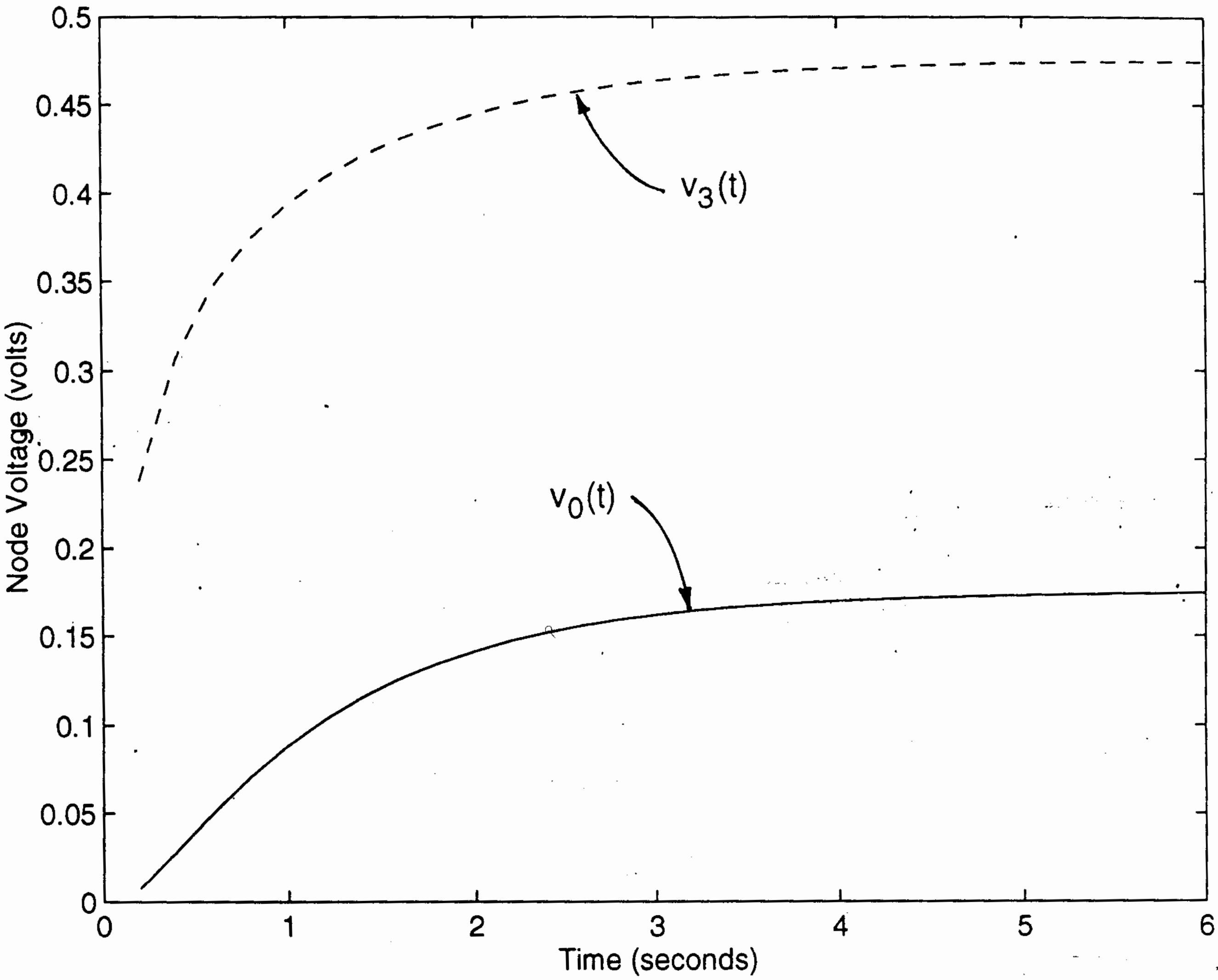


Fig. 7



0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
Node Voltage (volts)
0
1
2
3
4
5
6
Time (seconds)
$v_3(t)$
$v_0(t)$

Fig. 8