A LOCALLY COUNTABLE, CONVENTIONAL OR TRANSFINITE, TREE HAS UNCOUNTABLY MANY EXTREMITIES IF AND ONLY IF IT CONTAINS A BINARY TREE

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Abstract — A tree T^ν is a connected graph of transfiniteness rank ν having no loops (i.e., circuits), where $\nu = 0$ for a conventional tree and $\nu > 0$ for a transfinite tree. Its extremities are its end nodes of all ranks no larger than ν and also its infinite extremities of rank ν; the latter are designated by one-way infinite paths of rank ν. A section of rank β ($0 \leq \beta < \nu$) in T^ν is a subtree induced by a maximal set of branches that are connected through paths of rank β, and its external tips are essentially the connections it has to the rest of T^ν. A section of rank -1 is simply a branch. T^ν is called locally countable if each section of every rank has only countably many external tips. It is shown that a locally countable tree T^ν has uncountably many extremities if and only if it contains a binary tree, whose rank is any natural number up to ν.

Key Words: Transfinite trees, binary trees, graphical extremities, uncountably many extremities.

1 Introduction

Let an *extremity* of a conventionally infinite, locally countable tree be either an end node or a 0-tip (i.e., the latter being an equivalence class of one-way infinite paths where two such paths are taken to be equivalent if they are identical except for finitely many branches). We show that such a tree has uncountably many extremities if and only if the tree contains a binary tree or a subdivision of a binary tree. We then extend this result to locally countable transfinite trees. As is usual in transfinite graph theory, the needed arguments for the latter task are rather more complicated, and many more definitions are needed. Those definitions that are not presented herein, appear in [5], to which we refer with specific page references.

Transfinite graph theory was initially developed to attack certain problems in electrical network theory and therefore used conventional electrical terminology. We continue to do so in order to conform with all the prior works on transfinite graphs. Thus, we say "branch" instead of "edge" and "node" instead of "vertex."

$\boldsymbol{N}$ will denote the set $\{0, 1, 2, \ldots\}$ of all natural numbers. By a countable set, we will mean either a finite set or a denumerably infinite one. $\aleph_0$ denotes the cardinal number of a denumerably infinite set,[1] and $\aleph$ denotes the cardinal number of the continuum.

Let us indicate how we came upon the problem attacked herein. The number E of end nodes of a finite tree is given by the formula

$$E \;=\; 2 \,+\, \sum (d_x - 2) \tag{1}$$

where d_x is the degree of the non-end node x and the summation is over all the non-end nodes. This formula is easily established by starting with a path and appending branches one at a time in a connected fashion to create a tree. The formula holds at each step.

It also holds for a conventionally infinite, locally countable tree $\mathcal{T}$ having at most countably many 0-tips, where now E is the cardinality of the set of extremities of $\mathcal{T}$. Indeed, the set of nodes in $\mathcal{T}$ is countable [2, page 39]. This fact coupled with the countability of all the 0-tips yields $E = \aleph_0$. On the other hand, with x still denoting a non-end node, the summation in (1) consists of a countable set of terms each of which is either a natural number or $\aleph_0$. Therefore, the right-hand side of (1) also equals $\aleph_0$ [5, pages 380-381].

However, the formula can fail if no restriction is placed on the cardinality of the set of 0-tips for the locally countable tree. The binary tree provides an example. The set of 0-tips for the binary tree has the cardinality $\aleph$ of the continuum. Thus, $E = \aleph$. On the other hand, the summation on the right-hand side of (1) has countably many terms, and each

[1] $\aleph_0$ is the same as the first transfinite ordinal ω, but it is conventional to use $\aleph_0$ when dealing with the size of a denumerable set. Furthermore, the first transfinite ordinal ω has the same elements as the set N but has in addition the natural ordering explicitly imposed.

term is 1 except for a single 0 for the root node of the binary tree. Hence, that right-hand side equals $\aleph_0$.

Question: Is an embedded binary tree or a subdivision of it needed in order for a locally countable, conventionally infinite tree to have uncountably many 0-tips? The answer is "yes" (see Theorem 2.3 below). It follows that, if a locally countable tree has uncountably many 0-tips, it has at least $\aleph$-many of them. "Yes" remains the answer for transfinite trees as well, as is shown in Secs. 3 and 4.

2 Conventionally Infinite Trees

A *0-tree* $\mathcal{T}^0$ is a connected conventional graph having no loops (i.e., circuits). It is *locally countable* if the degree of each of its nodes is either a positive natural number or $\aleph_0$. Its *extremities* are its end nodes and its 0-tips, the latter occurring whenever $\mathcal{T}^0$ contains one-ended paths (i.e., one-way infinite paths). For infinite trees a 0-tip and an "end" [3, pages 62 and 230] are the same thing, but for infinite graphs in general they are different concepts.

Lemma 2.1. *Given any node x and any extremity of $\mathcal{T}^0$, there is a unique path starting at x and reaching that extremity.*

Proof. If the extremity is an end node, this is a standard property of trees. If the extremity is a 0-tip t^0, choose any representative path Q for t^0. If x is in Q, the sought-for path starts at x and follows Q toward t^0. If x is not in Q, choose any node y of Q, and let P be the unique path in $\mathcal{T}^0$ terminating at x and y. Then, the sought-for path starts at x, follows P until the first node of Q is reached, and then follows Q toward t^0. There is no other such path because $\mathcal{T}^0$ contains no loops. □

Now let x and y be any two nodes of $\mathcal{T}^0$, with y being a non-end node. Consider all the paths that start at x, pass through y, and reach extremities of $\mathcal{T}^0$. We say that, with respect to x and with the exception of y, the branches and nodes in the path between x and y are *before* y and all the other branches and nodes of those paths are *after* y. Moreover, any branch or node other than x is said to *lead to* (again with respect to x) all the extremities reached by the paths starting at x and passing through that branch or node.

We will also be dealing with binary 0-trees or subdivisions of them as subgraphs of

0-trees. It turns out to be inconvenient to reduce those subdivisions to binary 0-trees, especially when dealing with transfinite trees later on. For this reason, we shall use a more general definition of a binary 0-tree, one that encompasses those subdivisions. One of the properties imposed is that every node be of degree 2 or 3 with at least one node of degree 2. Moreover, a node of degree 3 will be called a *fork*. Given any node x of degree 2, which we shall call the *root*, the kth *fork from* x ($k \in \boldsymbol{N}, k \neq 0$) is a fork that is reached by a path starting at x and meeting exactly k forks (counting the kth fork as well).

A *binary 0-tree* is a 0-tree having the following three properties:

1. Each of its nodes is of degree 2 or 3.

2. There is at least one node of degree 2.

3. From any node x of degree 2, the number of kth forks from x is exactly 2^k.

In a binary 0-tree, condition 3 will hold for any choice of a node of degree 2 as the root. Moreover, the extremities of a binary 0-tree are all 0-tips; there are no end nodes.

It is also a fact that the cardinality of the set of 0-tips of a binary 0-tree is the cardinality $\aleph$ of the continuum. This can be seen as follows. Choose a root x, and for each fork consider the two incident branches coming after that fork with respect to x; label one of them 0 and label the other 1 . Then, any one-ended path starting at x can be identified by a sequence of 0's and 1's determined by its passage through forks. Moreover, that sequence of 0's and 1's can be identified with a real number in binary form lying in a unit interval. Furthermore, there is a bijection between the set of 0-tips and the set of such paths, and there is another bijection between the set of such paths and the numbers in the said interval, whence our assertion.

Theorem 2.2. *Let $\mathcal{T}^0$ be a locally countable 0-tree. $\mathcal{T}^0$ has uncountably many extremities if and only if $\mathcal{T}^0$ contains a binary 0-tree as a subgraph.*

Note. Since a locally countable 0-tree has only countably many end nodes [2, page 39], "extremities" can be replaced by "0-tips" in this theorem.

Proof. If: Every 0-tip of the binary 0-tree is a 0-tip of $\mathcal{T}^0$. Since a binary 0-tree has $\aleph$-many 0-tips, our conclusion follows.

Only if: Choose any node x_0 of $\mathcal{T}^0$. Suppose all but one of the branches incident to x_0 together lead to at most countably many extremities of $\mathcal{T}^0$. Let b_0 be that exceptional branch. (There may be no other branch incident to x.) Let x_1 be the other node of b_0. Suppose in addition that all but one of the branches incident to x_1 other than b_0 together lead to at most countably many extremities of $\mathcal{T}^0$. Let that exceptional branch be b_1, and let x_2 be the other node of b_1. Again suppose that all but one of the branches incident to x_2 other than b_1 together lead to at most countably many extremities of $\mathcal{T}^0$. Continue in this way to get a path

$$\{x_0, b_0, x_1, b_1, \ldots, x_k, b_k, \ldots\}, \tag{2}$$

and make the same supposition at each step, namely, other than the incident branch b_k just after x_k all the branches incident to and after x_k lead to (with respect to x_0) at most countably many extremities of $\mathcal{T}^0$.

Two possibilities arise: Either the path (2) terminates at an end node of $\mathcal{T}^0$, or it continues indefinitely reaching toward a unique 0-tip of $\mathcal{T}^0$. In either case, we have to conclude that all the extremities mentioned above comprise a countable set. But, every extremity of $\mathcal{T}^0$ will be so mentioned because every extremity of $\mathcal{T}^0$ will be reached by a unique path starting at x_0 (Lemma 2.1). This violates the hypothesis of the "only if" part of Theorem 2.2.

We must conclude that, at some node x_k in (2), all the branches incident to x_k together lead to uncountably many extremities of $\mathcal{T}^0$. Since the degree of x_k is either finite or $\aleph_0$ and since the union of a countable collection of countable sets is countable, at least one of the branches incident to x_k leads to uncountably many extremities of $\mathcal{T}^0$. However, we have eliminated the possibility that all but one of the branches incident to each x_k together lead to at most countably many extremities, whatever be x_k in (2). Therefore, there will be a first node x_k in (2) at which (at least) two branches incident to and after x_k each lead to uncountably many extremities of $\mathcal{T}^0$.

Let those two branches be b_{11} and b_{12}. Also, let $\mathcal{T}_{11}^0$ (resp. $\mathcal{T}_{12}^0$) be the subtree of $\mathcal{T}^0$ induced by all the branches in all the paths starting at x_k and passing through b_{11} (resp. b_{12}). Thus, $\mathcal{T}_{11}^0$ (resp. $\mathcal{T}_{12}^0$) is locally countable with uncountably many extremities.

Therefore, we can apply to it the same argument as that applied to $\mathcal{T}^0$. Thus, in $\mathcal{T}^0_{11}$ (resp. $\mathcal{T}^0_{12}$) we can find a path starting at x_k, passing through b_{11} (resp. b_{12}), and reaching a first node y_{11} (resp. y_{12}) at which there are two branches b_{21} and b_{22} (resp. b_{23} and b_{24}) incident to and after y_{11} (resp. y_{12}), each branch leading to uncountably many extremities of $\mathcal{T}^0$ through two subtrees $\mathcal{T}^0_{21}$ and $\mathcal{T}^0_{22}$ (resp. $\mathcal{T}^0_{23}$ and $\mathcal{T}^0_{24}$). This construction continues indefinitely, and we can thereby find the following structure in $\mathcal{T}^0$:

Let $P_{x,y}$ denote a path terminating at the nodes x and y. Set $y_0 = x_k$. We find two paths $P(y_0, y_{11})$ and $P(y_0, y_{12})$, then four paths $P(y_{11}, y_{21})$, $P(y_{11}, y_{22})$, $P(y_{12}, y_{23})$, $P(y_{12}, y_{24})$, then eight paths, and so forth. With respect to the subtree induced by those paths, each of those paths has terminal nodes of degree 3 except for y_0; y_0 and all internal nodes (if such exist) of those paths are of degree 2. We have indeed found a binary 0-tree as a subtree of $\mathcal{T}^0$. □

Corollary 2.3. *If a locally countable 0-tree has uncountably many extremities, then it has at least* $\aleph$*-many 0-tips.*

3 Transfinite Trees Having Natural-Number Ranks

The idea for extending Theorem 2.2 transfinitely is to let "sections" play at times the role that branches did previously and at other times the role that 0-nodes played. For this purpose, we need to explicate some preliminary concepts. Henceforth, α, β, γ, and μ will denote natural numbers with $\alpha, \beta, \gamma, \leq \mu$ and $\mu > 0$. They will represent ranks of graphical transfiniteness, with the rank 0 being for conventional graphs. A *transfinite* μ*-tree* $\mathcal{T}^\mu$ is a connected μ-graph having no loops. See [5, page 48] for the definition of "transfinite connectedness," [5, page 31] for a "μ-graph," and [5, page 35] for a "loop." See also [5, page 30] for the definitions of a "transfinite tip" and a "transfinite node," and see [5, page 31] for an "embraced tip." Unless the opposite is stated, every node we mention herein is understood to be "maximal" [5, page 32], that is, it is not contained in a node of higher rank.

It should be noted here that "nondisconnectable tips" [5, page 58] do not exist in $\mathcal{T}^\mu$ because the presence of such tips implies the presence of loops. As a result, Condition 3.5-1 of [5, page 71] is trivially satisfied, and all the results of [5, Section 3.5] hold for $\mathcal{T}^{\mu}$.

The *degree* of a transfinite node is the cardinality of the set of its embraced tips. An *end node* is a node of degree 1, and a *fork* is a node of degree 3. The extremities of a μ-tree are its end nodes and its μ-tips. Unlike the tips of lower ranks in $\mathcal{T}^{\mu}$, which are all embraced in nodes, the μ-tips of $\mathcal{T}^{\mu}$ are not so embraced because nodes of ranks higher than μ do not exist in $\mathcal{T}^{\mu}$ by definition.

Lemma 3.1. *Given any node x and any extremity of $\mathcal{T}^{\mu}$, there is a unique path in $\mathcal{T}^{\mu}$ starting at x and reaching the extremity.*

The proof of this lemma is the same as that of Lemma 2.1, but now we need to invoke [5, Corollary 3.5-5] to assert the existence of a first node at which the path P meets the path Q.

With $\beta < \mu$, a *β-section* $\mathcal{S}^{\beta}$ of $\mathcal{T}^{\mu}$ is a subtree (i.e., a subgraph[2]) induced by a maximal set of branches that are pairwise β-connected [5, page 49]. The β-sections partition $\mathcal{T}^{\mu}$ in the sense that each branch of $\mathcal{T}^{\mu}$ resides in one and only one β-section [5, Corollary 3.5-6]. An α-tip t^{α} ($\alpha \leq \beta$) is said to be *in* a β-section $\mathcal{S}^{\beta}$ and $\mathcal{S}^{\beta}$ is said to *have* t^{α} if the branches of any representative path of t^{α} are in $\mathcal{S}^{\beta}$; t^{α} is then said to be an *internal tip* of $\mathcal{S}^{\beta}$. A *boundary node* x^{γ} ($\gamma > \beta$) of $\mathcal{S}^{\beta}$ is a node that embraces a tip in $\mathcal{S}^{\beta}$ and a tip not in $\mathcal{S}^{\beta}$. A tip embraced by x^{γ} that is not in $\mathcal{S}^{\beta}$ is called an *external tip* of $\mathcal{S}^{\beta}$. Also, x^{γ} and $\mathcal{S}^{\beta}$ are said to be *incident* to each other.

Let x^{γ} be a boundary node of the β-section $\mathcal{S}^{\beta}$ (thus, $\beta < \gamma$) in $\mathcal{T}^{\mu}$. Remember that x^{γ} is a maximal node. All the tips embraced by x^{γ} will reside in $(\gamma-1)$-sections incident to x^{γ}, and, if $\beta < \gamma - 1$, a β-section containing a tip of x^{γ} will reside within such a $(\gamma-1)$-section.

Lemma 3.2. *Let x^{γ} be a (maximal) node in the μ-tree $\mathcal{T}^{\mu}$, and consider all the $(\gamma-1)$-sections incident to x^{γ}. Except possibly for one such $(\gamma-1)$-section, each such $(\gamma-1)$-section will contain as an internal tip exactly one tip embraced by x^{γ}, and that tip will be of rank $\gamma - 1$. The exceptional $(\gamma-1)$-section $\mathcal{S}_e^{\gamma-1}$, if it exists, may contain one or many tips embraced by x^{γ}, but then those tips will all be of ranks less than $\gamma - 1$ and will reside in a*

[2] In general, a subgraph of a transfinite graph is not a transfinite graph by itself, but it can be reduced to a transfinite graph by removing all the tips embraced by all the nodes of the subgraph whose representative paths do not lie in the subgraph. This yields the "reduced graph" of [4, page 143]. This reduction will be understood when referring to a subtree.

β-section within $\mathcal{S}_e^{\gamma-1}$, where β is the largest of the ranks of those tips.

Proof. Suppose a $(\gamma-1)$-section $\mathcal{S}^{\gamma-1}$ contains two or more $(\gamma-1)$-tips embraced by x^γ, then a branch of a representative path of one such $(\gamma-1)$-tip and a branch of a representative path of another such $(\gamma-1)$-tip will be $(\gamma-1)$-connected through some path P of rank less than γ lying within $\mathcal{S}^{\gamma-1}$. But, there will also be a γ-path connecting those branches and passing through x^γ by means of the said $(\gamma-1)$-tips. The union of those two paths will contain a loop [5, Corollary 3.5-5], in violation of the fact that $\mathcal{T}^\mu$ is a tree.

On the other hand, x^γ may embrace one or more tips of ranks less than $\gamma-1$, and, if there are two or more, those tips will be connected through paths that pass through x^γ by means of an embraced node of x^γ of rank less than γ. Thus, those tips will all lie in a single $(\gamma-1)$-section $\mathcal{S}_e^{\gamma-1}$. However, $\mathcal{S}_e^{\gamma-1}$ cannot contain a $(\gamma-1)$-tip, for that would again imply the presence of a loop in $\mathcal{T}^\mu$. □

A μ-tree will be called *locally countable* if every 0-node is of countable degree and if, for every β $(0 \leq \beta < \mu)$, every β-section has only countably many external tips. As a special case, we can view each branch as being a section of rank -1; then, the countable degree of each 0-node will follow from the next lemma for the case where $\gamma = 0$.

Lemma 3.3. *Assume $\mathcal{T}^\mu$ is a locally countable μ-tree. Then, every node x^γ of $\mathcal{T}^\mu$ has a countable degree.*

Proof. If x^γ is an end node, it contains exactly one tip. If x^γ is not an end node, it embraces two or more tips, at least one of which is of rank $\gamma-1$. Thus, by Lemma 3.2, it has at least two incident $(\gamma-1)$-sections. By local countability, every one of the $(\gamma-1)$-sections incident to x^γ will have at most countably many external tips, and every tip embraced by x^γ will be an external tip of at least one of those $(\gamma-1)$-sections. Furthermore, by Lemma 3.2, there can be only countably many $(\gamma-1)$-sections incident to x^γ, for otherwise any one of those $(\gamma-1)$-sections would have uncountably many external tips. Thus, the tips embraced by x^γ lie in the union of the countable sets of external tips of countably many $(\gamma-1)$-sections. This yields our lemma. □

Given any any two nodes x and y of $\mathcal{T}^\mu$, consider all the paths starting at x, passing through y, and reaching extremities of $\mathcal{T}^\mu$. We refer to branches and nodes in those paths as occurring *before* or *after* y exactly as in the preceding section, and we say that, with respect to x, y *leads to* all the branches, nodes, and extremities that occur after y. Similarly, any tip t of y whose representative-path branches occur after y is said to *lead away* from x and (again with respect to x) to *lead to* all branches, nodes, and extremities after y that are reached by paths that start at x and pass through t. Furthermore, given any β-section $\mathcal{S}^\beta$ of $\mathcal{T}^\mu$ with an external tip t, we can choose any node x of $\mathcal{S}^\beta$ to state in the same way what entities t *leads to* with respect to $\mathcal{S}^\beta$; this assertion will not depend upon the choice of x in $\mathcal{S}^\beta$ (Lemma 3.1). We also say that t *leads away from* $\mathcal{S}^\beta$.

More generally, given any external tip t_1^δ (resp. t_2^ζ) of $\mathcal{S}^\beta$, let $\mathcal{T}_1$ (resp. $\mathcal{T}_2$) be the subtree induced by all the branches to which t_1^δ (resp. t_2^ζ) leads with respect to $\mathcal{S}^\beta$. Then, $\mathcal{T}_1$ and $\mathcal{T}_2$ do not meet or share extremities except possibly when t_1^δ and t_2^ζ are embraced by the same boundary node x^γ of $\mathcal{S}^\beta$ (otherwise, $\mathcal{T}^\mu$ would contain a loop). In the latter case, $\mathcal{T}_1$ and $\mathcal{T}_2$ meet only at x^γ. Because any end node outside of $\mathcal{S}^\beta$ or any μ-tip of $\mathcal{T}^\mu$ is an extremity of exactly one subtree such as $\mathcal{T}_1$ for the given $\mathcal{S}^\beta$, we have part (a) of the following lemma. Part (b) is an immediate consequence of the fact that any path starting through an external tip of $\mathcal{S}^\beta$ and extending away from $\mathcal{S}^\beta$ as far as possible will either terminate at an end node or will become a representative one-ended path of a μ-tip.

Lemma 3.4. *Let $\mathcal{S}^\beta$ be any β-section in $\mathcal{T}^\mu$, where $0 \leq \beta < \mu$.*

(a) *Given any end node outside $\mathcal{S}^\beta$ or any μ-tip of $\mathcal{T}^\mu$, there is a unique external tip of $\mathcal{S}^\beta$ that leads to that end node or μ-tip.*

(b) *Each external tip of $\mathcal{S}^\beta$ leads to at least one extremity of $\mathcal{T}^\mu$.*

We also need the definition of a binary β-tree ($0 \leq \beta \leq \mu$). It is much like that of a binary 0-tree, but let us be specific. Consider now a β-tree $\mathcal{T}^\beta$ in which every node is of degree 2 or 3 and there exists at least one node x of degree 2.[3] Recall that a *fork* is a node of degree 3. For each positive natural number k, the kth *fork from* x is a fork that is reached by a path starting at x and meeting k forks (counting the kth fork as well). Here too, x is called the *root node* or simply the *root.*

[3]Later on, $\mathcal{T}^\beta$ will be a subtree of $\mathcal{T}^\mu$ whose rank is β, where $\beta \leq \mu$. In that case, when $\beta < \mu$, the nodes of ranks greater than β in $\mathcal{T}^\mu$ that embrace β-tips of $\mathcal{T}^\beta$ will not be considered to be part of $\mathcal{T}^\beta$ simply because, $\mathcal{T}^\beta$ being of rank β, has no such nodes.

A *binary β-tree* $\mathcal{T}^\beta$ is a β-tree satisfying the following three conditions:

1. Each of its nodes is of degree 2 or 3.

2. There exists at least one node of degree 2.

3. From any node of degree 2, the number of kth forks is 2^k.

Here too, in a binary tree Condition 3 will hold for any choice of a node of degree 2 as the root. Also, every extremity of a binary β-tree is a β-tip, there being no end nodes. Moreover, the cardinality of the set of β-tips is equal to the cardinality $\aleph$ of the continuum. Furthermore, any tracing of a path that is extended as far as possible (i.e., does not terminate at any node) will trace a representative of a β-tip.

Next, we restrict our attention to the $(\mu-1)$-sections in $\mathcal{T}^\mu$ to get some more needed results. The $(\mu-1)$-sections of $\mathcal{T}^\mu$ partition $\mathcal{T}^\mu$ [5, Corollary 3.5-6], and the boundary nodes between $(\mu-1)$-sections are all of rank μ. Two $(\mu-1)$-sections that share a boundary node share only that boundary node, and they are said to be *adjacent.*

Lemma 3.5. *A locally countable μ-tree has only countably many $(\mu-1)$-sections, and each $(\mu-1)$-section has only countably many adjacent $(\mu-1)$-sections.*

Proof. Let $\mathcal{S}^{\mu-1}$ be a $(\mu-1)$-section in $\mathcal{T}^\mu$. By Lemma 3.2, each of the external $(\mu-1)$-tips of $\mathcal{S}^{\mu-1}$ lies in an adjacent $(\mu-1)$-section, which in turn does not contain any other external or internal tips of $\mathcal{S}^{\mu-1}$ (again because a loop would otherwise exist). On the other hand, if an external tip t^γ of $\mathcal{S}^{\mu-1}$ is of rank γ less than $\mu-1$, then all the external tips of $\mathcal{S}^{\mu-1}$ of ranks less than $\mu-1$ embraced by the boundary node that embraces t^γ will lie in the same adjacent $(\mu-1)$-section, which again will not contain any external $(\mu-1)$-tips of $\mathcal{S}^{\mu-1}$. In particular, the number (i.e., the cardinality of the set) of adjacent $(\mu-1)$-sections for $\mathcal{S}^{\mu-1}$ cannot be greater than the number of external tips of $\mathcal{S}^{\mu-1}$. By the local countability of $\mathcal{T}^\mu$, we can conclude that $\mathcal{S}^{\mu-1}$ has only countably many adjacent $(\mu-1)$-sections. This is the second conclusion of our lemma.

Next, note that, given two nonadjacent $(\mu-1)$-sections, there is a two-ended μ-path P terminating at a node in each of those $(\mu-1)$-sections because of the μ-connectedness of $\mathcal{T}^\mu$. That path will pass through at least one and at most finitely many $(\mu-1)$-sections. Now, choose any $(\mu-1)$-section $\mathcal{S}_0^{\mu-1}$. Let C_0 be the countable set of $(\mu-1)$-sections adjacent to $\mathcal{S}_0^{\mu-1}$. Recursively, let C_k $(k \in \boldsymbol{N}, k \neq 0)$ be the countable set of $(\mu-1)$-section adjacent to the $(\mu-1)$-sections in C_{k-1}. Thus, $\cup_{k\in \boldsymbol{N}} C_k$ is the set of all $(\mu-1)$-sections in $\mathcal{T}^\mu$, and the first conclusion of our lemma follows. □

Theorem 3.6. *Let $\mathcal{T}^\mu$ be a locally countable μ-tree. $\mathcal{T}^\mu$ has uncountably many extremities if and only if $\mathcal{T}^\mu$ contains a binary β-tree for some β with $0 \leq \beta \leq \mu$.*

Proof. If: By Theorem 2.2, the "if" statement is true when $\mu = 0$. Now, consider a locally countable μ-tree $\mathcal{T}^\mu$ containing a β-tree $(0 \leq \beta \leq \mu)$. When $\beta = \mu$, the uncountably many μ-tips of the binary μ-tree are also extremities of $\mathcal{T}^\mu$ as well.

So, now assume that $\mathcal{T}^\mu$ contains a binary β-tree $\mathcal{T}_b^\beta$ for some β with $0 \leq \beta < \mu$. Each of the uncountably many β-tips of $\mathcal{T}_b^\beta$ either will be the sole member of an end $(\beta+1)$-node of $\mathcal{T}^\mu$ or will be embraced by a nonsingleton γ-node $(\beta+1 \leq \gamma \leq \mu)$ which will lead (with respect to any node of $\mathcal{T}_b^\beta$) to at least one extremity of $\mathcal{T}^\mu$. Moreover, that γ-node will embrace just one of those β-tips. So again, $\mathcal{T}^\mu$ has uncountably many extremities.

Only if: We now use strong induction on ranks, the first step $(\mu = 0)$ having been established by Theorem 2.2. Assume our present theorem is true for all ranks up to $\mu - 1$. and consider a locally countable μ-tree $\mathcal{T}^\mu$ having uncountably many extremities. Two possibilities arise:

1. There is a β-section $\mathcal{S}^\beta$ in $\mathcal{T}^\mu$ $(0 \leq \beta < \mu)$ having uncountably many extremities as a locally countable subtree of $\mathcal{T}^\mu$ with its β-tips serving as extremities: Then, by our inductive assumption applied to $\mathcal{S}^\beta$, there is a binary α-tree $\mathcal{T}_b^\alpha$ in $\mathcal{S}^\beta$ $(0 \leq \alpha \leq \beta)$ and therefore in $\mathcal{T}^\mu$ too.

2. Every β-section $\mathcal{S}^\beta$ in $\mathcal{T}^\mu$ $(0 \leq \beta < \mu)$ has at most countably many extremities: In particular, every $(\mu-1)$-section has only countably many extremities. By Lemma 3.5, there are only countably many $(\mu-1)$-sections. Therefore, there are only countably many such extremities for all the $(\mu-1)$-sections taken together. The only extremities of $\mathcal{T}^\mu$ that are not extremities of its $(\mu-1)$-sections are its μ-tips. We conclude that $\mathcal{T}^\mu$ has uncountably many μ-tips.

Now, choose any $(\mu-1)$-section $\mathcal{S}_0^{\mu-1}$ of $\mathcal{T}^\mu$. By Lemma 3.4(a), each μ-tip of $\mathcal{T}^\mu$ is reached through a unique external tip of $\mathcal{S}_0^{\mu-1}$. Suppose that all but one of its external tips lead to at most countably many μ-tips of $\mathcal{T}^\mu$. Let t_0 be that external tip, and let $\mathcal{S}_1^{\mu-1}$ be the unique $(\mu-1)$-section containing t_0 as an internal tip. Suppose that all but one of the external tips of $\mathcal{S}_1^{\mu-1}$ lead to at most countably many μ-tips of $\mathcal{T}^\mu$. Let t_1 be that external tip of $\mathcal{S}_1^{\mu-1}$; t_1 cannot be an internal tip of $\mathcal{S}_0^{\mu-1}$. Also, let $\mathcal{S}_2^{\mu-1}$ be the unique $(\mu-1)$-section containing t_1 as an internal tip. Suppose again that all but one of the external tips of $\mathcal{S}_2^{\mu-1}$ lead to at most countably many μ-tips of $\mathcal{T}^\mu$. Continue this way to get a unique sequence of $(\mu-1)$-sections $\mathcal{S}_0^{\mu-1}$, $\mathcal{S}_1^{\mu-1}$, $\mathcal{S}_2^{\mu-1}$, ... and a unique sequence of tips $t_0, t_1, t_2, \ldots$. If either sequence terminates, so too will the other. Assume they do. By local countability, every $(\mu-1)$-section has only countably many external tips. By virtue of Lemma 3.4(a), we thus have a countable collection of countable sets of μ-tips whose union is the set of μ-tips of $\mathcal{T}^\mu$. This contradicts the fact that $\mathcal{T}^\mu$ has uncountably many μ-tips.

The only other possibility is that the sequences of $(\mu-1)$-sections and external tips continue indefinitely. But now, the sequence of external tips defines a unique one-ended μ-path reaching a unique μ-tip of $\mathcal{T}^\mu$, and we conclude again in the same way that $\mathcal{T}^\mu$ has only countably many μ-tips—a contradiction once more.

So, at least one of our suppositions is false. This means that there is some $(\mu-1)$-section $\mathcal{S}_a^{\mu-1}$ whose external tips lead to uncountably many μ-tips, and more than one of them together lead to uncountably many μ-tips. Moreover, $\mathcal{S}_a^{\mu-1}$ has only countably many external tips by local countability. Therefore, (at least) two external tips t_{11} and t_{12} of $\mathcal{S}_a^{\mu-1}$ each lead to uncountably many μ-tips.

Now, as the second step of our construction, let $\mathcal{T}_{11}^\mu$ be the μ-subtree of $\mathcal{T}^\mu$ induced by all the branches that t_{11} leads to (with respect to $\mathcal{S}_a^{\mu-1}$). $\mathcal{T}_{11}^\mu$ is a locally countable μ-tree having uncountably many μ-tips. Let $\mathcal{S}_{11}^{\mu-1}$ be the $(\mu-1)$-section having t_{11} as an internal tip. Upon repeating the above argument with $\mathcal{S}_{11}^{\mu-1}$ replacing $\mathcal{S}_0^{\mu-1}$, we find the first $(\mu-1)$-section $\mathcal{S}_b^{\mu-1}$ with two external tips t_{21} and t_{22}, each leading to uncountably many μ-tips. Similarly, starting with t_{12} and defining the μ-subtree $\mathcal{T}_{12}^\mu$ using t_{12} in place of t_{11}, we find the first $(\mu-1)$-section $\mathcal{S}_c^{\mu-1}$ with two external tips t_{23} and t_{24}, each leading to uncountably many μ-tips. This ends the second step of our constructions.

In the same way, the four tips $t_{21}, t_{22}, t_{23}, t_{24}$ lead to eight tips, each of the latter leading to uncountably many μ-tips. This completes the third step of our construction. Continuing onward, for each $k \in \boldsymbol{N}, k \neq 0$ we find 2^k tips at the kth step of this construction.

Now, consider again the first two tips t_{11} and t_{12}. Either they are embraced by the same boundary node x^μ of $\mathcal{S}_a^{\mu-1}$ or there will be a path in $\mathcal{S}_a^{\mu-1}$ connecting the two boundary nodes of $\mathcal{S}_a^{\mu-1}$ that embrace t_{11} and t_{12} respectively. In the first case, we can take x^μ to be the root of the binary tree we seek. In the second case, any internal node of $\mathcal{S}_a^{\mu-1}$ on the said path can be taken as the root.

Next, from t_{11} we progress to t_{21} and t_{22} as external tips of $\mathcal{S}_b^{\mu-1}$. If t_{21} and t_{22} are embraced by the same boundary node y_a^μ of $\mathcal{S}_b^{\mu-1}$, there is a path starting through t_{11} and ending at y_a^μ, and moreover y_a^μ will be a fork. (That path may pass through one or many $(\mu-1)$-sections.) On the other hand, if t_{21} and t_{22} are in different boundary nodes y_1^μ and y_2^μ of $\mathcal{S}_b^{\mu-1}$, there will be two paths starting through t_{11} and ending at y_1^μ and y_2^μ. The subtree induced by the branches of those two paths will have a single fork, and all other nodes of the subtree will be of degree 2.

This construction can be continued indefinitely and will thereby yield the binary μ-tree we seek. □

Since a binary tree has $\aleph$-many extremities, we also have the following.

Corollary 3.7. *A locally countable μ-tree with uncountably many extremities has at least $\aleph$-many extremities.*

4 $\vec{\omega}$-Trees

$\vec{\omega}$ denotes the arrow rank immediately preceding the first transfinite-ordinal ω [5, page 36]. An *$\vec{\omega}$-tree* $\mathcal{T}^{\vec{\omega}}$ is a connected $\vec{\omega}$-graph [5, page 38] having no loops. This time, connectedness means that any two branches are μ-connected for some $\mu \in \boldsymbol{N}$ depending upon the branches. $\mathcal{T}^{\vec{\omega}}$ has sections of all natural-number ranks, with any μ-section being partitioned by the β-sections within it for each $\beta < \mu$. Moreover, $\mathcal{T}^{\vec{\omega}}$ is also partitioned by μ-sections for each $\mu \in \boldsymbol{N}$. Because there is no highest natural-number rank for the μ-sections, we have to alter the arguments of the preceding section substantially.

Let us first review and change were needed some prior definitions. The *degree* of a node is again the cardinality of the set of its embraced tips, where now we also have $\vec{\omega}$-nodes [5, page 37]; $\vec{\omega}$-nodes are perforce of degree $\aleph_0$ and thus cannot be end nodes. As before, *end nodes* and *forks* are nodes of degrees 1 and 3 respectively and can have any ranks other than arrow ranks. The *extremities* of an $\vec{\omega}$-tree are now its end nodes of any natural-number ranks and its $\vec{\omega}$-tips [5, page 42]. With μ replaced by $\vec{\omega}$, Lemma 3.1 holds again with this new idea of an extremity.

The *boundary nodes*, the *internal tips*, and the *external tips* of a μ-section are defined as before. Lemma 3.2 also continues to hold when a $(\gamma-1)$-section $\mathcal{S}^{\gamma-1}$ has a maximal γ-node as a boundary node. Now, however, $\mathcal{S}^{\gamma-1}$ may have an $\vec{\omega}$-node as a boundary node. We will also need the case where $\mathcal{S}^{\gamma-1}$ has a boundary node of rank μ where $\mu > \gamma$. In either case, the boundary node will embrace a nonmaximal γ-node x^γ that will serve the same role as it did in Lemma 3.2, but now the external tips of the $(\gamma-1)$-section may be of ranks greater than γ. Virtually, the same proof as before yields the following alteration of Lemma 3.2. Remember that all the internal tips of $\mathcal{S}^{\gamma-1}$ have ranks no greater than $\gamma - 1$.

Lemma 4.1. *Let $\mathcal{S}^{\gamma-1}$ be a $(\gamma-1)$-section having a maximal boundary node x^ν, where the rank ν either is a natural number no less than γ or is $\vec{\omega}$. Then, x^ν embraces a γ-node x^γ such that $x^\gamma = x^\nu$ if $\nu = \gamma$, and x^γ is a nonmaximal node if $\nu > \gamma$. Moreover, one of the two following cases holds.*

1. If $\mathcal{S}^{\gamma-1}$ is incident to x^ν through a $(\gamma-1)$-tip, it will do so only through that one internal tip, and the external tips of $\mathcal{S}^{\gamma-1}$ at x^ν may have any natural-number ranks.

2. If $\mathcal{S}^{\gamma-1}$ is incident to x^ν through a β-tip ($\beta < \gamma-1$), it may do so through one or many internal tips, but their ranks will all be less than $\gamma - 1$; moreover, the external tips of $\mathcal{S}^{\gamma-1}$ at x^ν will all be of ranks no less than $\gamma - 1$,

Now, choose any 0-section $\mathcal{S}^0$ in the $\vec{\omega}$-tree $\mathcal{T}^{\vec{\omega}}$. Let $\mathcal{S}^1$ be the 1-section whose branches are 1-connected to the branches of $\mathcal{S}^0$. Thus, $\mathcal{S}^0 \subset \mathcal{S}^1$ in the sense that $\mathcal{S}^0$ is a subgraph of $\mathcal{S}^1$. Recursively, let $\mathcal{S}^\mu$ ($\mu > 0$) be the μ-section whose branches are μ-connected to the branches of $\mathcal{S}^{\mu-1}$. Continue this construction through all the natural-numbers μ. We get a nested sequence

$$\mathcal{S}^0 \subset \mathcal{S}^1 \subset \ldots \subset \mathcal{S}^\mu \subset \ldots \qquad (3)$$

Since every two branches are μ-connected in $\mathcal{T}^{\vec{\omega}}$ for some μ depending on the choice of the two branches, they will reside in the same μ-section. This implies that $\mathcal{T}^{\vec{\omega}}$ is the union[4] of the sections in (3):

$$\mathcal{T}^{\vec{\omega}} = \cup_{\mu \in N} \mathcal{S}^\mu \qquad (4)$$

Lemma 4.2. *Given any two 0-sections $\mathcal{S}_1^0$ and $\mathcal{S}_2^0$ in $\mathcal{T}^{\vec{\omega}}$, the corresponding nested sequences (3) will eventually be identical in the sense that there will be some rank μ (depending on the choices of $\mathcal{S}_1^0$ and $\mathcal{S}_2^0$) beyond which the sequences will be the same.*

Proof. A branch of $\mathcal{S}_1^0$ and a branch of $\mathcal{S}_2^0$ will be μ-connected for some positive natural number μ. By the transitivity of μ-connectedness [5, Theorem 3.5-2], the same is true with the same μ for every choice of those two branches in those 0-sections. Thus, the branches of $\mathcal{S}_1^0$ and $\mathcal{S}_2^0$ will all lie in some μ-section $\mathcal{S}^\mu$. Moreover, μ-connectedness implies ν-connectedness for every $\nu > \mu$. This implies the conclusion. □

An *internal node* of a μ-section $\mathcal{S}^\mu$ is a node of rank less than or equal to μ whose embraced tips are all internal tips of $\mathcal{S}^\mu$. Thus, it is not a boundary node of $\mathcal{S}^\mu$.

Lemma 4.3. *With the nested sequence being fixed for some choice of $\mathcal{S}^0$, every (maximal) end node of $\mathcal{T}^{\vec{\omega}}$ is an internal node of $\mathcal{S}^\mu$ for some sufficiently large μ.*

Proof. Since no $\vec{\omega}$-node can be an end node, each end node x^α of $\mathcal{T}^{\vec{\omega}}$ has some natural-number rank α, and so its one and only tip has the rank $\alpha - 1$. Choose a representative path $P^{\alpha-1}$ for that tip. The branches of $P^{\alpha-1}$ will be $(\alpha - 1)$-connected and will therefore lie in an $(\alpha - 1)$-section $\mathcal{S}^{\alpha-1}$. Moreover, there will be a $(\mu - 1)$-section $\mathcal{S}^{\mu-1}$ in (3) with $\mu - 1 > \alpha$ whose branches are $(\mu - 1)$-connected to the branches of $P^{\alpha-1}$. Thus, $\mathcal{S}^\mu$ in (3) will contain not only $P^{\alpha-1}$ but also x^α as an internal node. □

An $\vec{\omega}$-tree $\mathcal{T}^{\vec{\omega}}$ will be called *locally countable* if every maximal 0-node is of countable degree and if every μ-section of every natural-number rank μ has only countably many external tips.

[4]The union of subgraphs is the subgraph induced by the branches in the union of the branch sets of the subgraphs.

Lemma 4.4. *Every node x^ρ $(0 \le \rho \le \vec{\omega})$ in a locally countable $\vec{\omega}$-tree $\mathcal{T}^{\vec{\omega}}$ has a countable degree.*

Proof. If ρ is a natural number, the proof of Lemma 3.3 can be applied again. So, let $\rho = \vec{\omega}$, and consider any one of the embraced nonmaximal nodes x^γ in $x^{\vec{\omega}}$, where γ is a natural number. x^γ is a maximal node in the γ-graph of $\mathcal{T}^{\vec{\omega}}$; this γ-graph [5, page 38] is the graph obtained by disallowing all nodes of ranks greater than γ. With respect to that γ-graph, Lemma 3.3 holds again, and we can conclude that x^γ embraces only countably many tips—even as a nonmaximal node of $\mathcal{T}^{\vec{\omega}}$. Furthermore, $x^{\vec{\omega}}$ embraces only countably many nodes of natural-number ranks. Therefore, $x^{\vec{\omega}}$ embraces only countably many tips. □

The words *before*, *after*, *leads away from*, and *leads to* will be employed precisely as in the preceding section, where now an extremity to which a tip leads may be an $\vec{\omega}$-tip. Virtually, the same arguments as those used for Lemma 3.4 establishes the following lemma.

Lemma 4.5. *Let $\mathcal{S}^\mu$ $(\mu \in \boldsymbol{N})$ be any μ-section in $\mathcal{T}^{\vec{\omega}}$.*

(a) *Given any end node outside of $\mathcal{S}^\mu$ or any $\vec{\omega}$-tip of $\mathcal{T}^{\vec{\omega}}$, there is a unique external tip of $\mathcal{S}^\mu$ that leads to that end node or $\vec{\omega}$-tip.*

(b) *Each external tip of $\mathcal{S}^\mu$ leads to at least one extremity of $\mathcal{T}^{\vec{\omega}}$.*

A binary $\vec{\omega}$-tree $\mathcal{T}_b^{\vec{\omega}}$ is defined much as before. We start with an $\vec{\omega}$-tree in which every node is of degree 2 or 3 and there is a node x of degree 2. We designate x as the *root.* That tree has no $\vec{\omega}$-node because the degree of an $\vec{\omega}$-node is $\aleph_0$. On the other hand, nodes of all natural-number ranks appear in that $\vec{\omega}$-tree. We also define a kth *fork from* x as before.

A *binary $\vec{\omega}$-tree $\mathcal{T}_b^{\vec{\omega}}$* is an $\vec{\omega}$-tree satisfying the following three conditions:

1. Each of its nodes is of degree 2 or 3.

2. There exists at least one node of degree 2.

3. From any node of degree 2, the number of kth forks is 2^k.

The properties listed just after the definition of a binary β-tree also hold for $\mathcal{T}_b^{\vec{\omega}}$ when "β" is replaced by "$\vec{\omega}$".

Theorem 4.6. *Let $\mathcal{T}^{\vec{\omega}}$ be a locally countable $\vec{\omega}$-tree. $\mathcal{T}^{\vec{\omega}}$ has uncountably many extremities if and only if $\mathcal{T}^{\vec{\omega}}$ contains a binary β-tree for some β with $0 \le \beta \le \vec{\omega}$.*

Proof. If: If $\mathcal{T}^{\vec{\omega}}$ contains a binary $\vec{\omega}$-tree $\mathcal{T}_b^{\vec{\omega}}$, each of the uncountably many $\vec{\omega}$-tips of $\mathcal{T}_b^{\vec{\omega}}$ will be extremities of $\mathcal{T}^{\vec{\omega}}$, which implies the desired conclusion. If $\mathcal{T}^{\vec{\omega}}$ contains a binary β-tree $\mathcal{T}_b^{\beta}$ with $\beta < \vec{\omega}$, each of the uncountably many β-tips of $\mathcal{T}_b^{\beta}$ will be embraced by a γ-node x^γ of $\mathcal{T}^{\vec{\omega}}$ with γ greater than β and depending in general on the β-tip and with x^γ embracing just one of those β-tips. If x^γ is a singleton, it is an end node and thus an extremity of $\mathcal{T}^{\vec{\omega}}$. If x^γ is a nonsingleton, it will lead (with respect to any node of $\mathcal{T}_b^{\beta}$) to at least one extremity of $\mathcal{T}^{\vec{\omega}}$. This yields the desired conclusion again.

Only if: We now hypothesize that $\mathcal{T}^{\vec{\omega}}$ has uncountably many extremities. Two possibilities arise:

1. There is a μ-section $\mathcal{S}^\mu$ for some $\mu \in \boldsymbol{N}$ having uncountably many extremities as a locally countable subtree of $\mathcal{T}^{\vec{\omega}}$. (Those extremities are its end nodes, which will also be end nodes of $\mathcal{T}^{\vec{\omega}}$, and its μ-tips.) By Theorem 3.6 applied to $\mathcal{S}^\mu$, there is a binary β-tree $(0 \leq \beta \leq \mu)$ in $\mathcal{S}^\mu$ and therefore in $\mathcal{T}^{\vec{\omega}}$ too.

2. For every $\mu \in \boldsymbol{N}$, every μ-section of $\mathcal{T}^{\vec{\omega}}$ has at most countably many extremities (again counting its μ-tips as extremities). Choose arbitrarily any 0-section $\mathcal{S}^0$ in $\mathcal{T}^{\vec{\omega}}$. By Lemma 4.5(a), all the external tips of $\mathcal{S}^0$ together lead to all the extremities of $\mathcal{T}^{\vec{\omega}}$ other than the end nodes of $\mathcal{S}^0$. There are only countably many of the latter. Suppose all but one of the external tips of $\mathcal{S}^0$ each lead to only countably many extremities of $\mathcal{T}^{\vec{\omega}}$ and therefore, by local countability, together lead to only countably many extremities of $\mathcal{T}^{\vec{\omega}}$. Let that exceptional tip be t^{α_0}. By Lemma 4.5(b), t^{α_0} leads to at least one extremity of $\mathcal{T}^{\vec{\omega}}$. (We presently make no assertion about the cardinality of the set of extremities to which t^{α_0} leads.)

Now, consider the representation (4) of $\mathcal{T}^{\vec{\omega}}$ as the union of the μ-sections ($\mu = 0, 1, 2, \ldots$) in the expanding sequence (3). Let $\mathcal{S}^1$ by the 1-section in (3) containing $\mathcal{S}^0$, and let T_1 be the set of external tips of $\mathcal{S}^1$ to which t^{α_0} leads. T_1 is either the singleton $\{t^{\alpha_0}\}$ (we say in this case that t^{α_0} leads to itself), or T_1 is a set all of whose elements are different from t^{α_0}. Suppose all but one of the tips in T_1 together lead to at most countably many extremities of $\mathcal{T}^{\vec{\omega}}$. (If $T_1 = \{t^{\alpha_0}\}$, there are no such tips different from t^{α_0}.) Let t^{α_1} be that exceptional tip. (The rank α_1 can be equal to, or greater than, or less than α_0.) Either $t^{\alpha_1} = t^{\alpha_0}$, or there is a unique path starting through t^{α_0} and reaching t^{α_1}. Also, t^{α_1} leads to at least one extremity of $\mathcal{T}^{\vec{\omega}}$ (Lemma 4.5(b) again).

Recursively, let $\mathcal{S}^{\mu-1}$ and $\mathcal{S}^{\mu}$ be consecutive sections in (3) with $\mu > 0$. As our recursive supposition, we take it that all but one of the external tips of $\mathcal{S}^{\mu-1}$ together lead to at most countably many extremities of $\mathcal{T}^{\vec{\omega}}$. Let $t^{\alpha_{\mu-1}}$ be the exceptional external tip of $\mathcal{S}^{\mu-1}$ that may or may not lead to uncountably many extremities of $\mathcal{T}^{\vec{\omega}}$. Let T_{μ} be the set of external tips of $\mathcal{S}^{\mu}$ to which $t^{\alpha_{\mu-1}}$ leads. Here too, T_{μ} is either the singleton $\{t^{\alpha_{\mu-1}}\}$, or T_{μ} is a set all of whose elements are different from $t^{\alpha_{\mu-1}}$. Suppose all but one of the tips in T_{μ} lead to at most countably many extremities of $\mathcal{T}^{\vec{\omega}}$. Let $t^{\alpha_{\mu}}$ be that exceptional tip. Either $t^{\alpha_{\mu}} = t^{\alpha_{\mu-1}}$ or there is a unique path starting through $t^{\alpha_{\mu-1}}$ and reaching $t^{\alpha_{\mu}}$. Also, $t^{\alpha_{\mu}}$ leads to at least one extremity of $\mathcal{T}^{\vec{\omega}}$.

In this way, we get a sequence of tips $t^{\alpha_0}, t^{\alpha_1}, t^{\alpha_2}, \ldots$. These tips determine a unique path that passes through all of them. That path may either terminate in $\mathcal{T}^{\vec{\omega}}$ at an end node of natural-number rank, or it may be one-ended with an $\vec{\omega}$-tip. In either case, we will have altogether a sequence of countable sets of external tips for the $\mathcal{S}^{\mu}$ ($\mu = 0, 1, 2, \ldots$), with each tip leading to only countably many extremities of $\mathcal{T}^{\vec{\omega}}$. Other than the countably many end nodes in $\mathcal{S}^0$, every extremity of $\mathcal{T}^{\vec{\omega}}$ is one to which an external tip of one of the $\mathcal{S}^{\mu}$ leads (Lemma 4.5(a)). Altogether then, this violates the "only if" hypothesis that $\mathcal{T}^{\vec{\omega}}$ has uncountably many extremities.

Thus, at least one of our suppositions is false. This means that there is a section, say $\mathcal{S}^{\mu_0}$ in (3) whose external tips together lead to uncountably many extremities. Moreover, since $\mathcal{S}^{\mu_0}$ has only countably many external tips and since it is not true that all but one of them lead to only countably many extremities, there must be (at least) two tips t_{11} and t_{12} of $\mathcal{S}^{\mu_0}$, each leading to uncountably many extremities.

Each end node resides in a μ-section in (3). Thus, by possibility 2 above, there can be only countably many end nodes in $\mathcal{T}^{\vec{\omega}}$. Therefore, each of t_{11} and t_{12} lead to uncountably many $\vec{\omega}$-tips.

Now, consider the union of the set of paths starting through t_{11} and proceeding away from $\mathcal{S}^{\mu_0}$. That union will be an $\vec{\omega}$-tree $\mathcal{T}_{11}^{\vec{\omega}}$ having uncountably many $\vec{\omega}$-tips with the node containing t_{11} being a singleton. Therefore, we can repeat our argument as applied to $\mathcal{T}_{11}^{\vec{\omega}}$ to conclude that we can find two tips t_{21} and t_{22} different from t_{11}, each leading to uncountably many $\vec{\omega}$-tips. Similarly, we have an $\vec{\omega}$-tree $\mathcal{T}_{12}^{\vec{\omega}}$ obtained as the union af all possible paths starting through t_{12} and proceeding away from $\mathcal{S}^{\mu_0}$. By the same argument, we can find in $\mathcal{T}_{12}^{\vec{\omega}}$ two tips t_{23} and t_{24} different from t_{12}, each leading to uncountably many $\vec{\omega}$-tips. From these four tips, we get eight tips with the same property—and so on for 2^k tips, $k = 1, 2, 3, \ldots$.

Now, if t_{11} and t_{12} are incident to two different boundary nodes of $\mathcal{S}^{\mu_0}$, there will be a unique internal node x in $\mathcal{S}^{\mu_0}$ from which two paths in $\mathcal{S}^{\mu_0}$ can be found that are disjoint except at x and which meet the two boundary nodes respectively. On the other hand, if t_{11} and t_{12} are incident to the same boundary node of $\mathcal{S}^{\mu_0}$, we let x be that boundary node. In either case, x will be the root of the binary $\vec{\omega}$-tree we seek. That tree is defined by the set of paths starting at x and passing along the tips mentioned above. Indeed, at x a binary choice can be made as to which of t_{11} and t_{12} to proceed through. Having made that choice, another binary choice can be made as to which of t_{12} and t_{22} or of t_{23} and t_{24} to proceed through. This implies that there is a fork between t_{11} (resp. t_{12}) and t_{21} and t_{22} (resp. t_{23} and t_{24}) through which the chosen path proceeds. In general, each such path is defined by making a binary choice between which of two tips to follow at each fork between the kth set of 2^k tips and the $(k+1)$st set of 2^{k+1} tips. Each fork will either lie between sets of tips or will be a node incident to two tips. Moreover, the paths so generated will be one-ended $\vec{\omega}$-paths because they pass through all of the μ-sections in (3) and thereby through boundary nodes of ever-increasing natural-number ranks. We have indeed found a binary $\vec{\omega}$-tree in $\mathcal{T}^{\vec{\omega}}$. □

Here too, Corollary 3.7 holds with μ replaced by $\vec{\omega}$.

5 Transfinite Trees of Higher ranks

The definitions of a locally countable tree and of a binary tree can be extended to ranks higher than $\vec{\omega}$ in obvious ways. Then, Theorem 3.6 can be extended to locally countable trees $\mathcal{T}^{\omega+\mu}$ of ranks $\omega + \mu$, where $\mu \in \boldsymbol{N}$, using much the same arguments as those of Secs.

2 and 3. For example, $\mathcal{T}^{\omega}$ now takes the role that $\mathcal{T}^{0}$ played before in Sec. 2; $\omega - 1$ now represents the rank $\vec{\omega}$, whereas $0 - 1$ was the rank of an elementary tip of a branch [5, page 9]. Next, Theorem 4.6 can be extended to an $(\omega \vec{\cdot} 2)$-tree by repeating the arguments of Sec. 4 in virtually the same way using $\mathcal{S}^{\omega}$ in place of $\mathcal{S}^{0}$. This inductive procedure can be continued to still higher ranks using the arguments of Sec. 3 for trees of ordinal ranks and the arguments of Sec. 4 for trees of arrow ranks. However, as is the case for transfinite graphs in general, we make no claim as to how far through the countable-ordinal ranks and their accompanying arrow ranks our proofs can be extended because it is unclear whether the needed assumptions for a general transfinite induction can be made.

References

[1] A. Abian *The Theory of Sets and Transfinite Arithmetic*, W.B. Saunders Company, London, 1965.

[2] R.J. Wilson, *Introduction to Graph Theory*, Academic Press, New York, 1972.

[3] W. Woess, *Random Walks on Infinite Graphs and Groups*, Cambridge University Press, New York, 2000.

[4] A.H. Zemanian, *Infinite Electrical Networks*, Cambridge University Press, New York, 1991.

[5] A.H. Zemanian, *Transfiniteness for Graphs, Electrical Networks, and Random Walks*, Birkhauser, Boston, 1996.