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SOLUTION OF A SYSTEM OF SIMULTANEOUS LINEAR EQUATIONS WITH A SPARSE COEFFICIENT MATRIX BY ELIMINATION METHODS

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SOLUTION OF A SYSTEM OF SIMULTANEOUS LINEAR EQUATIONS WITH A SPARSE COEFFICIENT MATRIX BY ELIMINATION METHODS

R. P. Tewarson*

1. Introduction. We consider the solution of the system of simultaneous linear equations of the form

$$AX = b \tag{1.1}$$

by the two well known methods viz., the Gaussian elimination and the Gauss-Jordan elimination [1]. In the system (1.1), A is a square non-singular sparse matrix of order n, X and b are both n element column vectors. It is evident that the positions where the pivots are chosen in A will affect the number of new non-zero elements created during the course of each of the elimination processes. In this paper, after showing how the methods for minimizing the density of the Product Form of Inverse of a Sparse Matrix given in [2,3] can be used for minimizing the number of new non-zero elements created during the course of the elimination methods, we shall state and prove some new theorems which in turn lead to new methods. This paper, in a way, can be regarded as a sequel to [3].

We recall that in both of the elimination processes, at each stage, a column of the transformed coefficient matrix is selected from those columns that remain to be pivoted. Then in this column a pivot is chosen and all the other non-zero elements of the column are reduced to zero by elementary row operations which are performed on this and the other columns that remain to be pivoted. In Gaussian elimination the row as

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well as the column corresponding to the chosen pivot are dropped from further consideration after being transformed as above. On the other hand, in the case of Gauss-Jordan elimination, in addition to the transformation of the other elements of the selected column to zero, the pivot is transformed to unity [1] and only the column (but not the row) corresponding to the chosen pivot is dropped. In view of the above facts it becomes important to select those columns of the transformed matrix that remain to be pivoted in such an order that on the elimination of each one of them a minimum number of new non-zero elements are created in the remaining rows and columns. It is easy to see that the Gauss-Jordan method for solving (1.1) is equivalent to computing the Product Form of Inverse, since in both the number of columns to be transformed decreased by unity at each stage [2]. Therefore, the methods for selecting the pivots, so as to minimize the creation of new non-zero elements, given in [3] can also be used in the case of Gauss-Jordan elimination. With trivial modifications these methods can also be used in the case of the Gaussian elimination. We shall associate an incidence matrix B with (1.1). This will help us not only in giving concise definitions of the various measures defined in [2,3] but also in the proofs of the new theorems given in this paper.

2. Definitions and notations. Let B be an incidence matrix of the variables $x_1, x_2, \ldots, x_n$ vs. the equations 1, 2, ..., n of (1.1); that is the i th row and j th column element of B is 0 or 1 according as x_i does or does not occur in equation j. In other words, B is obtained from A by replacing each non-zero element of A by unity. Let $A^{(k)}$ be the matrix of the unpivoted columns of A after k - 1 of its columns have been pivoted, where k = 1, 2, ..., n. In the case of Gauss-Jordan elimination $A^{(k)}$ is n x m and for Gaussian elimination it is m x m, where m = n - k + 1.

Let B_k be the incidence matrix corresponding to $A^{(k)}$. Let U_k denote a column vector with m ones and V a row vector with n ones, thus $U_1^T = V$. We are now in a position to define briefly, in the case of Gauss-Jordan elimination, the following (see [3] for additional details):

The row count vector $[r_i^{(k)}]$, each element of which gives the number of non-zero elements in the corresponding row of $A^{(k)}$ (the vector $[r_i^k]$ itself is a column vector with n elements):

$$[r_i^{(k)}] = B_k U_k. \tag{2.1}$$

The column count vector $[c_j^{(k)}]$, each element of which denotes the number of non-zero elements in the corresponding column of $A^{(k)}$ (the vector $[c_j^{(k)}]$ itself is a row vector having m elements):

$$[c_j^{(k)}] = V B_k. \tag{2.2}$$

Notice that we have found it convenient to make a slight change in the notation of [3] for this paper viz., $r_i^{(k)}$ is used instead of $C_i^{(k)}$ and $c_j^{(k)}$ is used in place of $R_j^{(k)}$.

In a similar manner the various other quantities given in [3] can be redefined more concisely as follows:

$$[D_j^{(k)}] = [r_i^{(k)}]^T B_k = U_k^T B_k^T B_k. \tag{2.3}$$

$$[D_i^{(k)}] = B_k [c_j^{(k)}]^T = B_k B_k^T V^T. \tag{2.4}$$

$$[\delta_j^{(k)}] = [D_i^{(k)}]^T B_k = V B_k B_k^T B_k. \tag{2.5}$$

$$[\tilde{c}_j^{(k)}] = V B_k - U_k^T. \tag{2.6}$$

$$[\tilde{r}_i^{(k)}] = B_k U_k - V^T \tag{2.7}$$

$$[\tilde{D}_j^{(k)}] = [D_j^{(k)}] - [c_j^{(k)}] = U_k^T B_k^T B_k - V B_k$$

$$= (U_k^T B_k^T - V) B_k. \tag{2.8}$$

If in the above results, wherever V occurs, it is replaced by U_k^T we get the corresponding formulas for the Gaussian elimination; of course, we recall that B_k in this case will be a square matrix of order m.

3. Column Selection for Pivoting. The column of $A^{(k)}$ can be selected, at stage k, for pivoting by using $\min_j D_j^{(k)}$, $\min_j \delta_j^{(k)}$, $\min_j \bar{D}_j^{(k)}$ or $\min_j \Delta_j^{(k)}$.

$\Delta_j^{(k)}$ was defined in theorem 3 of [3] as follows:

(3.1) $\Delta_j^{(k)} = \delta_j^{(k)} - D_j^{(k)} - (c_j^{(k)} - 1)^2$, we recall that in this paper $c_j^{(k)}$ is used in place of $R_j^{(k)}$ of [3]. Let P.Q denote the matrix obtained by elementwise product of any two matrices P and Q. Hence using (2.3), (2.5) and (2.6) in (3.1) we have:

$$(3.2) \quad [\Delta_j^{(k)}] = V B_k B_k^T B_k - U_k^T B_k^T B_k - (V B_k - U_k^T) \cdot (V B_k - U_k^T)$$

$$= (V B_k - U_k^T) B_k^T B_k - (V B_k - U_k^T) \cdot (V B_k - U_k^T).$$

$$= (V B_k - U_k^T) \{B_k^T B - \cdot (V B_k - U_k^T)\},$$

where $-\cdot$ denotes that each element of the row vector $V B_k - U_k^T$ is subtracted from the corresponding diagonal element of $B_k^T B$.

In the case of Gaussian elimination replacing V by U_k^T in (3.2) we have

$$(3.3) \quad [\Delta_j^{(k)}] = U_k^T (B_k - I) \{B_k^T B - \cdot U_k^T (B_k - I)\}.$$

So far we have discussed the column selection methods that were given in [3]. We shall now describe some new methods for selecting the columns 'a priori'. In such methods we decide in what order we will pivot all the columns before the elimination process starts. Let $\hat{T} = B^T B$ and t_{ij} denote the i th row and j th column element of $\hat{T}$, then we have the following:

Theorem 3.1. If all the non-zero elements in the s th column of $\hat{T}$ are equal to the diagonal element t_{ss}, then by the elimination process,

no new non-zero elements are created, no matter which element of the s th column of A is chosen as a pivot.

Proof: Let β_j denote the j th column of B. Then $t_{ss} = \beta_s^T \beta_s = c_s^{(1)}$ which is the number of non-zero elements in column s of A. Also $t_{js} = \beta_j^T \beta_s$ is the number of rows in which both the j th and s th columns simultaneously have non-zero elements. If $t_{ss} = t_{js}$, then to every non-zero element of column s there corresponds a non-zero element in column j. Hence no matter which element of the s th column of A is chosen as a pivot, no new non-zero elements are created in the j th column of A. On the other hand if $t_{js} = 0$ for some j, then again no new non-zero elements are created in such a column j when column s is pivoted. This completes the proof of the theorem.

Some additional definitions which will be needed now follow: Let us call $t_{ss} = \beta_s^T \beta_s$ as the 'length' of the s th column and $t_{js} = \beta_j^T \beta_s = \beta_s^T \beta_j = t_{sj}$ as the 'length of the intersection' of the s th and j th columns of B. From (2.3), we have $[D_j^{(k)}] = U_k^T B_k^T B_k$, thus $D_s^{(k)}$ is equal to the sum of the 'lengths of the intersections' of column s of $A^{(k)}$ with all the other columns. Also, if $t_{ss} = t_{js}$, then column s is said to have a 'contained intersection' with column j. Note that $0 \leq \beta_s^T \beta_j \leq \min (\beta_s^T \beta_s, \beta_j^T \beta_j)$, hence $\beta_s^T \beta_j = \beta_s^T \beta_s$ if and only if $\beta_s^T \beta_s \leq \beta_j^T \beta_j$. Thus a column can have a 'contained intersection' only with those columns that are greater than or equal to it in 'length'. If $t_{js} = 0$, then the j th and s th columns are said to be 'disjoint'. Thus theorem 3.1 asserts that those columns that have only a 'contained intersection' with the other columns when pivoted create no new non-zero elements. Now, we can state the following corollary to theorem 3.1:

Corollary 3.1. All those columns of A that have only one element when pivoted create no new non-zero elements.

Proof: If $t_{ss} = 1$ and $t_{js} \neq 0$, then since $t_{js} \leq t_{ss}$ implying that $t_{ss} = t_{js}$ for all $t_{js} \neq 0$ and the corollary follows from theorem 3.1.

In view of theorem 3.1 and corollary 3.1 we can say that all those columns of A that have a 'contained intersection' with the other columns (this includes the columns having only one element) should be pivoted first and this will give rise to no new non-zero elements. The following theorem which is subsidiary to theorem 3.1 is helpful in deciding if theorem 3.1 can be used:

Theorem 3.2. A necessary (but not sufficient) condition that column s of A will have only a contained intersection with all the other columns of A is that all those elements of the row count vector $[r_i^{(1)}]$ that correspond to non-zero elements of the s th column be equal to each other.

Proof: Because column s has contained intersection with all the other columns, therefore it follows that $t_{ss} = t_{js}$ whenever $t_{js} \neq 0$. Which implies that if columns s and j intersect then to each non-zero element of column s there corresponds a non-zero element in column j. Hence the theorem follows by summing up the number of all such non-zero elements in each row having a non-zero element in column s for all those columns which intersect with column s.

Thus unless among the elements of $[r_i^{(1)}]$ there exist one or more subsets such that the members of each subset are equal in magnitude, it will not be worthwhile to look for columns having contained intersections by making use of theorem 3.1.

The terms 'length', 'disjoint', 'length of intersection' and 'contained intersection' given earlier for the columns of B, can be defined similarly for the rows of B (e.g. by using columns of B^T). In order to describe some interesting consequences of the above, we shall need one additional definition, viz., 'properly contained intersection'. The s th column (row) of B is said to have a 'properly contained intersection'

with the j th column (row) if

$$\beta_s^T \beta_j = \beta_s^T \beta_s < \beta_j^T \beta_j. \tag{3.4}$$

Therefore, if

$$\beta_s^T \beta_j = \beta_s^T \beta_s = \beta_j^T \beta_j, \tag{3.5}$$

then column s does not have a 'properly contained intersection' with column j. In view of the above discussion, we have the following theorems:

Theorem 3.3. If a given column of B has only 'contained intersections' with the other columns and at least one of these intersections is a 'properly contained intersection', then no non-zero element of the given column can belong to any row that has only 'contained intersections' with the other rows of B.

Proof: Let us denote the element in the i th row and the j th column of B by β_{ij}. If s is the given column then, since it has at least one 'properly contained intersection', there exists a column u intersecting column s and a row i_1 such that $\beta_{i_1 s} = 0$ and $\beta_{i_1 u} \neq 0$. If we assume that there exists a row i_2 having only a 'contained intersection' with the other rows and that $\beta_{i_2 s} \neq 0$, then $\beta_{i_2 u} \neq 0$ since column s has a 'contained intersection' with column u. Now row i_2 can have only 'contained intersections' with the other rows and since it intersects row i_1 because $\beta_{i_1 u} \neq 0$, therefore $\beta_{i_1 s} \neq 0$. A contradiction, which concludes the proof of the theorem.

Theorem 3.4. If a given column of B has only 'contained intersections' with the other columns and if one of the non-zero elements of the given column belongs to a given row which has only 'contained intersections' with the other rows, then all the non-zero elements of every column which intersects with the given column lie only in those rows that intersect the given row. Furthermore, all such intersecting rows and columns are of equal 'length'.

Proof: Let s be the given column and i_1 the given row. Let the set of all columns which intersect with column s be denoted by S_c. Similarly,

let the set of all rows which intersect with row i be S_r. Clearly, $s \in S_c$ and $i_1 \in S_r$. Then, for all $j \in S_c$, $\beta_{i_1 s} \neq 0$ implies that $\beta_{i_1 j} \neq 0$, since column s has only 'contained intersections' with the other columns. Now, for some i, $\beta_{ij} \neq 0$ then evidently $i \in S_r$. But row i_1 has only 'contained intersections' with the other rows; therefore, it follows that $\beta_{is} \neq 0$. In other words, all the non-zero elements of all columns in S_c lie only in those rows that belong to S_r. Finally, from theorem 3.2 all the rows with non-zero elements in column s have the same length. Also from the preceding discussion all the columns in S_c are of the same length (since to each $\beta_{ij} \neq 0$ there corresponds a $\beta_{is} \neq 0$ and vice versa). In view of the above facts it is easy to see that the set $S_c \cap S_r$ consists of all ones and both the sets $S_c \cap S_r'$ and $S_r \cap S_c'$ consist only of zeros (prime denotes the complement of a set). Therefore, the set $S_c \cap S_r$ must be square since matrix A is non-singular. Hence all the rows in S_r and all the columns in S_c must be equal in length, which completes the proof of the theorem.

The preceding theorems suggest the following procedure for elimination: First, all those columns which have only 'contained intersections' with the other columns should be pivoted first. This will include all rows and columns associated with the sets of the type $S_c \cap S_r$ described in the proof of theorem 3.4. Second, if possible, all the columns in which pivots can be found in those rows which have only 'contained intersections' with other rows should be pivoted. Notice that all such rows have only 'properly contained intersections' with the other rows. Finally, the remaining columns can be pivoted by using the usual methods (e.g. in ascending order of $D_j^{(1)}$'s or $\Delta_j^{(1)}$'s etc.).

Let us now assume that A has no columns that have only contained intersections with the other columns. This entails no loss of generality, as all such columns can be pivoted first, without creating any new non-zero elements as shown above. We shall now give a criterion by which the columns

of A can be selected for pivoting. Let $W = B^T * B$, where * denotes that in place of the usual addition Boolean addition is used (viz., 1 + 1 = 1). Thus the i th and j th columns intersect if and only if i th row and j th column element w_{ij} of W is equal to unity. Let $W^{h+1} = W * W^h$ for h = 1, 2, ..., where once again Boolean addition is used. Let us denote the i th row j th column element of W^h by w_{ij}^h. We shall make use of the following theorem:

Theorem 3.5. In the matrix A, for all the columns a_j that can affect or be affected by a given column a_s in the course of the process of elimination, $w_{sj}^{n-1} = 1$.

Proof: With each column of B we associate a vertex of a graph. If two columns have a non-zero intersection then the corresponding vertices are connected by an edge. Since a column has a non-zero intersection with itself, we make the convention that each vertex is connected to itself by a loop. The column intersection graph so constructed evidently has W as its incidence matrix. (The reader is referred to [4] for the definitions of the terminology from graph theory.) Bearing in mind that all the diagonal elements of W are unity, it is easy to see that $w_{sj}^h = 1$ implies that there is a path of length h or less between the s th and j th vertices [5]. Since the path between two connected vertices can have a maximum length of n - 1, therefore $w_{sj}^{n-1} = 1$ implies that vertices s and j are connected by a path of length n - 1 or less. This fact along with the observation that in the process of elimination only the connected columns can possibily affect each other, completes the proof of the theorem.

In practice, we do not need to compute W^{n-1}, if for some $h < n - 1$, $W^{h+1} \equiv W^h$, then $W_{sj}^h = 1$ suffices to isolate all the connected components of the column intersection graph. It is not difficult to see that if

$W^{h+1} \equiv W^h$, then there exists a permutation matrix P such that $P\,W^h\,P^T$ consists only of square sub-matrices of all ones whose diagonals coincide with the diagonal of W. Notice that the proof of theorem 3.5 does not depend on the assumption that A has no columns that have only a contained intersection with the other columns. Hence we can say that such columns, if any, will be in one and only one of the connected components of the column intersection graph.

We shall now show how theorem 3.5 can be used in a practical manner. To this end we shall define a matrix M, such that if each element M_j of the column count vector of M is associated with the corresponding column a_j of A, then pivoting the columns of A in ascending values of M_j will minimize the creation of new non-zero elements. In the process of elimination those columns that intersect directly (there is a path of length one between the corresponding vertices) are more likely to affect each other than the other columns (vertices which are joined by paths of lengths greater than one). If we neglect the self-connecting loop of each vertex then the degree of each vertex is given by the corresponding element of the row vector $V(W - I)$. Notice that the degree of a given vertex is equal to the number of the columns directly intersecting the column associated with the given vertex. Similarly, $V(W^2 - W)$ is the row vector enumerating paths of length two between all the vertices and so on. Let α be a scalar such that $0 \leq \alpha < 1$. If we define M as follows:

$$M = (W-I) + \alpha(W^2-W) + \ldots + \alpha^{n-2}(W^{n-1}-W^{n-2}),$$

$$= (W+\alpha W^2+\alpha^2 W^2+ \ldots +\alpha^{n-2}W^{n-1})(1-\alpha) - I,$$

then

$$[M_j] = V(W+\alpha W^2+\alpha^2 W^3+ \ldots +\alpha^{n-2}W^{n-1})(1-\alpha) - V.$$

Since the row vector $[M_j]$ is used for ordering the columns of A and adding unity to each element of $[M_j]$ and the subsequent division by the common factor $1 - \alpha$, (since $\alpha < 1$), will not effect the ordering, we have

$$[\bar{M}_j] = V(W+\alpha W^2+\alpha^2 W^3+ \ldots +\alpha^{n-2}W^{n-1}). \tag{3.6}$$

Computational experiments to determine what values of α are reasonable are described later in section 5.

4. Pivot selection. We now consider the problem of selecting the pivots. They should be chosen so that the creation of new non-zero elements during the process of elimination is minimized. We will now state and prove some theorems related to the above problem.

Theorem 4.1. The maximum number of non-zero elements that can be created if the element $a_{ij}^{(k)}$ of $A^{(k)}$ is chosen as a pivot for the elimination process is equal to the corresponding element $p_{ij}^{(k)}$ of the matrix $P^{(k)}$ where

$$P^{(k)} = (B_k U_k - V^T)(VB_k - U_k^T). \tag{4.1}$$

Proof: If $a_{ij}^{(k)}$ is a pivot then the maximum number of new non-zero elements that can be created on the elimination of column j with $a_{ij}^{(k)}$ as pivot are given by

$$p_{ij}^{(k)} = (r_i^{(k)}-1)(c_j^{(k)}-1),$$

$$= \bar{r}_i^{(k)}\bar{c}_j^{(k)}.$$

Using (2.6) and (2.7) the above can be written as

$$P^{(k)} = (B_k U_k - V^T)(VB_k - U_k^T),$$

which proves the theorem.

We recall that in the case of Gaussian elimination V should be taken as U_k^T in (4.1), which in this case becomes

$$P^{(k)} = (B_k U_k - U_k)(U_k^T B_k - U_k^T)$$
$$= (B_k - I)\, U_k U_k^T\, (B_k - I),$$

or

(4.2) $$P^{(k)} = (B_k - I)\, N_k\, (B_k - I),$$

where N_k is a square matrix of all ones of order k.

The exact number of non-zero elements created by a given pivot can be enumerated as follows:

Theorem 4.2. The actual number of new non-zero elements created if the element $a_{ij}^{(k)}$ of $A^{(k)}$ is chosen as a pivot for the elimination process is equal to the corresponding element $p_{ij}^{(k)*}$ of the matrix $P^{(k)*}$ where

(4.3) $$P^{(k)*} = B_k\, \bar{B}_k^T\, B_k \text{ with } \bar{B}_k = V^T U_k^T - B_k.$$

Proof: The total number of elements of A (both zero and non-zero), which belong simultaneously to all the rows having non-zero elements in the j th column and also all the columns having non-zero elements in i th row, is equal to $r_i^{(k)} c_j^{(k)}$. The number of zero elements amongst the above gives the exact number of new non-zero elements that would be created if elimination is performed with $a_{ij}^{(k)}$ as the pivot. Therefore

$p_{ij}^{(k)*} = r_i^{(k)} c_j^{(k)} - \gamma_{ij}^{(k)}$, where $\gamma_{ij}^{(k)}$ is the total number of non-zero elements amongst the $r_i^{(k)} c_j^{(k)}$ elements. But $\gamma_{ij}^{(k)}$ is equal to the sum of the lengths of intersections of column j with all those columns that have a non-zero element in row i, viz.,

$$\gamma_{ij}^{(k)} = \sum_{s=1}^{m} \sum_{u=1}^{n} \beta_{is}^{(k)}\, \beta_{su}^{(k)}\, \beta_{uj}^{(k)}$$

where $\beta_{ij}^{(k)}$ denotes the i th row j th column element of B_k. Therefore we have

$$p_{ij}^{(k)*} = r_i^{(k)}\, c_j^{(k)} - \sum_{s=1}^{m} \sum_{u=1}^{n} \beta_{is}^{(k)}\, \beta_{su}^{(k)}\, \beta_{uj}^{(k)}.$$

In view of (2.1) and (2.2), the above equation is equivalent to the matrix

equation

$$P^{(k)*} = B_k U_k V B_k - B_k B_k^T B_k$$
$$= B_k(U_k V - B_k^T) B_k$$
$$= B_k \tilde{B}_k^T B_k,$$

which completes the proof of theorem 4.2.

Equation (4.3) provides us with an alternative way to find those columns of A that have only a 'contained intersection' with the other columns as follows:

Theorem 4.3. If all the non-zero elements of the s th column of $A^{(1)}$ are such that $p_{is}^{(1)*} = 0$, then the s th diagonal element of $B_1^T P^{(1)*}$ is zero.

Proof: The s th diagonal element is obtained as the inner product of the s th row of B_1^T (s th column of B_1) and the s th column of $P^{(1)*}$. Since all the elements of both the vectors are non-negative, hence a zero product implies that each $p_{is}^{(1)*}$ corresponding to every non-zero $\beta_{is}^{(1)}$ is zero. Therefore no matter which $a_{is}^{(1)} \neq 0$ is chosen as a pivot no new non-zero elements will be created since $p_{is}^{(1)*} = 0$, which completes the proof of the theorem.

For a given column s which has only 'contained intersections' (if any) with the other columns, it is easy to see that $p_{is}^{(1)*} = 0$ for all $a_{is}^{(1)} \neq 0$ and the s th diagonal element of $B_1^T P^{(1)*}$ is zero. Note that if the column s has $c_s^{(1)} = 1$, then the s th diagonal element of $B_1^T P^{(1)*}$ is necessarily zero and therefore all columns having only one element should be pivoted first. In passing we give the following theorem which is analogous to theorem 4.3.

Theorem 4.4. If all the non-zero elements of the s th row of $A^{(1)}$ are such that $p_{sj}^{(1)*} = 0$, then the s th diagonal element of $P^{(1)*} B_1^T$ is zero.

Proof: Same as in theorem 4.3.

Once again notice that the conditions of theorem 4.4 are trivially satisfied for all rows with $r_s^{(1)} = 1$, and therefore, if feasible, pivots should be chosen in all those rows of $A^{(1)}$ that have only one non-zero element.

If at every stage k (k = 1, 2, ..., n) it is possible to choose a pivot $a_{ij}^{(k)}$ such that $p_{ij}^{(k)*} = 0$, then evidently no new non-zero elements are created in the course of the entire process of elimination. For example, if A is a band type matrix viz.,

$$a_{ij} \neq 0, \text{ if } |i-j| \leq \theta$$
$$= 0, \text{ if } |i-j| > \theta,$$

(θ is the band width of A) and Gaussian elimination, with the diagonal elements chosen as pivots in the following order $a_{kk}^{(k)}$, k = 1, 2, ..., n, is used, then no new non-zero elements are created. Notice that in the above example $p_{kk}^{(k)*} = 0$, k = 1, 2, ..., n only for Gaussian elimination and not for the Gauss-Jordan method. Another example where Gaussian elimination creates no non-zero elements is given by Parter [6]. It is easily seen that for each end point say i of the tree defined in [6], $p_{ii}^{(k)*} = 0$. Instead of $p_{ij}^{(k)*}$ at times the use of $p_{ij}^{(k)}$ may be preferred especially since the computation of $P^{(k)}$ demands somewhat less work than that of $P^{(k)*}$. Therefore, if for each $a_{ij}^{(k)}$: k = 1, 2, ..., n, $p_{ij}^{(k)} = 0$, we have once again no new non-zero elements since $p_{ij}^{(k)} \geq p_{ij}^{(k)*} \geq 0$.

5. Some computational results. The program mentioned in [2] (written in FORTRAN IV for the IBM 7040/1401 system), which generates sparse coefficient matrices A, was used to construct three sets of 50 x 50 matrices $\{A_1\}$, $\{A_2\}$ and $\{A_3\}$. Set $\{A_1\}$ consists of 22 matrices of varying densities. On the other hand in $\{A_2\}$ and $\{A_3\}$ all the matrices within the same set have equal densities. The numerical values of the non-zero elements of all the matrices were taken in the range 99.00 to 0.01. This seemed to be a reasonable choice in view of the fact that various techniques for scaling a given coefficient matrix are available.

The Gauss-Jordan elimination was performed on each of the matrices in the sets $\{A_1\}$, $\{A_2\}$ and $\{A_3\}$ using four different methods given below. In all cases, as in [2], a pivot tolerance of 10^{-3} was used. The first method given below is similar to the first method described in [2].

Method M1.

(i) Columns of A were chosen sequentially for the elimination beginning with the first column.

(ii) The pivot was chosen as follows. Out of all those non-zero elements of the column chosen in step (i) above, we found the one that corresponded to an unpivoted row and had the least associated row count by using equation (2.1), provided that such an element was greater than 10^{-3} in absolute value. If this was not possible, then the maximum non-zero element of the chosen column corresponding to an unpivoted row was taken as the pivot, provided that it was greater than 10^{-5} in absolute value. If no pivot greater than 10^{-5} in absolute value could be determined, then the matrix under consideration was declared singular. The choice of 10^{-5} for the complete re-

jection of the pivots worked well for all the problems under consideration. Finally, each element of the row count vector that corresponded to a non-zero element of the column being eliminated was decreased by unity.

Method M_c.

The columns of A were eliminated in the following order. All those columns of A that were 'contained columns', as well as those that do not intersect with any other column, were chosen prior to all the other columns. Such contained and non-intersecting columns were determined by making use of theorem 3.1. The remaining columns were eliminated in the order of ascending values of the elements of $[D_j^{(1)}]$ of equation (2.3), where in computing $[D_j^{(1)}]$ the 'contained and isolated' columns of A were ignored. In case of 'contained' columns, the elements of maximum absolute value were chosen as pivots. We recall that elimination of a 'contained column' does not create any new non-zero elements. During the later part of the method viz., when the remaining columns of A were eliminated in ascending values of the elements of $[D_j^{(1)}]$, the row count vector was updated (modified) as in Method IV given in [2].

Method M_α.

The elimination on the columns of A was performed in the order of the ascending values of the elements of $[\bar{M}_j]$ given by equation (3.6). The row count vector was updated as in Method M_c above. Different values of $\alpha < 1$ were tried and it was found,

that for the matrices under consideration, $0.1 \leq \alpha \leq 0.5$ gave the best results.

Method M_a.

The pivots were chosen on the basis of

$$\min_{i,j} p_{ij}^{(k)*} \quad , \quad k = 1, 2, \ldots, n$$

where $p_{ij}^{(k)*}$ is defined in theorem 4.2. Thus at each step k the non-zero element $a_{ij}^{(k)}$ of $A^{(k)}$ having the minimum associated $p_{ij}^{(k)*}$ was chosen as a pivot, if possible. This method, in general, involves more work than M_α or M_c.

The number of new non-zero elements created when using each of the methods M_c, M_α and M_a was compared with the corresponding number for M1. The average results for the matrices in each set are given in table 1. For example, for the matrices in the set $\{A_2\}$, Method M_c on the average led to the creation of 32% fewer non-zero elements than Method M1.

TABLE 1

Set	Number of matrices in the set	Original density of matrices	In comparison with method M1 the percentages of fewer non-zero elements created by		
			M_c	M_α	M_a
$\{A_1\}$	22	.027-.15	36	35	51
$\{A_2\}$	28	.041	32	15	35
$\{A_3\}$	29	.070	47	45	66

We notice that Method M_a gives significantly better results than either of the Methods M_α or M_c. This is not surprising because M_a in general involves more work than the other two methods. Both M_c and M_α select

columns "a priori" viz., before the elimination process starts. On the other hand, for M_a we have to compute $P^{(k)*}$ according to equation (4.3) after each column has been eliminated. The somewhat unexpected value of 32% for $\{A_2\}$ with M_c, which is nearly as good as that for the more complicated method M_a can be explained as follows. The density of the matrices in $\{A_2\}$ is much less than the density of the matrices in $\{A_3\}$ as well as the average density of the matrices in $\{A_1\}$. Consequently, for such matrices a large number of columns tend to have a 'contained intersection' (every column with one element is a 'contained' column, see corollary 3.1) or no intersection at all with the other columns. Therefore, M_c does much better than M_α, since M_α depends on the column graph which in general has less "a priori" information in this particular case.

6. Concluding remarks. In this paper we have described several methods for minimizing the number of new non-zero elements created during the elimination methods. The structure of the coefficient matrix A, in general, affects the efficiency of a particular method, especially if the non-zero elements of A are not randomly distributed. Since in many large sparse systems the non-zero elements of A are not generally randomly distributed, it is not possible to single out one method as the best possible one. Therefore, it seems that the method must be selected on the basis of the type of A matrices one is most likely to have for the elimination processes.

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