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A NOTE ON SMALL AMPLITUDE VIBRATION

OF A

SUBLIMING SOLID DISK ABOVE A HEATED PLATE

by

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A Note on Small Amplitude Vibration of a Subliming
Solid Disk Above a Heated Plate

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The complicated peregrinations of a liquid droplet when sup-
ported over a heated plate by its own vapor are notorious. As a
prelude to more detailed analysis it appears advisable to isolate
the simplest component of the motion - rigid body oscillation - and
by simple linear theory to predict small amplitude vibrational fre-
quency as a function of the thermal parameters. To achieve this we
here analyse a subliming solid disk of large radius which is con-
strained to move only in the vertical direction. On the expecta-
tion, checked a posteriori, that the Reynolds number based on film
thickness will be small we assume that viscous forces dominate and
calculate the steady state film thickness, resonant frequency of
small amplitude oscillation and the damping characteristics.

Analysis

Consider a disk of weight W and radius a suspended over a hot
plate. If the disk is of subliming material an equilibrium situa-
tion will arise in which the disk will assume a mean position
slightly above the plate. The assumption is made, based on the ex-
pectation of a narrow vapor gap and low velocities that viscous
forces dominate the momentum equation and are responsible for main-
taining the disk in position. By linearizing the analysis we remove
the obvious asymmetry that finite amplitude motion of the disk must

display and it will be the task of further experimentation and analysis to define carefully the practical limitation of the model. It is expected however that at least frequency may be usefully predicted since this property is generally insensitive to subtle forces and of course the mean film thickness should also be reasonably well accounted for. The damping characteristics are generally quite sensitive to detailed force balances and there is no reason to assume that they will be adequately described by the linear theory.

In polar coordinates with the origin at the center of the surface of the heated plate and the z coordinate normal to and positive upward from the plate we describe the disk position by $\xi(z, t)$. Under the assumptions of no dependence on angle and z-velocity a function only of z, say $H(z)$, the radial momentum equation takes the form

$$\frac{2}{\rho r} \frac{\partial p}{\partial r} = \frac{\partial H'}{\partial t} + HH'' - \tfrac{1}{2}H'^2 - \nu H''' = \chi(z, t) \tag{1}$$

where use has been made of the continuity equation to obtain

$$u_r = -\tfrac{1}{2}rH'(z)$$

From (1) we obtain

$$p = \frac{\rho \chi(z,t)}{4}(r^2 - a^2)$$

and remarking that from the z momentum balance $\frac{\partial p}{\partial z}$ is at most a function only of z and t we can conclude that $\chi = \chi(t)$. Consequently we now write

$$\frac{\partial H'}{\partial t} + HH'' - \tfrac{1}{2}H'^2 - \nu H''' = \chi(t) = \frac{2}{\rho r} \frac{\partial p}{\partial r} \tag{2}$$

A force balance on the disk reveals

$$m\ddot{\xi} = -mg + \int_0^a \int_0^{2\pi} prdrd\theta$$

or

$$m\ddot{\xi} = -my - \frac{\rho\pi}{8}a^4\chi(t)$$

and (2) can be written

$$\frac{\partial H'}{\partial t} + HH'' - \frac{1}{2}H'^2 - \nu H''' = -\frac{8}{\rho\pi a^4}(m\ddot{\xi} + W) \tag{3}$$

with the boundary conditions

$$at \; z = 0, \; H = 0$$

$$at \; z = 0 \; and \; z = \xi, \; H' = 0$$

$$at \; z = \xi, \; H = \xi + V$$

where V is the fluid particle velocity at the solid vapour interface.

Infinitesimal oscillations

The probable narrowness of the vapour layer thickness leads to the boundary layer type postulates

$$\frac{\frac{\partial}{\partial r}}{\frac{\partial}{\partial z}} = 0(\epsilon') \qquad\qquad u_r = 0(\epsilon^0)$$

$$u_z = 0(\epsilon')$$

$$\frac{\partial}{\partial t} = 0(\epsilon^0)$$

Then the term $\frac{\partial^2 u_r}{\partial z^2} = 0(\epsilon^{-2})$ dominates the left hand side of Equation 3. This postulate will have to be examined a posteriori by comparing the unsteady term and the inertia terms with the computed viscous contribution and this will be taken up subsequently.

With the above approximation Equation (3) reduces to

$$\nu H''' = \frac{8}{\rho \pi a^4} \left[W + m \ddot{\xi} \right] \qquad (4)$$

and the boundary conditions enumerated previously. A straight-forward solution yields

$$H = \frac{8}{\mu \pi a^4} \left[\frac{z^3}{6} - \xi \frac{z^2}{4} \right] \left[W + m \ddot{\xi} \right] \qquad (5)$$

where the condition $H_z = \xi = \frac{d\xi}{dt} + V(\xi)$ is still to be applied.

Its use reduces Equation (5) to a description of the motion of the surface

$$- \frac{8}{\mu \pi a^4} \left[W + m \ddot{\xi} \right] \frac{\xi^3}{12} = \dot{\xi} + V(\xi) \qquad (6)$$

To obtain information from Equation (6) we need an estimate of the behavior of the final term of the equation. Using the approximate result from the energy equation

$$L\rho V = k \frac{\partial T}{\partial z}$$

and employing a linear temperature profile through the gap we obtain

$$V(\xi) = - \frac{k}{L\rho} \frac{\Delta T}{\xi}$$

where L is the latent heat of sublimation, ΔT is the temperature drop

across the gap and k is the thermal conductivity of the vapor.

A reasonable form for (6) may then be

$$W\xi^3 + m\ddot{\xi}\xi^3 = -\frac{3}{2}\mu\pi a4\dot{\xi} + \frac{3}{2}\frac{k}{L}\frac{\Delta T}{\rho}\frac{\mu\pi a^4}{\xi} \tag{7}$$

linearizing we assume $\xi = \xi_0(1 + \epsilon(t))$ where ξ_0 is the steady state solution given by, from (7),

$$\xi_0 = \left(\frac{3k\Delta T\mu\pi a^4}{2L\rho W}\right)^{\frac{1}{4}} \tag{8}$$

Thus the first order equation becomes

$$\ddot{\epsilon} + \frac{3}{2}\frac{\mu\pi a^4}{m\xi_0^3}\dot{\epsilon} + \frac{4g}{\xi_0}\epsilon = 0 \tag{9}$$

or in terms of thermodynamic quantities

$$\ddot{\epsilon} + \frac{L\rho\xi_0 g}{k\Delta T}\dot{\epsilon} + \frac{4g}{\xi_0}\epsilon = 0 \tag{10}$$

The damped harmonic nature of Equation (10) is evident and information about natural frequency and damping ratio can immediately be calculated. If we define a Reynolds number and a Froude number in the following way;

$$R = \xi_0\frac{V_0}{\nu}$$

$$F = \frac{V_0^2}{\xi_0 g}$$

where V_0 is the velocity of evaporation from the solid surface when it is at the steady state position ξ_0, then Equation (10) can be rewritten in non dimensional form as

$$\frac{d^2\epsilon}{d\tau^2} + \left(\frac{2}{F}\right)^{\frac{1}{2}} \frac{d\epsilon}{d\tau} + \epsilon = 0 \qquad\qquad (11)$$

where $\qquad \tau = t\left(\frac{4g}{\xi_0}\right)^{\frac{1}{2}}$.

In other words (11) describes the motion in terms of a time scale based on the oscillatory period of an undamped motion, a period which is not too sensitive to moderate damping. The coefficient of the damping term in (11) is a ratio between the gravitational forces on the plate and the inertia forces and there is no guarantee that it will be small compared to unity. In fact one can expect that to obtain a coefficient of $O(1)$ severe restrictions will have to be placed on the disk weight per unit area. A larger coefficient will produce strong damping and make experimental observation of the motion much more difficult.

It is pertinent to examine the two basic assumptions which lead to Equation (11). In the first place we need an a posteriori check on the smallness of R. Naturally the thermodynamic parameters which specify the problem such as the temperature drop across the gap, the latent heat of sublimation and the material properties are the significant data in which to express the Reynolds number.

We have $R = \dfrac{\xi_0 V_0}{\nu}$

and therefore $R = \dfrac{k\Delta T}{L\rho\tau} = \dfrac{k\Delta T}{L\mu} = \dfrac{1}{P_r}\dfrac{C_p\Delta T}{L} = \dfrac{\Delta H}{P_r L}$

Thus given the properties P_r and L of a subliming substance an adjustment of ΔT can in principle be made to reduce R to as small a value as necessary. In practice we find for carbon dioxide that ΔT

should be less than 350°C.

In the second place ignoring the unsteady momentum terms is equivalent to assuming instantaneous readjustment of velocity and temperature profile. In other words the time scale for vorticity diffusion across the gap is to be significantly less than the time scale of the period predicted by Equation (10).

For diffusion $\quad T_D = 0\left(\dfrac{\xi_0{}^2}{\nu}\right)$,

and for oscillation $\quad T_o = 0\left[(\dfrac{4g}{\xi_0})^{-\frac{1}{2}}\right]$,

Thus the ratio $\quad \dfrac{T_D}{T_o} = \dfrac{2R}{[F_1]^{\frac{1}{2}}}$. The small Reynolds number assumption and the achievement of a low damping coefficient assures that $T_D < T_o$.

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