THE COMPOSITION OF BANACH-SPACE-VALUED DISTRIBUTIONS

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1. Introduction. The idea of a time-varying Banach system, which was defined and analyzed in a previous work [1], led to a study of composition operators acting on spaces of distributions that take their values in Banach spaces. We call such distributions "Banach-space-valued distributions". Two types of composition were considered in [1]. The first, which was called "composition $\circ$", makes use of Schwartz's kernel theorem [2], [3] and its extension to Banach-space-valued distributions. It provides an explicit representation for every continuous linear mapping N of $D(A)$ into $[D; B]$. Here, A and B are complex Banach spaces, $D(A)$ is the space of infinitely differentiable (i.e., smooth) A-valued functions of compact support on the real line supplied with its customary topology, $D = D(C)$, C being the complex plane, and $[D; B]$ is the space of B-valued distributions equipped with the topology of uniform convergence on the bounded sets of D. A shortcoming of this representation for N is that it can be extended only onto certain spaces of suitably restricted continuous functions; it cannot be extended onto spaces containing singular distributions.

The second type of composition, which was called "composition $\circ$" in [1], is an extension to Banach-space-valued distributions of a composition product developed by Cristescu [4], Cristescu and Marinescu [5], Sabac [6], Wexler [7], Cioranescu [8], Pondelicek [9], and Dolezal [10]. In contrast to composition $\circ$, not all continuous linear mappings of $D(A)$

into [D; B] can be represented by a composition $\underline{\circ}$ operator. However, composition $\underline{\circ}$ has the virtue that, when it does exist, it can be applied to singular distributions.

The present work is aimed at this gap between composition $\underset{\sim}{\circ}$ and composition $\underline{\circ}$. A technique is developed for extending composition $\underset{\sim}{\circ}$, which we henceforth refer to simply as "composition", onto singular Banach-space-valued distributions in an explicit fashion. This is accomplished not for all but rather for certain suitably restricted composition operators. The resulting representation has a form similar to that for composition $\underline{\circ}$. In certain cases the conclusions of the present work are stronger than those of [1]. For instance, some of the composition operators studied herein are defined on certain finite-order Banach-space-valued distributions, but not on infinite-order distributions. In addition, our results imply an estimate on the order of the resulting composition product. The composition $\underline{\circ}$ operators studied in [1] were required to be defined on infinite-order, as well as finite-order, Banach-space-valued distributions,and no estimate on the order of any composition $\underline{\circ}$ product was acheived. Another generalization is that distributions on multidimensional euclidean spaces are now allowed, whereas [1] was restricted to distributions on the real line.

The principal conclusions of this work appear in Secs 4, 5, 6, and 7. Sec. 2 contains an explanation of our notations and a discussion of certain results pertaining to Banach-space-valued distributions. Sec 3 is devoted to some lemmas we shall need. Finally, two examples of our extended composition product appear in Sec. 8.

The author is indebted to R. Meidan for suggesting the proof of lemma 3.7.

2. Some notations and preliminary results. The notations of this work are similar to those used in [1], [11], and [12]. The definitions of any symbols used but not defined herein can be found in those works; see primarily [11; Secs 2 and 3].

A and B always denote complex Banach spaces with norms $\|\cdot\|_A$ and $\|\cdot\|_B$ respectively. R^s is the s-dimensional real euclidean space and C the complex plane. We denote any $t \in R^s$ by $t = \{t_1, t_2, \ldots, t_s\}$, the t_ν being the components of t. $|t|$ denotes the customary magnitude $[\Sigma_\nu |t_\nu|^2]^{1/2}$ of t. Throughout this work $m = \{m_\nu\}_{\nu=1}^{s}$ denotes an s-tuple each of whose components is ∞ or a nonnegative integer in R^1; for example, $m = \{2, \infty, 0\}$ is such a triplet. The set of all such s-tuples is denoted by $R_e^{\,s}$. Similarly, we will always have the symbol j denote an n-tuple in $R_e^{\,n}$. If $\tau \in R^1$ or $\tau = \infty$, the notation $[\tau]$ denotes that s-tuple all of whose components are equal to τ. For $q = \{q_\nu\}$ and $r = \{r_\nu\}$ in R^s (or in $R_e^{\,s}$) the inequality $q \leq r$ means $q_\nu \leq r_\nu$ for $\nu = 1, 2, \ldots, s$. A compact interval K in R^s is a set of the form $\{t: c \leq t \leq d\}$ where $c \in R^s$ and $d \in R^s$ are fixed. $\mathring{K}$ denotes the interior of K.

If U and V are two topological linear spaces, the symbol [U; V] denotes the linear space of all continuous linear mappings of U into V. Unless the opposite is explicitly indicated, we always assign to [U; V] the topology of uniform convergence on the bounded sets of U, which we also call the " bounded topology". Thus, for instance, [A; B] is assigned its operator-norm topology. At times, we will make use of the topology of pointwise convergence (also called the "pointwise topology") for the space [U; V], but in such cases we will always denote this space by $[U; V]^o$. (In [11] we referred to the latter topology as the "simple topology".)

Let ϕ be a function from R^s into some Banach space. When we say that ϕ is continuous or has a derivative, it will always be understood that the continuity and derivative are with respect to the norm topology of the Banach space. Thus, for example, if the Banach space happens to be [A; B], the said continuity is with respect to the operator-norm topology of [A; B]. Let $k = \{k_1, \ldots, k_s\}$ be a nonnegativeinteger in R^s (i.e., every component k_ν is a nonnegative integer). Any (partial)derivative

$$\frac{\partial^{|k|}\phi}{\partial t_1^{k_1} \cdots \partial t_s^{k_s}} \qquad |k| = k_1 + \cdots + k_s$$

of ϕ will be denoted by $D^k\phi = \phi^{(k)}$. We will also write $D_t^k\phi(t) = \phi^{(k)}(t)$. We shall refer to k (and not $|k|$) as the order of the differential operator D^k. The notation $|k|$ should not be confused with the magnitude notation for the members of R^s. ϕ is said to be smooth on an open set Ω (or compact interval K) if it has continuous derivatives of all orders at all points of Ω (respectively, at all points of $\mathring{K}$ and these derivatives have continuous extensions onto the boundary of K).

The support of any function or distribution f on R^s is denoted by supp f.

Let K be a compact interval in R^s. $D_K^m(A)$ denotes the linear space of all functions ϕ from R^s into A such that supp $\phi \subset K$ and, for every integer k in R^s with $0 \leq k \leq m$, $\phi^{(k)}$ is continuous on R^s. $D_K^m(A)$ is equipped with the topology generated by the collection $\{\gamma_k: 0 \leq k \leq m\}$ of seminorms where

$$(2.1) \qquad \gamma_k(\phi) \triangleq \sup_{t\in K} \|\phi^{(k)}(t)\|_A.$$

$D_K^m(A)$ is a Frechet space.

Now, choose any sequence $\{K_\nu\}_{\nu=1}^\infty$ of compact intervals in R^s such that

$K_\nu \subset K_{\nu+1}$ and $\bigcup_{\nu=1}^{\infty} K_\nu = R^s$. $D^m_{R^s}(A)$ is the strict inductive limit of the $D^m_{K_\nu}(A)$, $\nu = 1, 2, \ldots$; it is independent of the choice of $\{K_\nu\}_{\nu=1}^{\infty}$. We will use the simpler notation $D^m(A) = D^m_{R^s}(A)$ whenever there is no need to specify the euclidean space R^s on which the testing functions ϕ are defined. Moreover, when $m = [\infty]$, we drop the superscript notation and write $D^m(A) = D(A)$. The same convention is followed for all the other testing-function spaces used in this paper; that is, the lack of a superscript notation implies that $m = [\infty]$.

$[D^m(A); B]$ is a space of vector-valued distributions on R^s of order m. Some results we shall need concerning such distributions are stated in the next two lemmas.

Lemma 2.1: Let $f \in [D^m(A); B]$ (or $f \in [D^m; A]$), and let K be a compact interval in R^s. Then, there exist a (finite) integer $p \in R^s$ with $0 \leq p \leq m$ and a continuous $[A; B]$-valued (respectively, A-valued) function h on K such that, for all $\phi \in D_K^{m+[2]}(A)$ (respectively, $\phi \in D_K^{m+[2]}$),

$$(2.2) \qquad \langle f, \phi\rangle = \int_K h(t)\, \phi^{(p+[2])}(t)\, dt.$$

Proof: Let $f \in [D^m; A]$ and let J be a compact interval containing a neighborhood of K. It is a known result (see, for example, [13; theorem 4.1]) that this lemma holds true for all $\phi \in D_J$ when K is replaced by J. So, given any $\phi \in D_K^{m+[2]}$, we choose a sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ such that $\phi_\nu \in D_J$ and $\phi_\nu \to \phi$ in $D_J^{m+[2]}$ and therefore in D_J^m. This allows us to write

$$\langle f, \phi\rangle \leftarrow \langle f, \phi_\nu\rangle = \int_J h(t)\, \phi_\nu^{(p+[2])}(t)\, dt \to \int_K h(t)\, \phi^{(p+[2])}(t)\, dt,$$

and establish thereby (2.2). The same kind of argument also establishes this lemma for the case where $f \in [D^m(A); B]$.

Lemma 2.2: Let the sequence $\{v_\mu\}_{\mu=1}^{\infty}$ tend to zero in $[D^m; A]^\sigma$, and let K be a compact interval in R^s. Then, there exist a (finite) integer $p \in R^s$ with $0 \leq p \leq m$, which does not depend on μ, and a sequence $\{g_\mu\}_{\mu=1}^{\infty}$ of continuous A-valued functions on K such that $g_\mu(t) \to 0$ as $\mu \to \infty$ uniformly on K and in addition, for each μ,

$$(2.3) \qquad \langle v_\mu, \theta \rangle = \int_K g_\mu(t)\, \theta^{(p+[2])}(t)\, dt$$

for all $\theta \in D_K^{m+[2]}$.

Proof: This lemma was established in [1; theorem 4.4] for the case where $m = [\infty]$, $R^s = R^1$, and $\{v_\mu\}$ converges with respect to the bounded topology of $[D; A]$. The extension of that result to the case where $s > 1$ is straightforward. Now, for the given $m \in R_e^s$ assume that $\{v_\mu\}_{\mu=1}^{\infty}$ converges to zero in $[D^m; A]^\sigma$. By [14; p. 22, corollary 1] and the fact that D^m is complete, it follows that $\{v_\mu\}$ is a bounded set in $[D^m; A]$. Now, two cases arise. First assume that at least one component on f is ∞. Then, by the argument used in the proof of [11; lemma 3.1], we conclude that there exists a constant $M > 0$ and an integer $p \in R^s$ with $0 \leq p \leq m$ such that, for all $\phi \in D_K^m$,

$$\sup_\mu \| \langle v_\mu, \phi \rangle \|_A \leq M \max_{0 \leq k \leq p} \sup_{t \in R^s} |\phi^{(k)}(t)|.$$

On the other hand, if all the components of m are finite, then both D_K^m and A are Banach spaces. Since the bounded topology of $[D^m; A]$ is now the operator-norm topology, we have that the operator norm of each v_μ is bounded by some constant M that does not depend on μ. This implies the same inequality as above, where now we set $p = m$.

The said inequality is the same as [1; inequality (4.9)]. We now proceed exactly as in the proof of [1; theorem 4.4] to obtain lemma 2.2.

Next, $E^m(A) = E^m_{R^s}(A)$ is the space of all functions ϕ from R^s into A such that, for every integer k in R^s with $0 \le k \le m$, $\phi^{(k)}$ is continuous on R^s. $E^m(A)$ possesses the topology generated by the collection of seminorms $\{\gamma_{K_\nu,k}\}_{\nu,k}$ where $0 \le k \le m$, $\{K_\nu\}_{\nu=1}^\infty$ is a sequence of compact intervals with $K_\nu \subset K_{\nu+1}$ for every ν and $\bigcup_{\nu=1}^\infty K_\nu = R^s$ as before, and

$$\gamma_{K_\nu,k}(\phi) = \sup_{t\in K_\nu} \|\phi^{(k)}(t)\|_A.$$

$[E^m(A); B]$ turns out to be that subspace of $[D^m(A); B]$ consisting of the distributions of bounded support; it has a topology stronger than that induced on it by $[D^m(A); B]$. With $\{K_\nu\}_{\nu=1}^\infty$ defined as always, let $[E^m(A); B]_{K_\nu}$ be the linear space of all $f \in [E^m(A); B]$ with supp $f \subset K_\nu$. Assign to $[E^m(A); B]_{K_\nu}$ the topology induced on it by $[E^m(A); B]$. A sequence converges in $[E^m(A); B]$ if and only if it is contained and convergent in $[E^m(A); B]_{K_\nu}$ for some ν; the proof of this is the same as that for scalar distributions.

Given any compact interval $K \subset R^s$, $C^m_K(A)$ denotes the space of all functions ϕ from K into A such that, for every integer $k \in R^s$ with $0 \le k \le m$, $\phi^{(k)}$ is continuous on K. We assign to $C^m_K(A)$ the topology generated by the collection $\{\gamma_k: 0 \le k \le m\}$ of seminorms where γ_k is defined by (2.1). This makes $C^m_K(A)$ a Fréchet space.

For the next set of spaces, namely, $D^m_-(A)$ and $D^m_+(A)$, we require that $s = 1$. Hence, m is either ∞ or a nonnegative integer in R^1, and we will be dealing with testing functions and distributions defined on the real line R^1. Let $T \in R^1$ and $m \in R^1_e$ be fixed. $D^m_{-,T}(A)$ is the space of all A-valued functions ϕ on R^1 into A such that supp ϕ is contained in the closed interval $(-\infty, T]$ and, for every integer $k \in R^1$ with $0 \leq k \leq m$, $\phi^{(k)}$ is continuous. We equip $D^m_{-,T}(A)$ with the topology generated by the collection $\{\gamma_{\nu,k}\}_{\nu,k}$ where $0 \leq k \leq m$, $\nu = 1, 2, \ldots,$ and

$$\gamma_{\nu,k}(\phi) \triangleq \sup_{-\nu \leq t < \infty} \|\phi^{(k)}(t)\|_A.$$

Now consider the sequence of spaces obtained by setting $T = 1, 2, \ldots$ in $D^m_{-,T}(A)$. It can be seen that $D^m_{-,T}(A)$ is a closed subspace of $D^m_{-,T+1}(A)$ for every T, and the topology of $D^m_{-,T}(A)$ is the same as the topology induced on it by $D^m_{-,T+1}(A)$. $D^m_-(A)$ is defined as the strict inductive limit of the $D^m_{-,T}(A)$, $T = 1, 2, \ldots$.

$[D^m_-(A); B]$ turns out to be the subspace of $[D^m(A); B]$ consisting of the distributions whose supports are bounded on the left. Its topology is stronger than that induced on it by $[D^m(A); B]$. Let τ be any negative integer in R^1. $[D^m_-(A); B]_\tau$ denotes the linear subspace of $[D^m_-(A); B]$ consisting of the distributions whose supports are bounded on the left at τ and supplied with the topology induced by $[D^m_-(A); B]$. A sequence converges in $[D^m_-(A); B]$ if and only if it is contained and convergent in $[D^m_-(A); B]_\tau$ for some τ.

The spaces $D^m_{+,T}(A)$, $D^m_+(A)$, and $[D^m_+(A), B]$ are defined in a similar way and have similar properties. Now, however, the supports of the testing functions are bounded on the left, and the supports of the distributions are bounded on the right.

All the assertions made so far concerning the topologies and convergent sequences of $[E^m(A); B]$, $[D_-^m(A); B]$, and $[D_+^m(A); B]$ remain valid when these topologies as well as that of $[D^m(A); B]$ are replaced by the pointwise topologies.

Now, let $I^m(A)$ denote anyone of the following spaces: $D_K^m(A)$, $D^m(A)$, $E^m(A)$, $D_{-,T}^m(A)$, $E_-^m(A)$, $D_{+,T}^m(A)$, $D_+^m(A)$. As usual, when $A = C$, we write $I^m(C) = I^m$. $I^m \otimes A$ denotes the subspace of $I^m(A)$ consisting of all linear combinations of terms of the form ϕa where $\phi \in I^m$ and $a \in A$. If $m' \geq m$, then $I^{m'}(A)$ is a subspace of $I^m(A)$ and has a topology stronger than that induced on it by $I^m(A)$.

We define generalized differentiation D^k of order k on any $f \in [I^m(A); B]$ in the usual way:

$$(2.4) \qquad \langle D^k f, \phi \rangle \triangleq \langle f, \tilde{D}^k \phi \rangle \qquad \phi \in I^{m+k}(A),$$

where $\tilde{D}^k \triangleq (-1)^{|k|} D^k$. It follows that D^k is a continuous linear mapping of $[I^m(A); B]$ into $[I^{m+k}(A); B]$, as well as of $[I^m(A); B]^\sigma$ into $[I^{m+k}(A); B]^\sigma$. We also use the notation $D^k f = f^{(k)}$.

Another fact we shall make use of is the following: For $f \in [D^m(A); B]$ and $\phi \in D^m(A)$, $\langle f, \phi \rangle$ depends only on the values that ϕ assumes on a neighborhood of supp f. Thus, for example, if supp f is contained in the interior $\mathring{L}$ of a compact interval L, then $\langle f, \phi \rangle$ is uniquely defined for every $\phi \in C_L^m(A)$.

Before terminating this section, we state the extended form of the Schwartz kernel theorem referred to in Sec. 1. In the following n and s are both nonnegative integers in R^1.

Theorem 2.1: $\mathfrak{N}$ is a sequentially continuous linear mapping of $D_{R^s}(A)$ into $[D_{R^n}; B]^\sigma$ if and only if there exists an $f \in [D_{R^{n+s}}(A); B]$ such that $\mathfrak{N}v = f \underset{\sim}{\bullet} v$ for every $v \in D_{R^s}(A)$. Here, f is uniquely determined by $\mathfrak{N}$ and conversely. The composition product $f \underset{\sim}{\bullet} v$ is defined by

$$(2.5) \qquad \langle f \underset{\sim}{\bullet} v, \phi \rangle \triangleq \langle f(t, x), \phi(t)v(x) \rangle$$

$$\phi \in D_{R^n}, \; v \in D_{R^s}(A),$$

where $t \in R^n$ and $x \in R^s$.

A development of this result is given in [1; Sec. 4]. That discussion is restricted to the case where t and x are both in R^1; however, its extension to the present situation does not involve any difficulties. Moreover, we have merely assumed sequential continuity for $\mathfrak{N}$ in theorem 2.1 instead of continuity as in [1; Sec. 4] because, for any linear operator on $D_{R^s}(A)$, these two properties are equivalent. It is the composition operator $v \mapsto f \underset{\sim}{\bullet} v$ that we shall extend onto certain spaces such as $[E^m; A]$, $[D^m; A]$, and $[D^m_-; A]$.

3. Some lemmas. In this section we present some results needed in our discussion of composition operators.

Lemma 3.1: Let K and N be compact intervals in R^s such that $K \subset \mathring{N}$. Given any $\psi \in D^m_K(A)$, there exists a sequence $\{\psi_\mu\}_{\mu=1}^\infty$ such that all $\psi_\mu \in D_N \otimes A$ and, as $\mu \to \infty$, $\psi_\mu \to \psi$ in $D^m_N(A)$.

Proof: By a regularization procedure we can choose a sequence $\{\chi_k\}_{k=1}^\infty$ such that $\chi_k \in D_N(A)$, supp $\chi_k \subset \mathring{N}$, and, as $k \to \infty$, $\chi_k \to \psi$ in $D^m_N(A)$. Then, by virtue of [11; lemma 6.1], for each k we can choose a sequence $\{\zeta_{k,i}\}_{i=1}^\infty$ such that $\zeta_{k,i} \in D_N \otimes A$ and, as $i \to \infty$,

$\zeta_{k,i} \to \chi_k$ in $D_N(A)$ and therefore in $D_N^m(A)$. Since $D_N^m(A)$ satisfies the first axiom of countability, a sequence $\{\Psi_\mu\}$ that converges to Ψ in $D_N^m(A)$ can be picked out of the set of all $\zeta_{k,i}$.

Lemma 3.2: $E(A)$ is sequentially dense in $[D^m; A]^\sigma$. Also, if J and N are compact intervals in R^s with $J \subset \mathring{N}$ and if $v \in [E^m; A]_J$, then there exists a sequence $\{v_\nu\}_{\nu=1}^\infty$ with all $v_\nu \in D_N(A)$ such that $v_\nu \to v$ in $[E^m; A]_N^\sigma$. Finally, if $\tau \in R^1$ and $\zeta \in R^1$ satisfy $\tau < \zeta$ and if $v \in [D_-^m; A]_\zeta$, then there exists a sequence $\{v_\nu\}_{\nu=1}^\infty$ with all $v_\nu \in D_{+,\tau}(A)$ such that $v_\nu \to v$ in $[D_-^m; A]_\tau^\sigma$.
This lemma can be established by using regularization and [11; theorem 4.3].

Lemma 3.3: Given any $v \in [D^m; A]$, define q_v by

$$(3.1) \qquad \langle q_v, F\theta\rangle \triangleq F\langle v, \theta\rangle \qquad F \in [A; B],\ \theta \in D.$$

Then, there exists a unique $\hat{v} \in [D^{m+[2]}([A; B]); B]$ whose restriction to the elements of $D([A; B])$ of the form $F\theta$ coincides with q_v. Moreover, $v \mapsto \hat{v}$ is a sequentially continuous linear injection of $[D^m; A]$ into $[D^{m+[2]}([A; B]); B]$, as well as of $[D^m; A]^\sigma$ into $[D^{m+[2]}([A; B]); B]^\sigma$.

Proof: Let $\{N_\nu\}_{\nu=1}^\infty$ be a sequence of compact intervals in R^s such that $N_\nu \subset \mathring{N}_{\nu+1}$ for every ν and $\bigcup_{\nu=1}^\infty N_\nu = R^s$. Let P_ν denote the set of all members of $D_{N_\nu}([A; B])$ of the form $F\theta$ where $F \in [A; B]$ and $\theta \in D_{N_\nu}$. We have that $P_\nu \subset P_{\nu+1}$ for every ν. Let $P = \bigcup_{\nu=1}^\infty P_\nu$. Equation (3.1) defines q_v as a mapping of P into B.

By lemma 2.1, for every ν there exists a continuous A-valued function g_ν on N_ν and a finite nonnegative integer $p_\nu \in R^s$ such that $p_\nu \leq m$ and

$$(3.2) \quad \langle v, \theta \rangle = \int_{N_\nu} \theta^{(p_\nu + [2])}(t)\, g_\nu(t)\, dt \qquad \theta \quad D_{N_\nu}^{m+[2]}.$$

Note that the g_ν and p_ν can be so chosen that $p_\nu \leq p_{\nu+1}$ for every ν; this we do. We use g_ν and p_ν to define a mapping $\hat{v}_\nu$ of $D_{N_\nu}^{m+[2]}([A; B])$ into B by means of the expression:

$$(3.3) \quad \langle \hat{v}_\nu, \psi \rangle \triangleq \int_{N_\nu} \psi^{(p_\nu + [2])} g_\nu(t)\, dt \qquad \psi \in D_{N_\nu}^{m+[2]}([A; B]).$$

It follows that $\hat{v}_\nu \in [D_{N_\nu}^{m+[2]}([A; B]); B]$.

Observe that, for any $\Psi = F\theta \in P_\nu$,

$$(3.4) \quad \langle \hat{v}_\nu, F\theta \rangle = \int_{N_\nu} F\, \theta^{(p_\nu + [2])}(t)\, g_\nu(t)\, dt.$$

Since $F \in [A; B]$, we may take F outside the integral sign to obtain

$$F \int_{N_\nu} \theta^{(p_\nu + [2])}(t)\, g_\nu(t)\, dt = F\langle v, \theta \rangle = \langle q_v, F\theta \rangle.$$

Hence, $\hat{v}_\nu$ coincides with q_v on P and therefore on P_μ for all $\mu < \nu$.

The last result implies that, for every $\nu > 1$, $\hat{v}_{\nu+1}$ coincides with $\hat{v}_\nu$ on $D_{N_{\nu-1}}^{m+[2]}([A; B])$. Indeed, by lemma 3.1, given any $\psi \in D_{N_{\nu-1}}^{m+[2]}([A; B])$, we can choose a sequence $\{\psi_k\}_{k=1}^{\infty}$ such that $\psi_k \in D_{N_\nu} \otimes [A; B]$ and $\psi_k \to \psi$ in $D_{N_\nu}^{m+[2]}([A; B])$ and therefore in $D_{N_{\nu+1}}^{m+[2]}([A; B])$. Hence, as $k \to \infty$, we have that

$$\langle \hat{v}_{\nu+1}, \psi_k \rangle \to \langle \hat{v}_{\nu+1}, \psi \rangle \qquad k \to \infty.$$

But, ψ_k is a linear combination of elements from P_ν. Moreover, both $\hat{v}_{\nu+1}$ and $\hat{v}_\nu$ are linear on $D_{N_\nu}([A; B])$ and have the same restriction to P_ν, namely, q_v. Hence,

$$\langle \hat{v}_{\nu+1}, \psi_k \rangle = \langle \hat{v}_\nu, \psi_k \rangle \to \langle \hat{v}_\nu, \psi \rangle \qquad k \to \infty,$$

so that $\langle \hat{v}_{\nu+1}, \psi \rangle = \langle \hat{v}_\nu, \psi \rangle$, as was asserted.

Now, define $\hat{v}$ on $D^{m+[2]}([A; B])$ by setting

$$\langle \hat{v}, \psi \rangle = \langle \hat{v}_\nu, \psi \rangle \qquad \psi \in D^{m+[2]}_{N_{\nu-1}}([A; B])$$

for each ν. By virtue of the preceding paragraph, $\hat{v}$ is uniquely defined on $D^{m+[2]}([A; B])$. Moreover, $\hat{v}$ is linear. In addition, since $D^{m+[2]}([A; B])$ is the inductive limit of the $D^{m+[2]}_{N_{\nu-1}}([A; B])$, $\nu = 2, 3, 4, \ldots$, and since $\hat{v}$ has the continuous restrictions $\hat{v}_\nu$ to the subspaces $D^{m+[2]}_{N_{\nu-1}}([A; B])$, it follows that $\hat{v}$ is continuous. Thus, $\hat{v} \in [D^{m+[2]}([A; B]); B]$. Moreover, the restriction of $\hat{v}$ to P is q_v because, for each $\nu > 1$, its restriction to $P_{\nu-1}$ is the restriction of $\hat{v}_\nu$ to $P_{\nu-1}$, which in turn is q_v, as was noted above. There cannot be another member of $[D^{m+[2]}([A; B]); B]$ with q_v as its restriction to P because, as is implied by lemma 3.1, P is total in $D^{m+[2]}([A; B])$.

We now argue that $v \mapsto \hat{v}$ is injective. Assume that v and u are both members of $[D^m; A]$ and that $\langle v, \theta \rangle \neq \langle u, \theta \rangle$ for some $\theta \in D^m$. Then, there exists at least one $F \in [A; B]$ such that $F\langle v, \theta \rangle \neq F\langle u, \theta \rangle$. (Indeed, by the Hahn-Banach theorem [15; p. 187, corollary 2], there exists a continuous linear functional f on A such that $f\langle v, \theta \rangle \neq f\langle u, \theta \rangle$. Now, set $F = bf$ where b is any member of B other than the zero member.) Now, let $\hat{v}$ and $\hat{u}$ be the members of $[D^{m+[2]}([A; B]); B]$ defined from v and u respectively, as above. Since the restrictions of $\hat{v}$ and $\hat{u}$ to P are q_v and q_u respectively, we may write

$$<\hat{v}, F\theta> = <q_v, F\theta> = F<v, \theta> \neq F<u, \theta> = <q_u, F\theta> = <\hat{u}, F\theta>.$$

So truly, $v \mapsto \hat{v}$ is injective.

To see that $v \mapsto \hat{v}$ is linear, let $\alpha \in C$, $\beta \in C$, $v \in [D^m; A]$, and $u \in [D^m; A]$. Note that

$$<q_{\alpha v+\beta u}, F\theta> = F<\alpha v + \beta u, \theta> = \alpha F<v, \theta> + \beta F<u, \theta>$$

$$= \alpha<q_v, F\theta> + \beta<q_u, F\theta>$$

Thus,

$$<(\alpha v + \beta u)^\wedge, F\theta> = \alpha<\hat{v}, F\theta> + \beta<\hat{u}, F\theta> = <\alpha\hat{v} + \beta\hat{u}, F\theta>.$$

Since P is total in $D^{m+[2]}([A; B])$, we can conclude that $(\alpha v + \beta u)^\wedge = \alpha\hat{v} + \beta\hat{u}$.

It remains to prove that $v \mapsto \hat{v}$ is sequentially continuous. Let Ψ be a bounded set in $D^{m+[2]}([A; B])$. Consequently, Ψ is a bounded set in $D^{m+[2]}_{N_{\nu-1}}([A; B])$ for some sufficiently large $\nu > 1$. Let $\{v_k\}_{k=1}^{\infty}$ converge to zero in $[D^m; A]$, and let $\{\hat{v}_k\}$ be the corresponding sequence in $[D^{m+[2]}([A; B]); B]$. We now invoke lemma 2.2. With respect to the interval N_ν, there exist a finite integer $p_\nu \geq 0$ in R^n and a sequence $\{g_{\nu,k}\}_{k=1}^{\infty}$ of continuous A-valued functions on N_ν such that $g_{\nu,k} \to 0$ as $k \to \infty$ uniformly on N_ν and

$$(3.5)\quad <v_k, \theta> = \int_{N_\nu} g_{\nu,k}(t)\, \theta^{(p_\nu + [2])}(t)\, dt \qquad \theta \in D^{m+[2]}_{N_\nu}.$$

By the definition (3.3), we have that

$$<\hat{v}_{\nu,k}, \Psi> \triangleq \int_{N_\nu} \Psi^{(p_\nu + [2])}(t)\, g_{\nu,k}(t)\, dt \qquad \Psi \in D^{m+[2]}_N([A; B]).$$

Moreover, the restriction of $\hat{v}_k$ to $D_{N_{\nu-1}}^{m+[2]}([A; B])$ is by definition equal to the restriction of $\hat{v}_{\nu,k}$ to that space. In view of these results, we may write

$$\sup_{\psi \in \Psi} \|<\hat{v}_k, \psi>\|_B = \sup_{\psi\in\Psi} \|<\hat{v}_{\nu,k}, \psi>\|_B$$

$$= \sup_{\psi\in\Psi} \left\| \int_{N_{\nu-1}} \psi^{(p_\nu+[2])}(t)\, g_{\nu,k}(t)\, dt \right\|_B$$

$$\leq \sup_{\psi\in\Psi} \int_{N_{\nu-1}} \left\|\psi^{(p_\nu+[2])}(t)\right\|_{[A;\ B]} dt \sup_{t\in N_{\nu-1}} \|g_{\nu,k}(t)\|_A$$

The right-hand side tends to zero as $k \to \infty$. This establishes the sequential continuity of $v \mapsto \hat{v}$ from $[D^m; A]$ into $[D^{m+[2]}([A; B]); B]$. By restricting Ψ to sets of single elements, we arrive at the same conclusion with respect to the topologies of pointwise convergence. This completes the proof of lemma 3.3.

The next two lemmas are actually corollaries to lemma 3.3.

Lemma 3.4: Given any $v \in [E^m; A]$, define q_v by (3.1). Then, there exists a unique $\hat{v} \in [E^{m+[2]}([A; B]); B]$ whose restriction to the elements of $D([A; B])$ of the form $F\theta$, $F \in [A; B]$, $\theta \in D$, coincides with q_v. Moreover, supp v = supp $\hat{v}$. In addition, $v \mapsto \hat{v}$ is a sequentially continuous linear injection of $[E^m; A]$ into $[E^{m+[2]}([A; B]); B]$, as well as of $[E^m; A]^\sigma$ into $[E^{m+[2]}([A; B]); B]^\sigma$.

Proof: $[E^m; A]$ is a subspace of $[D^m; A]$ and has a topology stronger than that induced on it by $[D^m; A]$. So, upon defining $\hat{v}$ as in the proof of lemma 3.3, we see that $v \mapsto \hat{v}$ is a sequentially continuous linear injection of $[E^m; A]$ into $[D^{m+[2]}([A; B]); B]$. Moreover, $\hat{v}$ coincides with q_v on P. Thus, we have

(3.6) $\langle \hat{v}, F\theta \rangle = \langle q_v, F\theta \rangle = F\langle v, \theta \rangle \qquad F \in [A; B], \theta \in D.$

Now, let K be a compact interval in the null set Λ of v. Since Λ is open, there exists another compact interval $N \subset \Lambda$ with $K \subset \mathring{N}$. Then, for every $\theta \in D_N$, $\langle v, \theta \rangle = 0$. Hence, for every $F \in [A; B]$,

$$\langle \hat{v}, F\theta \rangle = F\langle v, \theta \rangle = 0.$$

But then, for every $\Psi \in D_K([A; B])$, we can choose a sequence $\{\Psi_k\}_{k=1}^{\infty}$ with elements in $D_N \otimes [A; B]$ and convergent to Ψ in $D_N([A; B])$ according to lemma 3.1. Consequently,

$$0 = \langle \hat{v}, \Psi_k \rangle \to \langle \hat{v}, \Psi \rangle \qquad k \to \infty.$$

Hence $\langle \hat{v}, \Psi \rangle = 0$. Since K can be any compact interval in Λ, we conclude that supp $\hat{v} \subset$ supp v.

On the other hand, let Ω be the null set of $\hat{v}$. Then,

$$F\langle v, \theta \rangle = \langle \hat{v}, F\theta \rangle = 0$$

for every $F \in [A; B]$ and every $\theta \in D$ with supp $\theta \subset \Omega$. By the Hahn-Banach theorem, $\langle v, \theta \rangle = 0$ for every such θ. Hence, supp v $\subset$ supp $\hat{v}$. Thus, supp v = supp $\hat{v}$.

In view of the results obtained so far, we have that $v \mapsto \hat{v}$ is a sequentially continuous linear injection of $[E^m; A]$ into $[D^{m+[2]}([A; B]); B]$ with range in $[E^{m+[2]}([A; B]); B]$. We now invoke that fact that,

if a sequence $\{v_k\}_{k=1}^{\infty}$ tends to zero in $[E^m; A]$, then there exists a compact interval K such that supp $v_k \subset K$ for all k. Hence, $\hat{v}_k \to 0$ in $[D^{m+[2]}([A; B]); B]$ with supp $\hat{v}_k \subset K$ for all k. Whence, $\hat{v}_k \to 0$ in $[E^{m+[2]}([A; B]); B]$. This argument leads to the same conclusion in regard to $[E^m; A]^\sigma$

and $[E^{m+[2]}([A; B]); B]^{\sigma}$.

In the next lemma it is understood that $R^n = R^1$.

Lemma 3.5: Given any $v \in [D_-^m; A]$, define q_v by (3.1). Then, there exists a unique $\hat{v} \in [D_-^{m+[2]}([A; B]); B]$ whose restriction to the elements of $D([A; B])$ of the form $F\theta$, $F \in [A; B]$, $\theta \in D$, coincides with q_v. Moreover, supp v = supp $\hat{v}$. In addition, $v \mapsto \hat{v}$ is a sequentially continuous linear injection of $[D_-^m; A]$ into $[D_-^{m+[2]}([A; B]); B]$, as well as of $[D_-^m; A]^{\sigma}$ into $[D_-^{m+[2]}([A; B]); B]^{\sigma}$.

The proof of this lemma is the same as that of lemma 3.4.

We now consider certain regular distributions and some of their properties. Let $v \in E^m(A)$. Then, v generates a unique member of $[D^0; A]$, which we also denote by v, through the definition:

$$\langle v, \phi \rangle \triangleq \int_{R^s} v(x)\phi(x)\, dx \qquad \phi \in D^0.$$

(Here, the superscript 0 in D^0 denotes the s-tuple [0].) Moreover, v defines an $[[A; B]; B]$-valued function v' on R^s by the definition:

$$v'(x)F = Fv(x) \qquad F \in [A; B],\ x \in R^s.$$

By the Hahn-Benach theorem, supp v' = supp v and, for each x, $v(x) \mapsto v'(x)$ is injective.

Moreover, for every integer k in R^s with $0 \le k \le m$, $D^k v'$ exists and is continuous from R^s into $[[A; B]; B]$. For example, fix $x \in R^s$, assume that m is such that $\partial_\nu \triangleq \partial/\partial x_\nu$ is of order less than or equal to m, and let $x + \Delta x|_\nu$ denote that element of R^s obtained from $x \in R^s$ by adding $\Delta x_\nu \in R^1$ to the νth component of x. For $\Delta x_\nu \neq 0$, consider

$$\sup_{\|F\|=1} \left\| \left[\frac{v'(x+\Delta x|_\nu) - v'(x)}{\Delta x_\nu} - (\partial_\nu v)'(x) \right] F \right\|_B$$

$$= \sup_{\|F\|=1} \left\| F \left[\frac{v(x+\Delta x|_\nu) - v(x)}{\Delta x_\nu} - (\partial_\nu v)(x) \right] \right\|_B$$

$$\leq \left\| \frac{v(x+\Delta x|_\nu) - v(x)}{\Delta x_\nu} - (\partial_\nu v)(x) \right\|_A \to 0 \qquad \Delta x_\nu \to 0.$$

So truly, for each $x \in R^s$, $\partial_\nu v'(x)$ exists and is equal to $(\partial_\nu v)'(x) \in [[A; B]; B]$. Hence, $\partial_\nu v'(x)F = (\partial_\nu v)'(x)F = F\partial_\nu v(x)$. To show the continuity of $\partial_\nu v'$, we need merely write, for any $x \in R^s$ and $\Delta x \in R^s$,

$$\sup_{\|F\|=1} \| [\partial_\nu v'(x+\Delta x) - \partial_\nu v'(x)]F \|_B = \sup_{\|F\|=1} \| F[\partial_\nu v(x+\Delta x) - \partial_\nu v(x)] \|_B$$

$$\leq \| \partial_\nu v(x+\Delta x) - \partial_\nu v(x) \|_A \to 0 \qquad \Delta x \to 0.$$

Continuing in this fashion to the higher-order derivatives, we obtain our assertion concerning $D^k v'$, as well as the equality $D^k v'(x)F = FD^k v(x)$.

Next, we note that v' generates a (regular) member of $[D^0([A; B]); B]$, which we also denote by v', through the definition:

$$\langle v', \Psi \rangle \triangleq \int_{R^s} v'(x)\Psi(x)\, dx \qquad \Psi \in D^0([A; B]).$$

This allows us to write, for any $F \in [A; B]$ and $\theta \in D^0$,

$$\langle v', F\theta \rangle = \int_{R^s} v'(x)F\theta(x)\, dx = \int_{R^s} Fv(x)\theta(x)\, dx.$$

Since F is continuous and linear on A, we may take it outside the integral sign to get

$$\langle v', F\theta\rangle = F\int_{R^s} v(x)\theta(x)\,dx = F\langle v, \theta\rangle.$$

Let us now compare the last result with lemma 3.3 and invoke the facts that $v \in E^m(A) \subset [D^0; A]$ and $v' \in [D^0([A; B]); B] \subset [D^{[2]}([A; B]); B]$. We conclude that $v' = \hat{v}$ in the sense of equality in $[D^{[2]}([A; B]); B]$. Consequently, supp v' = supp $\hat{v}$.

We summarize these results as follows.

Lemma 3.6: Let $v \in E^m(A)$. Define v' from v as above and let $\hat{v}$ correspond to v in accordance with lemma 3.3. Then, for every integer $k \in R^s$ with $0 \leq k \leq m$, we have that $D^k v'$ exists, is continuous from R^s into $[[A; B]; B]$, and therefore generates a regular member of $[D^0([A; B]); B]$. Moreover,

$$D^k v'(x)F = FD^k v(x)$$

for every $F \in [A; B]$. Finally, $v' = \hat{v}$ in the sense of equality in $[D^{[2]}([A; B]); B]$, and supp $\hat{v}$ = supp v' = supp v.

Lemma 3.7: Let I and L be compact intervals in R^n and R^s respectively. Assume that $h(t, x)$ is a continuous $[A; B]$-valued function on the compact interval $I \times L$ in R^{n+s} and is such that there exist an $m \in R_e^s$, a $j \in R_e^n$, and an integer $i \in R^n$ with $0 \leq i \leq j$ for which

$$(3.7) \qquad \chi_\phi \triangleq \int_I h(t, \cdot)\phi^{(i)}(t)\,dt \in C_L^m([A; B])$$

for all $\phi \in D_I^j$. Then, $\phi \mapsto \chi_\phi$ is a continuous linear mapping of D_I^j into $C_L^m([A; B])$.

Here, as well as elsewhere, the notation $h(t, \cdot)$ denotes the member of $C_L^0([A; B])$ defined by the mapping $x \mapsto h(t, x)$ and depending upon the choice of t.

Proof: We shall make use of the closed graph theorem. We first note that every $f \in C_L^m([A; B])$ defines an $\hat{f} \in [C_L^0(A); B]$ by means of the definition

$$\langle \hat{f}, \theta \rangle \triangleq \int_L f(x)\, \theta(x)\, dx \qquad \theta \in C_L^0(A).$$

In fact, it readily follows that $f \mapsto \hat{f}$ is a continuous linear injection of $C_L^m([A; B])$ into $[C_L^0(A); B]^\sigma$. Upon replacing $\hat{f}$ by χ_ϕ, we obtain

$$\langle \chi_\phi, \theta \rangle = \int_L \int_I h(t, x)\, \phi^{(i)}(t)\, dt\, \theta(x)\, dx.$$

This implies that $\phi \mapsto \chi_\phi$ is a continuous linear mapping of D_I^j into $[C_L^0(A); B]^\sigma$.

Now, choose a sequence $\{\phi_\nu\}$ such that $\phi_\nu \to \phi$ in D_I^j and $\chi_{\phi_\nu} \to \Psi$ in $C_L^m([A; B])$. We have that $\chi_{\phi_\nu} \to \chi_\phi$ in $[C_L^0(A); B]^\sigma$ as well. But, since $[C_L^0(A); B]^\sigma$ is separated and since $C_L^m([A; B])$ can be identified with a subspace of $[C_L^0(A); B]^\sigma$ as above, we have that $\Psi = \chi_\phi$. This implies that the linear operator $\phi \mapsto \chi_\phi$ is closed because both D_I^j and $C_L^m([A; B])$ are Fréchet spaces. The closed graph theorem [15; p. 173] completes the proof.

4. Composition operators on $[E^m_{R^s}; A]$ into $[D^j_{R^n}; B]$. Let $f \in [D_{R^{n+s}}(A); B]$. Choose any two compact intervals $I \subset R^n$ and $L \subset R^s$. Upon appealing to lemma 2.1, we see that there exist a $j \in R^n_e$ not depending on I, an $h \in C^0_{I \times L}([A; B])$, and two nonnegative integers $i \in R^n$ and $l \in R^s$ with $i \leq j$ such that, for all $\phi \in D^j_I$ and $v \in D_L(A)$,

$$\langle f \circledast v, \phi\rangle = \langle f(t, x), \phi(t)v(x)\rangle$$

$$= \int_I \int_L h(t, x)\, \phi^{(i)}(t)\, v^{(l)}(x)\, dt\, dx.$$

We may now convert the integral on the right-hand side into a repeated integral:

$$\int_L \left[\int_I h(t, x)\, \phi^{(i)}(t)\, dt\right] v^{(l)}(x)\, dx.$$

Assume now that the inner integral has continuous derivatives on L of at least order $l + [2]$. Then, we may integrate by parts $|l|$ times to obtain

$$\int_L \left[\tilde{D}^l_x \int_I h(t, x)\, \phi^{(i)}(t) dt\right] v(x)\, dx \qquad \tilde{D}^l_x \triangleq (-1)^{|l|} D^l_x.$$

(The last two steps can be justified in just the same way as for scalar Riemann integrals.) Let us now assume in addition that $\operatorname{supp} v \subset \mathring{L}$. By virtue of lemma 3.6 and its notation, we may write

$$\langle f \circledast v, \phi\rangle = \int_L v'(x)\, \tilde{D}^l_x \int_I h(t, x)\, \phi^{(i)}(t)\, dt\, dx$$

$$= \left\langle \hat{v}(x), \tilde{D}^l_x \int_I h(t, x)\, \phi^{(i)}(t)\, dt \right\rangle$$

This result motivates the following assumption and definition, which provide a means of extending the definition (2.5) of the composition operator $f\circledast: v \mapsto f \circledast v$ onto the space $[E^m_{R^s}; A]$.

Assumption 4.1: Assume that corresponding to a given $f \in [D_{R^{n+s}}(A); B]$ there is a $j \in R^n_e$ and an $m \in R^s_e$ for which the following conditions are satisfied. For every choice of the compact intervals $I \subset R^n$ and $L \subset R^s$, there exist an $h \in C^0_{I \times L}([A; B])$ and two nonnegative integers $i \in R^n$ and $l \in R^s$ with $i \leq j$ such that, for all $\phi \in D^j_I$ and all $v \in D_L(A)$,

$$(4.2) \qquad \langle f(t, x), \phi(t)v(x)\rangle = \int_I \int_L h(t, x)\, \phi^{(i)}(t)\, v^{(l)}(x)\, dt\, dx$$

and, in addition,

$$(4.3) \qquad \int_I h(t, \cdot)\, \phi^{(i)}(t)\, dt \in C_L^{m+l+[2]}([A; B]).$$

Definition 4.1: Let $f \in [D_{R^{n+s}}(A); B]$ satisfy assumption 4.1. Given a $\phi \in D^j_{R^n}$ and a $v \in [E^m_{R^s}; A]$, choose I and L such that supp $\phi \subset I$ and supp $v \subset \mathring{L}$. Finally, choose h, i, and l in accordance with assumption 4.1. We define $\langle f \circledast v, \phi\rangle$ by

$$(4.4) \qquad \langle f \circledast v, \phi\rangle \triangleq \left\langle \hat{v}(x), \tilde{D}^l_x \int_I h(t,x)\, \phi^{(i)}(t)\, dt \right\rangle,$$

where $\hat{v}$ is that member of $[E^{m+[2]}_{R^s}([A; B]); B]$ corresponding to v in accordance with lemma 3.4.

Theorem 4.1: Under assumption 4.1 and definition 4.1, $\langle f \circledast v, \phi\rangle$ is independent of the choices of the parameters I, L, h, i, and l and is consistent with (2.5). Moreover, the operator $f\circledast: v \mapsto f \circledast v$ is a sequentially continuous linear mapping of $[E^m_{R^s}; A]$ into $[D^j_{R^n}; B]$, as well as of $[E^m_{R^s}; A]^\sigma$ into $[D^j_{R^n}; B]^\sigma$.

Proof: The consistency of the definition 4.1 with that of (2.5) is established by the manipulations leading up to (4.1). Now, fix I and L for the moment. Upon applying $\tilde{D}^1$ to the integral in (4.3), we get a member of $C_L^{m+[2]}([A; B])$. Since supp $\hat{v}$ = supp $v \subset \mathring{L}$, the right-hand side of (4.4) has a meaning and is a member of B. Therefore, $f \circledast v$ maps D_I^j into B. This mapping is linear. To see its continuity, let $\{\phi_\nu\}_{\nu=1}^{\infty}$ converge to zero in D_I^j. By assumption 4.1, lemma 3.7, and the continuity of $\tilde{D}^1$ as a mapping of $C_L^{m+1+[2]}([A; B])$ into $C_L^{m+[2]}([A; B])$, we have that

$$\tilde{D}^1_{\cdot} \int_I h(t, \cdot)\, \phi_\nu^{(1)}(t)\, dt \to 0 \qquad \nu \to \infty$$

in $C_L^{m+[2]}([A;B])$. By lemma 3.4 and the fact that supp $\hat{v} \subset \mathring{L}$, the right-hand side of (4.4) tends to zero in B. Thus, we conclude that $f \circledast v \in [D_I^j; B]$.

Now, let J be any compact interval in R^s with $J \subset \mathring{L}$. By what we have shown so far, the mapping $v \mapsto f \circledast v$ carries $[E^m_{R^s}; A]_J$ into $[D_I^j; B]$. That it is linear follows from the linearity of the mapping $v \mapsto \hat{v}$ (see lemma 3.4). To show its sequential continuity, let Φ be a bounded set in D_I^j. By lemma 3.7 again, we have that, as ϕ traverses Φ,

$$\tilde{D}^1_{\cdot} \int_I h(t, \cdot)\, \phi^{(1)}(t)\, dt$$

traverses a bounded set, say, Θ in $C_L^{m+[2]}([A; B])$. Next, let $\lambda \in D_L$ be such that $\lambda \equiv 1$ on a neighborhood of J. Extend $\lambda\theta$, where $\theta \in \Theta$, outside J as the zero function. Then, $\lambda\theta$ traverses a bounded set in $E^{m+[2]}_{R^s}([A; B])$ as θ traverses Θ. By (4.4),

$$(4.5) \qquad \sup_{\phi \in \Phi} \|\langle f \circledast v, \phi\rangle\|_B = \sup_{\theta \in \Theta} \|\langle \hat{v}, \theta\rangle\|_B = \sup_{\theta \in \Theta} \|\langle \hat{v}, \lambda\theta\rangle\|_B .$$

Now, let V denote the image in $[E^{m+[2]}_{R^s}([A;\ B]);\ B]_J$ of $[E^m_{R^s};\ A]_J$ under the linear mapping $v \mapsto \hat{v}$. V is a linear subspace, and we equip it with the induced topology. Equation (4.5) shows that the linear mapping $\hat{v} \mapsto f \circledast v$ is continuous from V into $[D^j_I;\ B]$. In view of lemma 3.4 again, we can conclude that the composite mapping $v \mapsto \hat{v} \mapsto f \circledast v$ is a sequentially continuous linear mapping of $[E^m_{R^s};\ A]_J$ into $[D^j_I;\ B]$, where J may be any compact interval in R^s with $J \subset \mathring{L}$. A similar argument wherein Φ is restricted to sets of single elements shows that the same conclusion holds with respect to $[E^m_{R^s};\ A]^\sigma_J$ and $[D^j_I;\ B]^\sigma$.

We will now show that given any $v \in [E^m_{R^s};\ A]_J$ and $\phi \in D^j_I$, the definition of $\langle f \circledast v,\ \phi\rangle$ is independent of the choices of I, L, h, i, and l, where it is understood that $J \subset \mathring{L}$. Let $\langle f \circledast v,\ \phi\rangle_1$ and $\langle f \circledast v,\ \phi\rangle_2$ be two definitions corresponding to two different choices of the set of parameters I, L, h, i, and l. Also, let N be a compact interval in R^s contained in the interiors of both choices of L and containing a neighborhood of J. Choose a sequence $\{v_\nu\}_{\nu=1}^\infty$ such that all $v_\nu \in D_N(A)$ and $v_\nu \to v$ in $[E^m_{R^s};\ A]^\sigma_N$ (lemma 3.2). By the manipulations leading up to (4.1), we have, for each v_ν,

$$(4.6) \quad \langle f \circledast v_\nu,\ \phi\rangle_1 = \langle f(t,\ x),\ \phi(t)v_\nu(x)\rangle = \langle f \circledast v_\nu,\ \phi\rangle_2.$$

But, by the last sentence of the preceding paragraph, the left-hand side (right-hand side) of (4.6) tends to $\langle f \circledast v,\ \phi\rangle_1$ (respectively, $\langle f \circledast v,\ \phi\rangle_2$) in B. This shows that $\langle f \circledast v,\ \phi\rangle_1 = \langle f \circledast v,\ \phi\rangle_2$.

Next, with $v \in [E^m_{R^s};\ A]$ remaining fixed but ϕ allowed to vary throughout $D^j_{R^n}$, we have from the last paragraph that $f \circledast v$ is uniquely defined on all of $D^j_{R^n}$. Moreover, it is linear on

$D^j_{R^n}$. Since its restriction to each D^j_I is continuous, we have that $f \circledast v \in [D^j_{R^n}; B]$.

Finally, let v be arbitrary in $[E^m_{R^s}; A]$. By what we have just shown, the operator $f \circledast$ is uniquely defined on all of $[E^m_{R^s}; A]$. Moreover, it follows from lemma 3.4 once again that $f \circledast$ is linear on $[E^m_{R^s}; A]$. To show its continuity, let the sequence $\{v_\nu\}_{\nu=1}^\infty$ tend to zero in $[E^m_{R^s}; A]$.

Hence, $v_\nu \to 0$ in $[E^m_{R^s}; A]_J$ for some J. Therefore, $f \circledast v_\nu \to 0$ in $[D^j_I; B]$ for every I. But, any bounded set in $D^j_{R^n}$ is contained and bounded in D^j_I for some I. This implies that $f \circledast v_\nu \to 0$ in $[D^j_{R^n}; B]$. A similar argument shows that, when $v_\nu \to 0$ in $[E^m_{R^s}; A]^\sigma$, $f \circledast v_\nu \to 0$ in $[D^j_{R^n}; B]^\sigma$. This completes the proof of theorem 4.1.

We now relate our definition of $f \circledast v$ to the composition $\underset{\sim}{\circ}$ product discussed in [1; Sec. 4]. In accordance with definition 4.1, let f, v, and ϕ be given and choose the parameters I, L, h, i, and l appropriately. In view of (4.4), we may write

$$(4.7) \qquad \langle f \circledast v, \phi \rangle = \langle \hat{v}, \Psi_\phi \rangle$$

where

$$(4.8) \qquad \Psi_\phi \triangleq \tilde{D}^l_{\cdot} \int_I h(t, \cdot)\, \phi^{(i)}(t)\, dt \in C^{m+[2]}_L([A; B])$$

by virtue of (4.3). Let us set

$$(4.9) \qquad \langle y_x, \phi \rangle \triangleq \Psi_\phi(x) \qquad \phi \in D^j_I, \quad x \in \mathring{L}.$$

For each $x \in \mathring{L}$, this defines y_x as a linear mapping of D^j_I into $[A; B]$. That y_x is continuous on D^j_I follows from lemma 3.7. Thus, for each $x \in \mathring{L}$, $y_x \in [D^j_I; [A; B]]$.

Now assume that $v \in D_{R^s}(A)$. By virtue of (4.7) and lemma 3.6, we may write

$$(4.10) \quad \langle f \circledast v, \phi \rangle = \langle \hat{v}, \Psi_\phi \rangle = \int_L v'(x)\, \Psi_\phi(x)\, dx$$

$$= \int_L \Psi_\phi(x)\, v(x)\, dx.$$

We have shown in the proof of theorem 4.1 that (4.10) does not depend on the choice of the parameters. Moreover, because of the continuity of Ψ_ϕ as a function of $x \in L$ and because of our freedom to choose L as large as we wish, a knowledge of the right-hand side of (4.10) for every $v \in D_{R^s}(A)$ and any fixed ϕ uniquely determines $\Psi_\phi(x) \in [A; B]$ for every $x \in R^s$. Since (4.8) is true for every L, $\Psi_\phi \in E^{m+[2]}_{R^s}([A; B])$. Thus, for any fixed $x \in R^s$, y_x is a uniquely determined mapping of $D^j_{R^n}$ into $[A; B]$, and, since its restriction to each D^j_I is linear and continuous, we have that $y_x \in [D^j_{R^n}; [A; B]]$. Thus, we have shown that y_x satisfies the following conditions, which are the same as those in [1; Sec. 4] except that in [1] it was assumed that $n = s = 1$, $j = \infty$, and $m = \infty$.

Conditions G:

G1. For each $x \in R^s$, $y_x \in [D^j_{R^n}; [A; B]]$.

G2. $\phi \mapsto \Psi_\phi$ is a mapping of $D^j_{R^n}$ into $E^{m+[2]}_{R^s}([A; B])$.

We summarize these conclusions as follows:

Corollary 4.1: Let f satisfy assumption 4.1 and define $\langle f \circledast v, \phi \rangle$ in accordance with definition 4.1. Also, define y_x from f through (4.8) and (4.9). Then, y_x is independent of the choice of the parameters I, L, h, i, and l, fulfills conditions G, and satisfies

(4.11) $\langle f \circledast v, \phi \rangle = \langle \hat{v}, \Psi_\phi \rangle, \quad \Psi_\phi(x) \triangleq \langle y_x, \phi \rangle$

for every $\phi \in D^j_{R^n}$ and $v \in [E^m_{R^s}; A]$.

The right-hand side of (4.11) is the definition of $\langle v \circledast y_x, \phi \rangle$ given in [1; equation (4.18)].

5. Composition operators on $[D^m_{R^s}; A]$ into $[D^j_{R^n}; B]$. The composition operator $f\circledast$ can be extended onto the space $[D^m_{R^s}; A]$ if, in addition to assumption 4.1, a condition is placed upon supp f.

Assumption 5.1: $f \in [D_{R^{n+s}}(A); B]$ satisfies assumption 4.1 and in addition the following condition on its support. For every choice of the compact interval $I \subset R^n$, the set

$$\Xi_I \triangleq (I \times R^s) \cap \text{supp } f$$

is bounded in R^{n+s}.

Since both $I \times R^s$ and supp f are closed, Ξ_I is also closed and therefore compact. As before, we let $t \in R^n$, $x \in R^s$, and $(t, x) \in R^n \times R^s = R^{n+s}$. In the following, $P_x \Xi_I$ denotes the projection of Ξ_I onto the x-space. $P_x \Xi_I$ is a compact set.

Definition 5.1: Let f satisfy assumption 5.1. Given a $\phi \in D^j_{R^n}$, choose a compact interval I in R^n such that supp $\phi \subset I$. Then, choose a compact interval L in R^s such that $P_x \Xi_I \subset \mathring{L}$. Finally, choose a $\lambda \in D_L$ such that $\lambda \equiv 1$ on a neighborhood of $P_x \Xi_I$. Then, for any $v \in [D^m_{R^s}; A]$, define $\langle f \circledast v, \phi \rangle$ by

(5.1) $\langle f \circledast v, \phi \rangle \triangleq \left\langle \hat{v}(x), \lambda(x) \tilde{D}^l_x \int_I h(t, x) \phi^{(i)}(t)\, dt \right\rangle$

where h, i, and l, are chosen to satisfy assumption 4.1 and $\hat{v}$ is the member of $[D^{m+[2]}_{R^s}([A; B]); B]$ corresponding to v in accordance with lemma 3.3. It is understood here that the testing function in the right-hand side of (5.1) has been extended outside the interval L as the zero function.

Theorem 5.1: Under assumption 5.1 and definition 5.1, $\langle f \circ v, \phi \rangle$ is independent of the choices of the parameters I, L, λ, h, i, and l and is consistent with (2.5). Moreover, the operator $f\circ : v \mapsto f \circ v$ is a sequentially continuous linear mapping of $[D^m_{R^s}; A]$ into $[D^j_{R^n}; B]$, as well as of $[D^m_{R^s}; A]^\sigma$ into $[D^j_{R^n}; B]^\sigma$.

Proof: We first fix upon appropriate choices of the parameters I, L, λ, h, i, and l for the given f and ϕ. By lemma 3.7 and assumption 4.1,

$$(5.2) \qquad \phi \mapsto \lambda(\cdot)\, \tilde{D}^l_{\cdot} \int_I h(t, \cdot)\, \phi^{(i)}(t)\, dt$$

is a continuous linear mapping of $\tilde{D}^j_I$ into $D^{m+[2]}_L([A; B])$. (The "dots" in the right-hand side of (5.2) indicate the places where the independent variable for that function in $D^{m+[2]}_L([A; B])$ isto be inserted.) It now follows from lemma 3.3 and the definition (5.1) that $f \circ v$ is a continuous linear mapping of D^j_I into B whatever be the choice of $v \in [D^m_{R^s}; A]$.

Thus, $f\circ$ maps $[D^m_{R^s}; A]$ into $[D^j_I; B]$. Moreover, it follows easily from the linearity of the mapping $v \mapsto \hat{v}$ that $f\circ$ is linear on $[D^m_{R^s}; A]$. To show its sequential continuity, let Φ be a bounded set in D^j_I. In view of our assertion concerning (5.2), as ϕ traverses Φ, the right-hand side of (5.2) traverses a bounded set, say, Θ in $D^{m+[2]}_L([A; B])$. Thus, Θ is also a bounded set in $D^{m+[2]}_{R^s}([A; B])$. Moreover, we may write

$$(5.3) \qquad \sup_{\phi \in \Phi} \| \langle f \circ v, \phi \rangle \|_B = \sup_{\theta \in \Theta} \| \langle \hat{v}, \theta \rangle \|_B .$$

Now, let V denote the image in $[D^{m+[2]}_{R^s}([A; B]); B]$ of $[D^m_{R^s}; A]$ under the mapping $v \mapsto \hat{v}$. V is a linear subspace, which we equip with the induced topology. By virtue of (5.3), the linear mapping $\hat{v} \mapsto f \underset{\sim}{\circledast} v$ is continuous from V into $[D^j_I; B]$. Thus, by the sequential continuity of $v \mapsto \hat{v}$, we conclude that the composite mapping $v \mapsto \hat{v} \mapsto f \underset{\sim}{\circledast} v$ is a sequentially continuous linear mapping of $[D^m_{R^s}; A]$ into $[D^j_I; B]$. A slight modification of this argument shows that the same conclusion holds with respect to the spaces $[D^m_{R^s}; A]^\sigma$ and $[D^j_I; B]^\sigma$.

We now argue that, for the given $f \in [D_{R^{n+s}}(A); B]$, for any given $\phi \in D^j_{R^n}$, and for all $v \in [D^m_{R^s}; A]$, the definition of $\langle f \underset{\sim}{\circledast} v, \phi \rangle$ does not depend on the choices of the parameters I, L, λ, h, i, and l so long as the conditions of definition 5.1 are satisfied. We first observe that, if $v \in E_{R^s}(A)$ and $\phi \in D^j_{R^n}$, then, by lemma 3.6, some integrations by parts, and the use of a Fubini-type relation for our Riemann integrals, we may write

$$\begin{aligned}
\langle f \underset{\sim}{\circledast} v, \phi \rangle &\triangleq \left\langle \hat{v}(x),\ \lambda(x)\, \tilde{D}^l_x \int_I h(t, x)\, \phi^{(i)}(t)\, dt \right\rangle \\
&= \int_L v'(x)\, \lambda(x)\, \tilde{D}^l_x \int_I h(t, x)\, \phi^{(i)}(t)\, dt\, dx \\
&= \int_L \left[\tilde{D}^l_x \int_I h(t, x)\, \phi^{(i)}(t)\, dt \right] v(x)\, \lambda(x)\, dx \\
&= \int_L \int_I h(t, x)\, \phi^{(i)}(t)\, dt\, D^l_x [v(x)\, \lambda(x)]\, dx \\
&= \int_I \int_L h(t, x)\, \phi^{(i)}(t)\, D^l_x [v(x)\, \lambda(x)]\, dt\, dx \\
&= \left\langle f(t, x),\ \phi(t)\, v(x)\, \lambda(x) \right\rangle \\
&= \left\langle f(t, x),\ \phi(t)\, v(x) \right\rangle .
\end{aligned}$$

The last equality is justified by assumption 5.1 and the fact that $\langle f, \theta \rangle$ depends only on the values that θ assumes on an arbitrarily small neighborhood of supp f. Since the right-hand side does not depend upon the aforementioned parameters, our assertion is established at least for all $v \in E_{R^s}(A)$. It also shows that the definition 5.1 is consistent with (2.5).

Next, for f and ϕ as above, make two different choices of the parameters. Also, fix upon an arbitrarily chosen $v \in [D^m_{R^s}; A]$. Let $\langle f \circledast v, \phi \rangle_1$ and $\langle f \circledast v, \phi \rangle_2$ denote the two resulting definitions of $\langle f \circledast v, \phi \rangle$. Now, choose a sequence $\{v_\nu\}_{\nu=1}^{\infty}$ such that $v_\nu \in E_{R^s}(A)$ and $v_\nu \to v$ in $[D^m_{R^s}; A]^\sigma$ (lemma 3.2). In view of the preceding paragraph and the sequential continuity of $v \mapsto f \circledast v$ on $[D^m_{R^s}; A]^\sigma$, we have that, as $\nu \to \infty$,

$$\langle f \circledast v, \phi \rangle_1 \leftarrow \langle f \circledast v_\nu, \phi \rangle_1 = \langle f \circledast v_\nu, \phi \rangle_2 \rightarrow \langle f \circledast v, \phi \rangle_2 .$$

So truly, the definition of $\langle f \circledast v, \phi \rangle$ is independent of the (appropriate) choices of the parameters I, L, λ, h, i, and l.

The last result implies that $f \circledast v$ is uniquely defined as a member of $[D^j_{R^n}; B]$. It has already been noted that the operator $f\circledast$ is linear on $[D^m_{R^s}; A]$. We now invoke the fact that a set bounded in $D^j_{R^n}$ is contained and bounded in D^j_I for some I. In view of the definition of the bounded topology of $[D^j_{R^n}; B]$ and the sequential continuity of $f\circledast$ from $[D^m_{R^s}; A]$ into $[D^j_I; B]$ for every I, we can conclude that $f\circledast$ is a sequentially continuous linear mapping of $[D^m_{R^s}; A]$ into $[D^j_{R^n}; B]$. The same conclusion holds for $[D^m_{R^s}; A]^\sigma$ and $[D^j_{R^n}; B]^\sigma$. This completes the proof.

6. Composition Operators on $[D^m_-; A]$ into $[D^j_-; A]$. Throughout this section it is understood that n = s = 1 so that $t \in R^1$,

$x \in R^1$, and j and m are either nonnegative integers in R^1 or ∞. Our objective is to extend the composition operator $f\circ$ in an explicit way onto the space $[D_-^m; A]$. In the following, $(-\infty, T]$ and $[X, \infty)$ represent semiinfinite closed intervals in R^1 with right-hand and respectively left-hand endpoints $T \in R^1$ and $X \in R^1$ Similarly, (X, ∞) is a semiinfinite open interval.

Assumption 6.1: $f \in [D_{R^2}(A); B]$ satisfies assumption 4.1 and in addition the following condition on its support. For every $T \in R^1$ and $X \in R^1$,

$$(6.1) \qquad \Xi \triangleq \{(-\infty, T] \times [X, \infty)\} \cap \text{supp } f$$

is a bounded and therefore compact set in R^2.

Definition 6.1: Let f satisfy assumption 6.1. Given a $\phi \in D_-^j$ and a $v \in [D_-^m; A]$, choose $T \in R^1$ and $X \in R^1$ such that supp $\phi \subset (-\infty, T]$ and supp $v \subset (X, \infty)$. Define Ξ by (6.1) and select two compact intervals I and L in R^1 such that $I \times L$ contains a neighborhood of Ξ. Finally, choose $\zeta \in D_I$ and $\lambda \in D_L$ such that $\zeta(t)\lambda(x) = 1$ on a neighborhood of Ξ. Define $\langle f\circ v, \phi\rangle$ by

$$(6.2) \qquad \langle f\circ v, \phi\rangle \triangleq \left\langle \hat{v}(x),\ \lambda(x)\, \tilde{D}_x^l \int_I h(t, x)\, D_t^i[\zeta(t)\, \phi(t)]\, dt \right\rangle$$

where h, i, and l are chosen to satisfy assumption 4.1 and $\hat{v}$ is the member of $[D_-^{m+[2]}([A; B]); B]$ corresponding to v in accordance with lemma 3.5. Here again, it is understood that the testing function in the right-hand side of (6.2) has been extended outside the interval L as the zero function.

Theorem 6.1: Under assumption 6.1 and definition 6.1, $\langle f\circ v, \phi\rangle$ is independent of the choices of the parameters T, X, I, L, ζ, λ, h, i, and l and is consistent with (2.5). Moreover, the operator $f\circ: v \mapsto f\circ v$ is a sequentially continuous

<u>linear mapping of $[D_-^m; A]$ into $[D_-^j; B]$, as well as of $[D_-^m; A]^\sigma$ into $[D_-^j; B]^\sigma$.</u>

<u>Proof</u>: With respect to the given f, v, and ϕ, make an appropriate choice of the aforementioned nine parameters. By lemma 3.7 and assumption 4.1,

(6.3) $$\phi \mapsto \lambda(\cdot)\, \tilde{D}_\circ^1 \int_I h(t, \cdot)\, D_t^i [\zeta(t)\, \phi(t)]\, dt$$

is a continuous linear mapping of $D_{-,T}^j$ into $D_L^{m+[2]}([A; B])$. Consequently, by lemma 3.5 and definition 6.1, $f \bullet v \in [D_{-,T}^j; B]$. In other words, $f\bullet$ maps $[D_-^m; A]_\tau$ into $[D_{-,T}^j; B]$ for every $\tau \in R^1$ with $\tau > X$. That $f\bullet$ is linear on $[D_-^m; A]_\tau$ follows easily from lemma 3.5. To establish the sequential continuity of $f\bullet$, let Φ be a bounded set in $D_{-,T}^j$. Then, as ϕ traverses Φ, the right-hand side of (6.3) traverses a bounded set, say, Θ in $D_L^{m+[2]}([A; B])$. Thus, upon invoking (6.2), we may write

(6.4) $$\sup_{\phi\in\Phi} \|\langle f \bullet v,\ \phi\rangle\|_B = \sup_{\theta\in\Theta} \|\langle \hat{v},\ \theta\rangle\|_B .$$

Now, let V be the image in $[D_-^{m+[2]}([A; B]); B]_\tau$ of $[D_-^m; A]_\tau$ under the mapping $v \mapsto \hat{v}$. V is a linear subspace, which we equip with the induced topology. The mapping $\hat{v} \mapsto f \bullet v$ is easily seen to be linear from V into $[D_{-,T}^j; B]$, and its continuity follows from (6.4). It now follows from the linearity and sequential continuity of $v \mapsto \hat{v}$ that the composite mapping $v \mapsto \hat{v} \mapsto f \bullet v$ is a sequentially continuous linear mapping of $[D_-^m; A]_\tau$ into $[D_{-,T}^j; B]$ for every $\tau > X$. A somewhat similar argument leads to the same conclusion in regard to the spaces $[D_-^m; A]_\tau^\sigma$ and $[D_{-,T}^j; B]^\sigma$.

We now show that the definition (6.2) does not depend on the (appropriate) choices of the nine parameters. In the special case where $v \in D_+(A)$, we may invoke lemma 3.6 to rewrite (6.2) as

$$\langle f \circledast v, \phi\rangle = \int_L v'(x)\, \lambda(x)\, \tilde{D}_x^{l} \int_I h(t, x)\, D_t^{i}[\zeta(t)\, \phi(t)]\, dt\, dx$$

$$= \int_L \left\{\tilde{D}_x^{l} \int_I h(t, x)\, D_t^{i}[\zeta(t)\, \phi(t)]\, dt\right\} \lambda(x)\, v(x)\, dx.$$

We now integrate by parts $|l|$ times and then convert the repeated integral intoa double integral.

$$\langle f \circledast v, \phi\rangle = \int_I \int_L h(t, x)\, D_t^{i}[\zeta(t)\, \phi(t)]\, D_x^{l}[\lambda(x)\, v(x)]\, dt\, dx.$$

In view of (4.2), we finally get

$$(6.5) \qquad \langle f \circledast v, \phi\rangle = \langle f(t, x),\ \zeta(t)\phi(t)\lambda(x)v(x)\rangle$$

$$= \langle f(t, x),\ \phi(t)v(x)\rangle .$$

The second equality in (6.5) is justified by assumption 6.1 and the fact that $\langle f, \theta\rangle$ depends only on the values that θ assumes on an arbitrarily small neighborhood of supp f. Since the right-hand side of (6.5) does not depend on the aforementioned parameters, our assertion is certainly true when $v \in D_+(A)$. This also shows that definition 6.1 is consistent with (2.5).

Next, assume that $v \in [D_-^m; A]$. Make two different choices of the set of nine parameters and denote the two resulting definitions by $\langle f \circledast v, \phi\rangle_1$ and $\langle f \circledast v, \phi\rangle_2$. Choose $\tau \in R^1$ and $\xi \in R^1$ such that $\tau < \xi$, τ is greater than both choices of X, and supp $v \subset (\xi, \infty)$. By lemma 3.2, there exists a sequence $\{v_\nu\}_{\nu=1}^{\infty}$ with $v_\nu \in D_{+,\tau}(A)$ which converges to v in $[D_-^m; A]_\tau^\sigma$. By virtue of the preceding paragraph and the sequential continuity of $v \mapsto f \circledast v$ on $[D_-^m; A]_\tau^\sigma$, we have that, as $\nu \to \infty$,

$$\langle f \circ v, \phi \rangle_1 \leftarrow \langle f \circ v_\nu, \phi \rangle_1 = \langle f \circ v_\nu, \phi \rangle_2 \rightarrow \langle f \circ v, \phi \rangle_2 .$$

So trùly, definition 6.1 does not depend upon the choices of the parameters so long as these parameters are chosen to satisfy the conditions of assumption 4.1 and definition 6.1.

The last result implies that $f \circ v$ is uniquely defined on all of D^j_-. Since D^j_- is the strict inductive limit of the sequence of spaces $D^j_{-,T}$, $T = 1, 2, \ldots$, and since $f \circ v$ is linear and continuous on each $D^j_{-,T}$, it follows that $f \circ v \in [D^j_-; B]$.

The penultimate paragraph also implies that $f\circ$ is uniquely defined on all of $[D^m_-; A]$. The linearity of $f\circ$ follows easily from lemma 3.5. In regard to its sequential coninutiy, assume that the sequence $\{v_\nu\}_{\nu=1}^{\infty}$ tends to zero in $[D^m_-; A]$.

Hence, $\{v_\nu\}$ tends to zero in $[D^m_-; A]_\tau$ for some τ. Therefore, by what has already been shown, $f \circ v_\nu \rightarrow 0$ in $[D^j_{-,T}; B]$ whatever be the choice of T. But, in view of the definition of the bounded topology of $[D^j_-; B]$, we can conclude that $f \circ v_\nu \rightarrow 0$ in $[D^j_-; B]$ because each bounded set in D^j_- is contained and bounded in $D^j_{-,T}$ for some T. This argument also leads to the same conclusion for $[D^m_-; A]^\sigma$ and $[D^j_-; B]^\sigma$. The proof of theorem 6.1 is therefore complete.

7. Some Other Composition Operators. In this section we state some results concerning a few other types of composition operators. Their proofs, being so similar to the arguments already presented, are omitted.

For the next two theorems we allow n and s to be greater than or equal to 1. The notation $t \in R^n$, $x \in R^s$, and $(t, x) \in R^{n+s}$

is again used. Theorem 7.1 is concerned with a class of composition operators that map $[E^m_{R^s}; A]$ into $[E^j_{R^n}; B]$.

Assumption 7.1: $f \in [D_{R^{n+s}}(A); B]$ satisfies assumption 4.1 and in addition the following condition on its support. For every compact interval $L \subset R^s$, the set

$$\Xi_L \triangleq (R^n \times L) \cap \text{supp } f$$

is bounded (and therefore compact) in R^{n+s}.

In the following definition, $P_t \Xi_L$ denotes the projection of Ξ_L onto the t-space.

Definition 7.1: Let f satisfy assumption 7.1. Given a $v \in [E^m_{R^s}; A]$, choose a compact interval $L \subset R^s$ such that supp $v \subset \mathring{L}$. Then, choose a compact interval $I \subset R^n$ such that $P_t \Xi_L \subset \mathring{I}$. Finally, choose a $\zeta \in D_I$ such that $\zeta \equiv 1$ on a neighborhood of $P_t \Xi_L$. Then, for any $\phi \in E^j_{R^n}$, define $\langle f \circ v, \phi \rangle$ by

$$\langle f \circ v, \phi \rangle = \left\langle \hat{v}(x), \tilde{D}^l_x \int_I h(t, x)\, D^i_t[\zeta(t)\, \phi(t)]\, dt \right\rangle$$

where h, i, and l are chosen to satisfy assumption 4.1 and $\hat{v}$ is the member of $[E^{m+[2]}_{R^s}([A; B]); B]$ corresponding to v in accordance with lemma 3.4.

Theorem 7.1: Under assumption 7.1 and definition 7.1, $\langle f \circ v, \phi \rangle$ is independent of the choices of the parameters I, L, ζ, h, i, and l and is consistent with (2.5). Moreover, the operator $f\circ ; v \mapsto f \circ v$ is a sequentially continuous linear mapping of $[E^m_{R^s}; A]$ into $[E^j_{R^n}; B]$, as well as of $[E^m_{R^s}; A]^\sigma$ into $[E^j_{R^n}; B]^\sigma$.

Next, we impose the strongest assumption on f made in this paper and obtain the strongest conclusion concerning $f\underset{\sim}{\circ}$.

Assumption 7.2: $f \in [D_{R^{n+s}}(A); B]$ satisfies assumption 4.1 and has a bounded support.

Definition 7.2: Let f satisfy assumption 7.2. Choose two compact intervals $I \subset R^n$ and $L \subset R^s$ such that $I \times L$ contains a neighborhood of supp f. Then choose $\zeta \in D_I$ and $\lambda \in D_L$ such that $\zeta(t)\lambda(x) = 1$ on a neighborhood of supp f. For any $\phi \in E^j_{R^n}$ and $v \in [D^m_{R^s}; A]$, define $\langle f\underset{\sim}{\circ} v, \phi\rangle$ by

$$(7.1)\qquad \langle f\underset{\sim}{\circ} v, \phi\rangle = \left\langle \hat{v}(x),\ \lambda(x)\, \tilde{D}^l_x \int_I h(t, x)\, D^i_t[\zeta(t)\, \phi(t)]\, dt \right\rangle$$

where h, i, and l are chosen to satisfy assumption 4.1 and $\hat{v}$ is the member of $[D^{m+[2]}_{R^s}([A; B]); B]$ corresponding to v in accordance with lemma 3.3. As before, the testing function in the right-hand side of (7.1) is extended outside L as the zero function.

Theorem 7.2: Under assumption 7.2 and definition 7.2, $\langle f\underset{\sim}{\circ} v, \phi\rangle$ is independent of the parameters I, L, ζ, λ, h, i, and l and is consistent with (2.5). Moreover, $f\underset{\sim}{\circ}$ is a sequentially continuous linear mapping of $[D^m_{R^s}; A]$ into $[E^j_{R^n}; B]$, as well as of $[D^m_{R^s}; A]^\sigma$ into $[E^j_{R^n}; B]^\sigma$.

Finally, we turn to the results of Sec. 6 and indicate what they become when the supports of v and $f\underset{\sim}{\circ} v$ are bounded either on the right or on the left. Actually, four cases arise when we also allow the situation discussed in Sec. 6. We shall state our results in terms of the quantities Z^j, Y^m, K_T, and H_X indicated in Table 1, thereby dealing with the four cases simultaneously. The first row of that table corresponds to the

TABLE 1

Row	Z^j	Y^m	K_T	H_X
1	D^j_-	D^m_-	$(-\infty, T]$	$[X, \infty)$
2	D^j_+	D^m_-	$[T, \infty)$	$[X, \infty)$
3	D^j_+	D^m_+	$[T, \infty)$	$(-\infty, X]$
4	D^j_-	D^m_+	$(-\infty, T]$	$(-\infty, X]$

situation of Sec. 6.

Once again we require that $n = s = 1$ and that j and m be either nonnegative integers in R^1 or ∞. The general result can be stated as follows. Theorem 6.1 remains valid when the following changes are made in lemma 3.5, assumption 6.1, definition 6.1, and theorem 6.1. D_-^j is replaced by Z^j, D_-^m by Y^m, $D_-^{m+[2]}$ by $Y^{m+[2]}$, $(-\infty, T]$ by K_T, $[X, \infty)$ by H_X, and (X, ∞) by $\mathring{H}_X$.

8. Examples. In this section we present two illustrations of our results. We now require that $n = s \geq 1$ and denote D_{R^n} by D and E_{R^n} by E.

Example 8.1: Let τ be a fixed member of R^n, p a nonnegative integer in R^n, and $m \in R_e^n$. Consider the operator $c\sigma_\tau D^p$, where $c \in E^{m+p+[4]}([A; B])$. Also, σ_τ is the shifting operator defined on any $v \in D(A)$ by

$$(\sigma_\tau v)(x) \triangleq \sigma_\tau(x)v(x) \triangleq v(x - \tau)$$

and on any $v \in [D; A]$ by

$$\langle \sigma_\tau v, \phi \rangle = \langle v, \sigma_{-\tau}\phi \rangle \qquad \phi \in D.$$

Thus, $c\sigma_\tau D^p$ denotes the composite operator consisting of first a pth-order differentiation, then a shift through a displacement τ, and finally a multiplication by c. Consequently, $c\sigma_\tau D^p$ is a sequentially continuous linear mapping of $D(A)$ into $[D; B]$. It was indicated in [1; examples 4.1 and 4.2] that, in accordance with theorem 2.1, this operator has the representation:

$$c\sigma_\tau D^p v = \underset{\sim}{f} \bullet v \qquad v \in D(A) \tag{8.1}$$

where

$$f(t, x) = c(t)\, \sigma_{-\tau}(x)\, \tilde{D}_x^p\, I(t, x)$$

and $I(t, x) \in [D^0_{R^{2n}}(B); B]$ is defined by

$$\langle I(t, x), \theta(t, x)\rangle \triangleq \int_{R^n} \theta(t, t)\, dt \in B \qquad \theta \in D^0_{R^{2n}}(B).$$

It also possesses the representation (4.2); namely, for $v \in D(A)$ and $\phi \in D^j$, where $j = m + p + [4]$, we have that

$$(8.3) \qquad \langle f \circledast v, \phi\rangle = \int_{R^n} \int_{R^n} h(t, x)\, \phi(t)\, v^{(p+[2])}(x)\, dt\, dx,$$

where

$$(8.4) \qquad h(t, x) = c(t)\, J_0(t - \tau - x) \in E^0_{R^{2n}}([A; B])$$

and $J_0(t)$ is defined by

$$J_0(t) \triangleq t1_+(t) \triangleq t_1 1_+(t_1) \cdots t_n 1_+(t_n),$$

$$1_+(t_\nu) = \begin{cases} 0 & t_\nu < 0 \\ 1/2 & t_\nu = 0 \\ 1 & t_\nu > 0. \end{cases}$$

Indeed, some integrations by parts show that the right-hand side of (8.3) is equal to

$$\int_{R^n} c(t)\, \phi(t)\, v^{(p)}(t - \tau)\, dt = \langle c\sigma_\tau D^p v, \phi\rangle .$$

Note that, in this particular example, the parameters h, $i = 0$, and $l = p + [2]$ are independent of the choices of I and L, and therefore both I and L can be and have been replaced by R^n.

Moreover, for $\phi \in D^j$, where $j = m + p + [4]$,

$$\chi_\phi(x) \triangleq \int_{R^n} h(t, x)\, \phi(t)\, dt = \int_{R^n} c(t+x)\, \phi(t+x)\, J_0(t-\tau)\, dt.$$

Some differentiations under the integral sign on the right-hand side show that $D^k \chi_\phi$ is continuous so long as $0 \leq k \leq m + p + [4]$. This result coupled with (8.3) shows that f satisfies assumption 4.1 when we choose i = 0, l = p + [2], and h as in (8.4). Thus, by theorem 4.1, the operator $f\circ$ can be extended onto $[E^m; A]$ through definition 4.1, and it maps $[E^m; A]$ into $[D^j; B]$ in a sequentially continuous and linear fashion.

We finally, note that supp f is contained in the set $\Lambda \triangleq \{(t, x): x = t - \tau\}$. Indeed, for $\theta \in D_{R^{2n}}(A)$,

$$\langle f(t, x)\, \theta(t, x)\rangle = \langle I(t, x),\, c(t)\, D_x^p\, \theta(t, x-\tau)\rangle$$

$$= \int_{R^n} c(t)\, D_t^p\, \theta(t, t - \tau)\, dt,$$

and the last integral is the zero member of B if supp θ does not meet Λ. This implies that f satisfies assumptions 5.1, 6.1, and 7.1, as well as the two assumptions corresponding to rows 1 and 3 of Table 1. Thus, theorems 5.1, 6.1, and 7.1 as well as the theorems corresponding to rows 1 and 3 of Table 1 also apply to $f\circ$.

Example 8.2: We now develop a representation such as (4.4) for the convolution operator $f\circ = y*$ where $y \in [D(A); B]$. We first note that, since $y*$ is a continuous linear mapping of D(A) into [D; B] (see [11; theorem 4.1]), it must possess the representation (2.5). To obtain it explicitly, we define y(t - x) as a member of $[D_{R^{2n}}(A); B]$ by

$$\langle y(t-x),\, \Psi(t, x)\rangle \triangleq \left\langle y(t),\, \int_{R^n} \Psi(t+x, x)\, dx\right\rangle \qquad \Psi \in D_{R^{2n}}(A).$$

Upon setting $f(t, x) = y(t - x)$, we obtain, for every $\phi \in D$ and $v \in D(A)$,

$$\langle f \circ v, \phi \rangle = \langle y(t-x), \phi(t)v(x) \rangle$$

$$= \left\langle y(t), \int_{R^n} \phi(t+x)v(x)\, dx \right\rangle = \langle y * v, \phi \rangle .$$

This is the representation (2.5) for $y*$.

Now, let us choose the compact intervals I and L in R^n arbitrarily. Then, there exists a compact interval $N \subset R^n$ such that

$$\int_L \phi(\cdot + x)\, v(x)\, dx \in D_N(A)$$

where $\phi \in D_I$ and $v \in D_L(A)$. But then, by virtue of lemma 2.1, there exists a continuous [A; B]-valued function on N and an integer $i \geqq 0$ in R^n for which

$$\left\langle y(t), \int_L \phi(t+x)v(x)\, dx \right\rangle = \int_N h(t)\, D_t^i \int_L \phi(t+x)v(x)\, dx\, dt$$

$$\phi \in D_I, \quad v \in D_L(A).$$

Some differentiations under the integral sign and a change of variables yield

$$(8.5) \quad \langle f \circ v, \phi \rangle = \int_I \int_L h(t-x)\, \phi^{(i)}(t)\, v(x)\, dt\, dx.$$

Moreover, for x restricted to L,

$$\int_I h(t-x)\, \phi^{(i)}(t)\, dt = \int_N h(t)\, \phi^{(i)}(t+x)\, dt$$

is a smooth [A; B]-valued function of x. This fact and (8.5) show that $f(t, x) = y(t-x)$ satisfies assumption 4.1 when we set $j = [\infty]$, $m = [\infty]$, and $l = 0$. Hence, theorem 4.1 holds when we define $f\circ$ on any $v \in [E; A]$ by

$$\langle f \underset{\sim}{*} v, \phi \rangle = \left\langle \hat{v}(x), \int_I h(t-x)\, \phi^{(1)}(t)\, dt \right\rangle .$$

We finally note that, if supp y is bounded, then assumptions 5.1 and 7.1 are fulfilled. Also, for the case n = 1, if supp y is bounded on the left(right), then the appropriate version of assumption 6.1 corresponding to row 1 (respectively, row 3) of Table 1 is also fulfilled.

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