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WALK-BASED TRANSFINITE GRAPHS AND NETWORKS

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Abstract — The theory of transfinite graphs developed so far has been based on the ideas that connectedness is accomplished through paths and that the infinite extremities of the graph are specified through one-way infinite paths. As a result, a variety of difficulties arise in that theory, leading to the need to restrict such path-based graphs in various ways in order to obtain certain results. In this work, we present a more general theory of transfinite graphs wherein connectedness and the designation of extremities are accomplished through walks rather than paths. This leads to a simpler and yet more general theory wherein new kinds of transfinite extremities are also encompassed. For instance, an ordinal-valued distance function can now be defined on all pairs of walk-connected nodes, in contrast to the path-based theory wherein no distance function is definable for those pairs of nodes that are not path-connected even though they are walk-connected. Some results concerning eccentricities, radii, and diameters are presented in this more general walk-based graph theory. Another new result herein is the development of an electrical network theory for networks whose graphs are walk-based. A unique voltage-current regime is established under certain conditions. The current regime is built up from current flows in closed transfinite walks—in contrast to a prior theory based upon flows in transfinite loops. A notable advantage of the present approach is that node voltages with respect to a given ground node are always unique whenever they exist. The present approach is more general in that it provides nontrivial voltage-current regimes for certain networks for which the prior approach would only provide trivial solutions having only zero currents and voltages everywhere.

Key Words: Transfinite graphs, transfinite walks, walk-based graphs, eccentricities, radii, diameters, transfinite electrical networks, node voltages.

1 Introduction

The theory of transfinite graphs introduced in [3] and explored more thoroughly in [4] and [6] used transfinite paths as the basic construct, rather than transfinite walks. This was a natural extension of finite graphs because connectedness for finite graphs is fully characterized by paths; indeed, any walk terminating at two nodes of a finite graph contains a path doing the same. However, this is no longer the case for transfinite graphs, as we will show by example later on. This has restricted the theory of transfinite graphs. For instance, path-connectedness need not be transitive as a binary relationship among transfinite nodes, and special conditions have to be imposed to ensure such transitivity. As a result, distances as defined by paths will not exist between certain pairs of nodes. This limitation is also reflected in the theory of transfinite electrical networks by the fact that node voltages need not be uniquely determined when they are defined along paths to a chosen ground node.

These troubles disappear when transfinite walks are used as the basic construct, but the resulting, more general kinds of transfinite graphs now encompass some strange structures that stress intuition based upon one's familiarity with finite graphs. Moreover, electrical network theory is similarly ensnarled. Such complications should not be surprising because infinite entities are mathematical abstractions involving a variety of counterintuitive phenomena, as for example the antinomies of infinite sets [2]. Following theory wherever it may lead, we now propose to explore walk-based transfinite graphs.

By the end of Sec. 7, we will have accomplished one of the objectives of this paper, namely, to define and develop recursively transfinite graphs based upon walk-defined extremities. This opens up the possibility of establishing a variety of more general results analogous to those already proven for path-based transfinite graphs. Such will be done in Secs. 8 through 12, where we examine distances in walk-based transfinite graphs, and in Secs. 13 through 18, where we examine the electrical behavior of resistive networks having walk-based graphs. Much of those analyses are identical to the corresponding arguments for path-based graphs, but there are significant differences. So, we shall present definitions and arguments where the walk-based analysis differs substantially from the path-based analysis, but we will simply refer to prior works when there are no significant differences between the two.

We assume throughout this work all the definitions and results in [4]. In this regard, the Errata available under "Books" in the URL: "www.ee.sunysb.edu/~zeman" should be noted. On the other hand, we will be defining a variety of entities in this work that are analogous to the tips, nodes, sections, graphs, and path-connectedness of [3], [4], [6] but are based upon transfinite walks rather than on transfinite paths. So as not to introduce new terminology and some consequent unnecessary confusion, we will again use the path-based terminology but will employ the letter"w" as a prefix to indicate that our entities are now walk-based. Thus, we write "wtip," "wnode," "wsection," "wgraph," and "wconnectedness."[1]

Let us explicate another bit of terminology: A one-ended infinite sequence $\{a_0, a_1, \ldots, a_m, \ldots\}$ or a two-way infinite sequence $\{\ldots, a_{-m}, \ldots, a_{-1}, a_0, a_1, \ldots, a_m, \ldots\}$ will be said to possess *eventually* a certain property if there exists a natural number m_0 such that the subsequences having indices m with $|m| > m_0$ possess that property. Similarly, two one-way infinite sequences will be said to be *eventually identical* if, with an appropriate index renumbering of one of them, their elements are eventually the same.

2 Some Examples

nomena, as for example the antinomies of infinite sets [2]. Following theory wherever it may
[4].

Example 2.1. The simplest example of a transfinite graph exhibiting the difference between path-connectedness and walk-connectedness is the 1-graph of Fig. 1. Here we have an infinite series circuit of pairs of parallel connected branches, a_1 and b_1, a_2 and b_2, ... The 0-tip t_a^0 (resp. t_b^0) has as a representative the one-ended path along the branches a_k (resp. b_k) for $k = 1, 2, 3, \ldots$. x^1 (resp. y^1) is a singleton 1-node containing t_a^0 (resp. t_b^0). There is a 1-path along the branches a_k connecting x^1 to the 0-node n^0 and another 1-path along the b_k branches connecting y^1 to n^0. But, there is no 1-path connecting x^1 and y^1. However, there is a 1-walk passing along the branches $\ldots, a_3, a_2, a_1, b_1, b_2, b_3 \ldots$ that terminates at x^1 and y^1. Thus, x^1 and y^1 are walk-connected but not path-connected.

[1] One may pronounce wtip as "walk-tip" or simply say "tip" when the prefix "w" in understood—and similarly for the other "wentities."

Note that there are uncountably many 0-tips, each having a representative path that passes back and forth between the a_k and b_k branches infinitely often but not necessarily in a uniform pattern. The cardinality of the set of those 0-tips is that of the continuum. 1-nodes can be defined containing those 0-tips, either as singleton 1-nodes or as shortings of several such 0-tips. □

Actually, the 1-graph of Fig. 1 appears in modified form within many transfinite graphs, and, as a result, this nontransitivity of path-connectedness is a ubiquitous difficulty. Indeed, consider the following.

Example 2.2. Fig. 2 shows a one-way infinite ladder, each of whose 0-tips is the sole member of a singleton 1-node. x_a^1 (resp. x_b^1) is the 1-node containing the 0-tip of the one-ended 0-path along the branches a_k (resp. b_k), $k = 1, 2, 3, \ldots$. There are, however, infinitely many other 0-tips corresponding to paths that pass back and forth between the a_k and b_k branches via the branches c_k but doing so in different ways. x_{abc}^1 is a 1-node containing one such 0-tip. The others are not shown, but the set of them has the cardinality of the continuum. There is a two-ended 1-path terminating at x_a^1 and x_b^1; it passes along the branches $\ldots, a_2, a_1, c_0, b_1, b_2, \ldots$. Thus, x_a^1 and x_b^1 are path-connected. However, for no other pair of 1-nodes does path-connectedness hold. But, walk-connectedness holds for all such pairs. □

Thus, we are emboldened to search for greater generality in the theory of transfinite graphs by employing walk-connectedness.

3 The Infinite Extremities of 0-graphs and the Resulting 1-graphs

A 0-graph $\mathcal{G}^0 = \{\mathcal{B}, \mathcal{X}^0\}$, where $\mathcal{B}$ denotes the set of branches, and $\mathcal{X}^0$ denotes the set of 0-nodes, is a conventional graph, but we use the unconventional definition given in [4, page 10] just to conform with the definitions of transfinite graphs. It may or may not contain a one-ended 0-path, but, when it does, it possesses at least one infinite extremity, called as above a "0-tip" [4, page 20]. Can we define these or other kinds of extremities by using one-ended 0-walks instead of 0-paths?

A *nontrivial 0-walk* W^0 is an alternating sequence of 0-nodes x_m^0 and branches b_m:

$$W^0 = \{\ldots, x_{m-1}^0, b_{m-1}, x_m^0, b_m, x_{m+1}^0, \ldots\} \qquad (1)$$

where the indices m traverse a set of consecutive integers and, for each m, the branch b_m is incident to the two 0-nodes x_m^0 and x_{m+1}^0. In contrast to a path, nodes and branches may repeat in the sequence. We allow any branch to be a self-loop, in which case x_m^0 and x_{m+1}^0 are the same 0-node. If the sequence (1) terminates on either side, it terminates at a node. The 0-walk W^0 is called *two-ended* or *finite* if it terminates on both sides, *one-ended* if it terminates on just one side, and *endless* if it terminates on neither side. We assign to W^0 an *orientation* determined by the direction in which the indices m increase.

A *trivial 0-walk* is a singleton containing just one 0-node.

A superfluity arises when we try to define the infinite extremities of a 0-graph $\mathcal{G}^0$ by using 0-walks. Assume that $\mathcal{G}^0$ does have at least one such extremity—as witnessed by the presence of a one-ended 0-path, and consider a one-ended 0-walk that keeps returning to a fixed 0-node x^0 infinitely often. For the purpose of finding an infinite extremity , it appears intuitively that each time the walk returns to x^0, the walk may as well be reduced by discarding the loop just traced. But, if we keep doing so, the entire walk will reduce to a trivial walk. More generally, if we keep discarding loops as they are traced by any one-ended 0-walk W^0, W^0 may either be reduced to a finite path or to a one-ended path. In the latter case, the walk can be used to define an infinite extremity, but it will be none other than the 0-tip defined by that one-ended 0-path. We conclude that we may as well stick with one-ended 0-paths when defining the infinite extremities of a 0-graph. This we do.

It is only when we turn to the construction of graphs of higher ranks of transfiniteness that walks provide greater generality than do paths, as we shall see. At this point of our discussion, 0-tips and then 1-nodes and 1-graphs are defined exactly as they are in [4, pages 20-23]. Thus, a 1-*graph* is a triplet:

$$\mathcal{G}^1 = \{\mathcal{B}, \mathcal{X}^0, \mathcal{X}^1\}$$

where $\mathcal{X}^1$ is the set of 1-nodes and $\mathcal{B}$ and $\mathcal{X}^0$ are as before. Moreover, a 0-*section* of any transfinite graph also remains as before; it is the subgraph induced by a maximal set of branches that are 0-connected through two-ended 0-paths [4, page 49]. The 0-sections of any transfinite graph partition that graph because 0-connectedness is transitive; this is a special case of [4, Corollary 3.5-6].

How a one-ended 0-path *reaches* a 1-node is defined in [4, page 23]. That idea can be extended to a 0-walk as follows: A one-ended 0-walk

$$W^0 = \{x_0^0, b_0, x_1^0, b_1, x_2^0, b_2, \ldots\}$$

will be called *extended* if its 0-nodes x_m^0 are eventually pairwise distinct. Thus, W^0 is extended if it eventually is identical to a one-ended 0-path. We say that W^0 *traverses* a 0-tip if it is extended and is eventually identical to a representative of that 0-tip. Finally, W^0 is said to *reach* a 1-node x^1 if it traverses a 0-tip embraced by x^1. In the same way, an endless 0-walk can *reach* two 1-nodes (or possibly reach the same node) by traversing two 0-tips, one toward the left and the other toward the right. On the other hand, if a 0-walk terminates at a 0-node that is embraced by a 1-node, we again say that the walk *reaches* both of those nodes and does so *through* an elementary tip of its branch that is incident to that 0-node.

4 The Infinite Extremities of 1-graphs and the Construction of 2-wgraphs

Matters become more complicated when the infinite extremities of 1-graphs are considered.

Example 4.1. As was noted with respect to Fig. 1, there is a 1-walk connecting the 1-nodes x^1 and y^1, but no such 1-path. We can connect infinitely many such 1-graphs "in series," as shown in Fig. 3, to obtain a 1-graph with an extremity not definable through paths. That extremity is reached through the "one-ended 1-walk" that starts at x_1^1, passes to x_2^1 through an endless 0-walk, then to x_3^1 through another endless 0-walk, and so on. More specifically, from x_1^1 it passes along the left side of the first 0-section to, say, the 0-node x_1^0, then along the right side of that 0-section to x_2^1, then along the left side of the second 0-section to, say, the 0-node x_2^0, then along the right side of that 0-section to x_3^1, and so forth. On the other hand, there is no one-ended 1-path that passes through $x_1^1, x_2^1, x_3^1, \ldots$.

Thus, we have a new kind of extremity, something that was not considered in [3], [4], and [6].

Before leaving this example, let us consider a variation of it. Consider the 1-graph obtained from that of Fig. 3 by shorting together all the 0-nodes x_k^0 ($k = 1, 2, 3, \ldots$) along the bottom of Fig. 3 to obtain a 0-node x^0 of infinite degree. Then, the "one-ended 1-walk" described just before becomes one that keeps returning to the same 0-node x^0. This time, we will allow such a walk as a means of identifying an infinite extremity of the 1-graph. So long as the 1-nodes through which the walk passes become eventually pairwise distinct, we will take it that an infinite extremity of the 1-graph is being identified. Compare this with the situation considered in the preceding Section were a 0-walk that keeps returning to the same 0-node is reduced to a 0-path by removing loops. The important difference is that that 0-walk never passed through 1-nodes and therefore was unsuitable for the identification of the extremity of a 1-graph. □

Example 4.2. Extremities like this appear inherently in more familiar kinds of 1-graphs. For example, a 1-walk reaching such a walk-based extremity is indicated in Fig. 4 for a 1-graph consisting of two infinite checkerboard 0-graphs connected together through certain 1-nodes. x_1^1 is a singleton 1-node containing one 0-tip having as a representative a one-ended 0-path that passes horizontally toward the left and alternating up and down through single vertical branches, as shown. For $m = 2, 3, 4, \ldots$, each x_m^1 consists of two 0-tips, one for a similar 0-path toward the left and another for such a 0-path toward the right. Pairs of such consecutive 0-paths pass through infinitely many horizontal branches in opposite directions. Altogether, these infinitely many 1-nodes with these infinitely many 0-paths form a "one-ended 1-walk" whose extremity cannot be reached through any one-ended 1-path because those pairs of consecutive 0-paths intersect infinitely often. However, each walk between two consecutive 1-nodes x_m^1 and x_{m+1}^1 is the conjunction of two one-ended 0-paths forming an endless 0-walk passing through a 0-section (one of the checkerboards) and reaching those 1-nodes through 0-tips. So, perhaps, one-ended 1-walks that can be decomposed into endless 0-walks between 1-nodes that are eventually pairwise distinct will suffice in identifying a new kind of transfinite extremity. □

Let us implement this idea in the following precise way: Because 0-tips and 1-nodes are now defined exactly as they are for path-based transfinite graphs [4, pages 20 and 22] and because 0-connectedness is transitive, every 1-graph $\mathcal{G}^1$ is partitioned into 0-sections, that is, each branch belongs to exactly one 0-section. Every maximal bordering node [4, page 81] of any 0-section $\mathcal{S}^0$ of $\mathcal{G}^1$ is a 1-node, and every maximal boundary node [4, page 49] is a maximal bordering node of two or more 0-sections. To say this another way, a 1-node x^1 is a bordering node of a 0-section $\mathcal{S}^0$ in $\mathcal{G}^1$ if x^1 embraces an elementary tip or a 0-tip of $\mathcal{S}^0$; also, x^1 is a boundary node of $\mathcal{S}^0$ if it is a bordering node of $\mathcal{S}^0$ and also a bordering node of at least one other 0-section.

A given 0-section $\mathcal{S}^0$ can be entered from one of its bordering nodes either along a branch in $\mathcal{S}^0$ (that is, along an elementary tip) or along a representative of a 0-tip of $\mathcal{S}^0$. That representative is a one-ended 0-path. So, given the bordering nodes x^1 and y^1 of $\mathcal{S}^0$ (possibly $x^1 = y^1$), there exists a 0-walk W^0 in $\mathcal{S}^0$ leaving x^1 and reaching y^1. W^0 enters $\mathcal{S}^0$ along a branch or a representative of a 0-tip, proceeds along a 0-path, and then leaves $\mathcal{S}^0$ along a branch or representative of a 0-tip; but, nodes and/or branches may have to repeat along the walk W^0 from x^1 to y^1. We will say that W^0 is a 0-walk *through* $\mathcal{S}^0$, that w^0 *starts at* x^1 and *stops at* y^1, and that W^0 *reaches* x^1 and y^1 *through* either elementary tips or 0-tips. It can happen, of course, that a particular 0-walk through $\mathcal{S}^0$ is in fact a 0-path.

A special case occurs when x^1 and y^1 are connected through a single branch whose 0-nodes are embraced by x^1 and y^1. In this case, the branch set of $\mathcal{S}^0$ consists of that branch alone, and that branch along with its 0-nodes will also be called a 0-walk *through* $\mathcal{S}^0$.

Let s (resp. t) be the elementary tip or 0-tip through which W^0 reaches x^1 (resp. y^1). In general, there will be many 0-walks through $\mathcal{S}^0$ reaching x^1 and y^1, but any two of them having the same tips s and t will be considered *equivalent*. In this way, it is only how a 0-walk W^0 enters and leaves a 0-section that will be significant; the particular internal nodes and branches it passes through within $\mathcal{S}^0$ will not be important. We will make use of this equivalence idea later on.

A *nontrivial 1-walk* W^1 is an alternating sequence of 1-nodes and nontrivial 0-walks through 0-sections:

$$W^1 = \{\ldots, x^1_{m-1}, W^0_{m-1}, x^1_m, W^0_m, x^1_{m+1}, \ldots\} \qquad (2)$$

where, for every m, x^1_m and x^1_{m+1} are incident to the same 0-section (possibly $x^1_m = x^1_{m+1}$) and W^0_m is a 0-walk through that 0-section reaching x^1_m and x^1_{m+1}; it is also required that, for each m, at least one of W^0_{m-1} and W^0_m reaches x^1_m through a 0-tip (not an elementary tip). If the sequence in (2) terminates on either side, it is furthermore required that it terminate at a 0-node or 1-node. We refer to that node as a *terminal node* of W^1; in this case, the elementary tip (resp. 0-tip) with which the 0-walk in (2) adjacent to the terminal 0-node (resp. 1-node) reaches that terminal node will be called a *terminal* tip of W^1. (If (2) extends infinitely in either direction, we shall later on define another kind of "terminal wtip" for W^1.)

Note that, if W^1 terminates on either side at an internal 0-node of a 0-section, the adjacent 0-walk in the sequence (2) can be reduced to a 0-path by removing loops.

A 1-walk is said to *embrace* itself, all its elements, and all the elements that its elements embrace.

A *trivial* 1-walk is a singleton whose sole member is a 1-node.

A nontrivial 1-walk W^1 is called *two-ended* (resp. *one-ended*, resp. *endless*) if it terminates on both sides (resp. terminates on exactly one side, resp. terminates on neither side). If W^1 terminates at a 0-node, the adjacent 0-walk in (2) will be either one-ended or two-ended (not endless). However, W^1 may terminate at a 1-node that embraces a 0-node; in this case, we may also say that W^1 terminates at that 0-node.

As with 0-tips, 1-nodes, and 1-graphs, a 1-section is defined exactly as before [4, page 49]; it is a subgraph [4, page 23] induced by a maximal set of branches that are pairwise 1-connected through 1-paths. Unfortunately, 1-sections can overlap because 1-connectedness is not in general transitive. For example, consider Fig. 3 again; for each $m = 1, 2, 3, \ldots$, the subgraph between any x^1_m and x^1_{m+2} is a 1-section, and thus every two consecutive 1-sections overlap.

However, we can define a more general kind of connectedness, namely, "1-wconnectedness". Two branches (resp. two nodes) will be called 1-*wconnected* if there exists a two-ended 0-

walk or 1-walk that terminates at those branches, that is, at a 0-node of each branch (resp. that terminates at those two nodes). We will say that two 1-walks form a *conjunction* if a terminal node of one 1-walk embraces or is embraced by a terminal node of another 1-walk. In this case, the two 1-walks taken together form another 1-walk, obviously. It follows that 1-wconnectedness is a transitive binary relationship for the branch set $\mathcal{B}$ and is in fact an equivalence relationship.

Furthermore, if every two branches in the 1-graph $\mathcal{G}^1$ are 1-wconnected, we shall say that $\mathcal{G}^1$ is 1-*wconnected.* The 1-graphs of Figs 3 and 4 are 1-wconnected. Also, a subgraph induced by a maximal set of branches that are pairwise 1-wconnected will be called a 1-*wsection* and also a *component.*[2] Thus, if two branches or two nodes are 1-connected, they are also 1-wconnected because a 1-path is a special case of a 1-walk.

Our next objective is to define the new kind of extremity of a 1-graph illustrated in Examples 4.1 and 4.2. A one-ended 1-walk will be called *extended* if its 1-nodes x_m^1 are eventually pairwise distinct.[3] Now, consider an extended one-ended 1-walk starting at the node x_0:

$$W^1 = \{x_0, W_0^0, x_1^1, W_1^0, x_2^1, W_2^0, \ldots\} \tag{3}$$

For each $m \geq 0$, there exist two tips s_m and t_m (elementary tips or 0-tips) with which W_m^0 reaches x_m^1 and x_{m+1}^1 respectively. (W_0^0 reaches x_0 with s_0.) Thus, we have a sequence of tips:

$$\{s_0, t_0, s_1, t_1, s_2, t_2, \ldots\} \tag{4}$$

corresponding to (3), where at least one of t_m and s_{m+1} is a 0-tip, whatever be m. When W^1 is extended, the tips in (4) are eventually pairwise distinct.

Two extended one-ended 1-walks will be called *equivalent* if their tip sequences are eventually identical. This is truly an equivalence relationship for the set of all such 1-walks,

[2]Presently, a 1-wsection $\mathcal{G}^1$ is the same as a component of $\mathcal{G}^1$, but, later on, when we consider wgraphs of higher ranks, 1-wsections will be in general different from components. In fact, 1-wsections will partition those wgraphs in the sense that each branch will lie in exactly one 1-wsection. Also, different 1-wsections may be wconnected through walks of higher ranks than 1. In any case, 1-wsections do not overlap. In contrast, a component will be a maximal subgraph whose branches are wconnected by walks of arbitrary ranks; thus, different components will not be wconnected to each other.

[3]Equivalently, we need merely require that each 1-node x_m^1 appear only finitely many times in (2), for, by removing loops, we can obtain an extended 1-walk.

and thus that set is partitioned into equivalence classes, each of which will be called a 1-*wtip*. Specifically, with $\mathcal{W}^1$ denoting the set of all extended one-ended 1-walks, we have $\mathcal{W}^1 = \cup_{j \in J_1} \mathcal{W}_j^1$, where J_1 is an index set for the partition of $\mathcal{W}^1$ into 1-wtips $\mathcal{W}_j^1$. For any extended 1-walk W^1 in $\mathcal{W}_j^1$, we refer to $\mathcal{W}_j^1$ as the *terminal* 1-*wtip* of W^1, or simply as the 1-*wtip* of W^1.

Note that this equivalence relationship depends only on the tips s_m and t_m for all m sufficiently large. The equivalence relationship is not disturbed by changing some branches and 0-nodes in the 0-walks W_m^0 so long as the tips s_m and t_m remain unchanged except possibly for finitely many of them. The reason for this choice of definition is that we feel intuitively that the infinite extremities of the 1-walks illustrated in Figs. 3 and 4 should not depend on how each 0-walk between consecutive 1-nodes extends into its 0-section; the same infinite extremity should ensue. All this stands in contrast to the definition of 1-tips in [4, page 30], which depended upon the choice of the 0-path between consecutive 1-nodes. We could have developed a theory of transfinite graphs based upon the tip sequences of one-ended 1-paths, but did not do so in [4]. There is nothing God-given in our choice of definitions. All that is required (and hoped for) is that the definitions produce a self-consistent mathematical structure. We believe that such has been accomplished in both [4] and herein.

A 1-walk is said to *traverse* an elementary tip (resp. 0-tip or 1-wtip) if the 1-walk embraces the branch for that elementary tip (resp. embraces the branches of a representative of that 0-tip or 1-wtip).

The next step is to define a "2-wnode." First, we arbitrarily partition the set $\mathcal{P}^1 = \{\mathcal{W}_j^1\}_{j \in J_1}$ of 1-wtips, assuming there are 1-wtips, into subsets $\mathcal{P}_i^1$: $\mathcal{P}^1 = \cup_{i \in I_1} \mathcal{P}_i^1$, where I_1 is the index set for the partition, $\mathcal{P}_i^1 \neq \emptyset$ for all i, and $\mathcal{P}_i^1 \cap \mathcal{P}_k^1 = \emptyset$ if $i \neq k$. Next, for each $i \in I_1$, let $\mathcal{N}_i^1$ be either the empty set or a singleton whose only member is a 0-node or 1-node. Then, for each $i \in I_1$, we define the 2-wnode x_i^2 as

$$x_i^2 = \mathcal{P}_i^1 \cup \mathcal{N}_i^1 \tag{5}$$

so long as the following condition is satisfied: Whenever $\mathcal{N}_i^1$ is not empty, its single α-node $(0 \leq \alpha \leq 1)$ is not a member of any other 1-node or any other 2-wnode.

We say that a 2-wnode x_i^2 *embraces* itself, all its elements including the node of $\mathcal{N}_i^1$ if $\mathcal{N}_i^1$ is not empty, the 0-node of the 1-node in $\mathcal{N}_i^1$ if such a 1-node and 0-node exist, and the elementary tips in that 0-node. More concisely, we say that x_i^2 embraces itself, all its elements, and all the elements embraced by its elements.) We also say that x_i^2 *shorts together* all its embraced elements. Furthermore, we say that an α-walk ($0 \leq \alpha \leq 1$) *reaches* a 2-wnode if the α-walk traverses an α-wtip embraced by the 2-wnode.[4]

We now define a 2-wgraph $\mathcal{G}_w^2$ as the quadruplet:

$$\mathcal{G}_w^2 = \{\mathcal{B}, \mathcal{X}^0, \mathcal{X}^1, \mathcal{X}^2\} \tag{6}$$

where $\mathcal{B}$ is the set of branches, $\mathcal{X}^0$ and $\mathcal{X}^1$ are the sets of 0-nodes and 1-nodes respectively as defined in [4], and $\mathcal{X}^2$ is the set of 2-wnodes.

To save words later on, we shall also refer to the 0-nodes, 1-nodes, 0-graphs, and 1-graphs as 0-*wnodes*, 1-*wnodes*, 0-*wgraphs*, and 1-*wgraphs*, respectively.

Let us note at this point that a 2-graph, as defined in [4], is a 2-wgraph if the following condition always holds: Let t_a^1 and t_b^1 be two 1-tips of $\mathcal{G}^2$ as defined in [4], and assume that t_a^1 and t_b^1 have representative 1-paths $\mathcal{P}_a^1$ and $\mathcal{P}_b^1$ whose sequences of tips, as indicated in (4), are eventually identical; then, t_a^1 and t_b^1 are shorted together (i.e., are members of the same 2-node). Note also, that $\mathcal{G}^2$ may have 1-wtips (as defined above) that are not 1-tips (as defined in [4]); these are left open, that is, they are members of singleton 2-wnodes.

As was observed above, 1-wconnectedness is an equivalence relationship for the branch set $\mathcal{B}$ of $\mathcal{G}_w^2$. Consequently, the 1-wsections partition $\mathcal{G}_w^2$, in contrast to the possibly overlapping 1-sections.

Again to conform with general terminology, we will refer to tips of ranks -1 or 0 as "(-1)-wtips" (alternatively, "elementary wtips") and "0-wtips", respectively. Similarly, a (-1)-wsection (resp. a 0-wsection) is understood to be a branch along with its incident 0-nodes (resp. a 0-section). An α-wtip ($-1 \leq \alpha \leq 1$) is said to be *embraced by* or to be *in* or to *belong to* a β-wsection $\mathcal{S}_w^\beta$ ($\alpha \leq \beta \leq 1$) if the branches of any one (and therefore of all) of its representative paths are all in $\mathcal{S}_w^\beta$.

[4] For the special case of a single branch connected between two 2-wtips, we take that branch and its two 0-nodes as comprising a 0-walk, in which case the 0-walk *reaches* the 2-wnode through a (-1)-tip of the branch.

A *bordering wnode* x^β of an α-wsection $\mathcal{S}_w^\alpha$ $(-1 \leq \alpha \leq 1)$ is a wnode of rank greater than α (i.e., $\beta > \alpha$) that embraces a wtip of $\mathcal{S}_w^\alpha$. A wnode of $\mathcal{S}_w^\alpha$ that is not embraced by a bordering wnode of $\mathcal{S}_w^\alpha$ is called an *internal* wnode of $\mathcal{S}_w^\alpha$. A *boundary wnode* of $\mathcal{S}_w^\alpha$ is a bordering wnode of $\mathcal{S}_w^\alpha$ that also embraces a wtip not belonging to $\mathcal{S}_w^\alpha$. Thus, a boundary β-wnode must be incident to two or more α-sections, but a bordering β-wnode may be incident to only one α-wsection.

An α-walk $(0 \leq \alpha \leq 1)$ *through* a 1-wsection $\mathcal{S}_w^1$ of $\mathcal{G}_w^2$ or simply a walk *through* $\mathcal{S}_w^1$ from one bordering wnode x^2 of $\mathcal{S}_w^1$ to another bordering wnode y^2 of $\mathcal{S}_w^1$ is an α-walk whose branches are all in $\mathcal{S}_w^1$ and that reaches x^2 and y^2.

A *nontrivial 2-walk* is an alternating sequence of 2-wnodes x_m^2 and nontrivial α_m-walks $W_m^{\alpha_m}$ $(0 \leq \alpha_m \leq 1)$ through 1-wsections:

$$W^2 \;=\; \{\ldots, x_{m-1}^2, W_{m-1}^{\alpha_{m-1}}, x_m^2, W_m^{\alpha_m}, x_{m+1}^2, \ldots\} \qquad (7)$$

such that, for each m, x_m^2 and x_{m+1}^2 are bordering 2-wnodes of the 1-wsection through which $W_m^{\alpha_m}$ passes, $W_m^{\alpha_m}$ reaches x_m^2 and x_{m+1}^2, and either $W_{m-1}^{\alpha_{m-1}}$ or $W_m^{\alpha_m}$ (perhaps both) reaches x_m^2 through a 1-wtip. It is also required that, if the sequence (7) terminates on either side, it does so at a wnode x of rank 2 or less. Again, we say that (7) *terminates at* a wnode y if it terminates at a wnode x that embraces or is embraced by y. *Terminal wnodes* and *terminal wtips* are defined as before for 1-walks that terminate on either side. (If (7) extends infinitely in either direction, W^2 has another kind of terminal wtip—a 2-wtip, which will be defined when we discuss the general case of a wgraph of rank higher than 2.)

A *trivial 2-walk* is a singleton containing a 2-wnode.

As usual, a 2-walk is either *two-ended, one-ended* or *endless* whenever (7) terminates on both sides, just on one side, or on neither side, respectively. A one ended 2-walk will be called *extended* if its 2-wnodes x_m^2 are eventually pairwise distinct.

Let $s_m^{\sigma_m}$ (resp. $t_m^{\tau_m}$) be the σ_m-wtip (resp. τ_m-wtip) with which $W_m^{\alpha_m}$ reaches x_m^2 (resp. x_{m+1}^2) in (7). Thus, $-1 \leq \sigma_m, \tau_m \leq 1$, and at least one of τ_m and σ_{m+1} equals 1 for every m. When (7) terminates on the left, we may write the corresponding sequence of wtips as

$$\{s_0^{\sigma_0}, t_0^{\tau_0}, s_1^{\sigma_1}, t_1^{\tau_1}, s_2^{\sigma_2}, t_2^{\tau_2}, \ldots\}. \qquad (8)$$

Wtips can repeat in this sequence, but, if (7) is extended, the wtips eventually do not repeat except possibly $t_m^{\tau_m} = s_{m+1}^{\sigma_{m+1}}$ for various m. Two extended one-ended 2-walks will be called *equivalent* if their wtip sequences are eventually identical. This, too, is an equivalence relationship, and it partitions the set of extended one-ended 2-walks into subsets, which we refer to as the 2-*wtips* of $\mathcal{G}_w^2$. We now take these 2-wtips to be the "infinite extremities" of $\mathcal{G}_w^2$.

5 The Infinite Extremities of μ-wgraphs and the Construction of a $(\mu+1)$-wgraph

We are now ready to construct recursively a wgraph of any natural-number rank. We assume that ρ-wgraphs $\mathcal{G}_w^\rho$ have been constructed for all natural-number ranks ρ up to and including some natural-number rank μ. We have done so explicitly for $\mu = 2$, but remember that 0-wgraphs and 1-wgraphs are the same as 0-graphs and 1-graphs as defined in [4].

Our recursive assumptions have it that, given any ranks α and β with $0 \leq \alpha < \beta \leq \mu$, the α-wsections partition each β-wsection. We also have recursively the following definition: An α-walk *through* a ρ-wsection $\mathcal{S}_w^\rho$ of $\mathcal{G}_w^\mu$ (where now $0 \leq \alpha \leq \rho \leq \mu-1$) from one bordering wnode x to another bordering wnode y of $\mathcal{S}_w^\rho$ (the ranks of x and y being greater than ρ) is an α-walk whose branches are all in $\mathcal{S}_w^\rho$ and that reaches x and y through the terminal wtips of the α-walk. (Possibly, $x = y$.) Here, too, as a special case we may have a single branch comprising all of $\mathcal{S}_w^\rho$ and incident to x and y, in which case that branch along with its incident 0-nodes is a 0-walk through $\mathcal{S}_w^\rho = \mathcal{S}_w^0$; those 0-nodes are embraced by bordering nodes x and y of $\mathcal{S}_w^\rho$ of ranks greater than ρ and are reached through the (-1)-tips of the branch.

A *nontrivial μ-walk* W^μ is an alternating sequence of μ-wnodes x_m^μ and nontrivial α_m-walks $W_m^{\alpha_m}$ $(0 \leq \alpha_m \leq \mu - 1)$ through $(\mu-1)$-wsections:

$$W^\mu = \{\ldots, x_{m-1}^\mu, W_{m-1}^{\alpha_{m-1}}, x_m^\mu, W_m^{\alpha_m}, x_{m+1}^\mu, \ldots\} \tag{9}$$

where, for each m, x_m^μ and x_{m+1}^μ are bordering wnodes of the $(\mu-1)$-wsection through which $W_m^{\alpha_m}$ passes, $W_m^{\alpha_m}$ reaches x_m^μ and x_{m+1}^μ through its terminal wtips, and either $W_{m-1}^{\alpha_{m-1}}$ or $W_m^{\alpha_m}$ (perhaps both) reaches x_m^μ through a $(\mu-1)$-wtip; it is also required that, if the sequence (9) terminates on either side, it terminates at a wnode x of rank μ or less. We call that wnode x a *terminal wnode* of (9), and we call the wtip with which the adjacent walk in (9) reaches that terminal wnode x a *terminal wtip* of (9). We shall also say that (9) *terminates* at a wnode y if the wnode x embraces or is embraced by y. (Here, too, if (9) extends infinitely on either side, W^μ will have a "terminal μ-wtip" on that side; it will be defined below.)

A μ-walk is said to *embrace* itself, all its elements, and all the elements its elements embrace.

A nontrivial μ-walk is called *two-ended*, or *one-ended*, or *endless* if the sequence (9) terminates on both sides, or just on one side, or on neither side, respectively.

A one-ended μ-walk is called *extended* if its μ-wnodes are eventually pairwise distinct.

A *trivial μ-walk* is a singleton containing a μ-wnode.

We now assume that $\mathcal{G}_w^\mu$ has at least one extended one-ended μ-walk. Corresponding to that μ-walk, we have a sequence of wtips just like (8):

$$\{s_0^{\sigma_0}, t_0^{\tau_0}, s_1^{\sigma_1}, t_1^{\tau_1}, s_2^{\sigma_2}, t_2^{\tau_2}, \ldots\} \tag{10}$$

except that now $-1 \leq \sigma_m, \tau_m \leq \mu - 1$ and at least one of τ_m and σ_{m+1} equals $\mu - 1$ for every m. Two extended one-ended μ-walks will be considered *equivalent* if their wtip sequences are eventually identical. This, too, is an equivalence relationship, and it partitions the set $\mathcal{W}^\mu$ of all extended one-ended μ-walks into subsets $\mathcal{W}_j^\mu$, which we refer to as the *μ-wtips* of $\mathcal{G}_w^\mu$. More specifically, with J_μ denoting an index set for the partition, $\mathcal{W}^\mu = \cup_{j \in J_\mu} \mathcal{W}_j^\mu$, where $\mathcal{W}_j^\mu \neq \emptyset$ for all j and $\mathcal{W}_j^\mu \cap \mathcal{W}_k^\mu = \emptyset$ if $j \neq k$. Also, any member of a μ-wtip is called a *representative* of that μ-wtip.

A one-ended μ-walk is said to *traverse* a μ-wtip or to *have* that μ-wtip as a *terminal μ-wtip* if that μ-walk is extended and is a member of that μ-wtip. Similarly, an endless μ-walk is said to traverse two μ-wtips (possibly the same μ-wtip) if it is extended on both sides and the two one-ended μ-walks obtained by separating the endless μ-walk into two one-ended μ-walks traverse those μ-wtips.

Next, we arbitrarily partition the set $\mathcal{P}^\mu = \{\mathcal{W}_j^\mu\}_{j \in J_\mu}$ of μ-wtips into subsets $\mathcal{P}_i^\mu$: $\mathcal{P}^\mu = \cup_{i \in I_\mu} \mathcal{P}_i^\mu$, where I_μ is an index set for the partition, $\mathcal{P}_i^\mu \neq \emptyset$ for all $i \in I_\mu$, and $\mathcal{P}_i^\mu \cap \mathcal{P}_q^\mu = \emptyset$

if $i \neq q$. Furthermore, for each $i \in I_\mu$, let $\mathcal{N}_i^\mu$ be either the empty set or a singleton whose sole member is an α-wnode where $0 \leq \alpha \leq \mu$. For each $i \in I_\mu$, we define a $(\mu+1)$-wnode $x_i^{\mu+1}$ by

$$x_i^{\mu+1} = \mathcal{P}_i^\mu \cup \mathcal{N}_i^\mu \tag{11}$$

so long as the following condition is satisfied: Whenever $\mathcal{N}_i^\mu$ is nonempty, its single α-wnode x_i^α is not a member of another β-wnode $(\alpha < \beta \leq \mu+1)$.

We define "embrace" exactly as in [4]. In particular, the sole α-wnode of $\mathcal{N}_i^\mu$, if it exists, is called the *exceptional element* of $x_i^{\mu+1}$, and that α-wnode may contain an exceptional element (an α_1-wnode with $0 \leq \alpha_1 < \alpha$) of lower rank, which in turn may contain an exception element of still lower rank, and so on through finitely many decreasing ranks. We say that $x_i^{\mu+1}$ *embraces* itself, its exceptional element x^α if that exists, the exceptional element x^{α_1} contained in x^α if that exists, and so on down through finitely many exceptional elements. We also say that $x_i^{\mu+1}$ *embraces* its μ-wtips as well as all the wtips in all those exceptional elements. Furthermore, we say that $x_i^{\mu+1}$ *shorts together* all its embraced elements. If $x_i^{\mu+1}$ is a singleton, its sole μ-wtip is said to be *open*. Also, any ρ-wnode x^ρ $(0 \leq \rho \leq \mu+1)$ is said to be *maximal* if x^ρ is not embraced by a wnode of higher rank.[5]

It follows exactly as in the proof of [4, Lemma 2.2-1] that, if x^α and y^β are an α-wnode and a β-wnode respectively with $0 \leq \alpha \leq \beta$ and if x^α and y^β embrace a common wnode, then y^β embraces x^α, and moreover $x^\alpha = y^\beta$ if $\alpha = \beta$.

Next, we define the $(\mu+1)$-wgraph to be

$$\mathcal{G}_w^{\mu+1} = \{\mathcal{B}, \mathcal{X}^0, \mathcal{X}^1, \ldots, \mathcal{X}^{\mu+1}\} \tag{12}$$

where $\mathcal{B}$ is a branch set and, for each $\rho = 0, \ldots, \mu+1$, $\mathcal{X}^\rho$ is a nonempty set of ρ-wnodes built up from the wnodes and walks of lower ranks as stated. Also, for each ρ, the subset $\mathcal{G}_w^\rho = \{\mathcal{B}, \mathcal{X}^0, \ldots, \mathcal{X}^\rho\}$ is called the *ρ-wgraph of* $\mathcal{G}_w^{\mu+1}$.

Just as certain 2-graphs are 2-wgraphs (see the preceding Section), certain μ-graphs as defined in [4, page 31] are μ-wgraphs.

[5] Of course, $x_i^{\mu+1}$ is automatically maximal because there are no nodes of higher rank at this stage of our recursive development.

An α-walk $(0 \le \alpha \le \mu)$ is said to *reach* a $(\mu+1)$-wnode if the α-walk traverses an α-wtip embraced by the $(\mu+1)$-wnode, in which case we say that the α-walk *reaches* the $(\mu+1)$-wnode *through* that α-wtip. As a special case, we view a branch as traversing two (-1)-tips; thus, a branch *reaches* a $(\mu+1)$-wnode *through* a (-1)-tip if one of its (-1)-tips is embraced by the $(\mu+1)$-wnode.

We now complete this cycle of our recursive development with a few more definitions. Let $\mathcal{B}_s$ be a subset of the the branch set $\mathcal{B}$ of $\mathcal{G}_w^{\mu+1}$. For each $\rho = 0, \ldots, \mu+1$, we let $\mathcal{X}_s^\rho$ be the set of ρ-wnodes in $\mathcal{X}^\rho$ each of which contains a $(\rho-1)$-wtip having a representative all of whose branches are in $\mathcal{B}_s$. (These wnodes may also contain other $(\rho-1)$-wtips.) Then,

$$\mathcal{G}_s^{\mu+1} = \{\mathcal{B}_s, \mathcal{X}_s^0, \mathcal{X}_s^1, \ldots, \mathcal{X}_s^{\mu+1}\} \tag{13}$$

is called the *wsubgraph of* $\mathcal{G}_w^{\mu+1}$ *induced by* $\mathcal{B}_s$. Any one of the $\mathcal{X}_s^\rho$ may be empty, but there will be a maximum rank λ for which the $\mathcal{X}_s^\rho$ are nonempty for $\rho = 0, \ldots, \lambda$ and empty for $\rho = \lambda+1, \ldots, \mu+1$.[6] It can also happen that $\mathcal{X}_s^0$ is empty or that $\mathcal{X}_s^{\mu+1}$ is nonempty. In general, a wsubgraph is not a wgraph because its wnodes may contain certain wtips having no representatives with all branches in $\mathcal{B}_s$.

For each $\rho = 0, \ldots, \mu$, two branches (resp. two wnodes) in $\mathcal{G}_w^{\mu+1}$ are said to be *ρ-wconnected* if there exists a two-ended α-walk $(0 \le \alpha \le \rho)$ that terminates at a 0-node of each branch (resp. at the two wnodes). Two walks W^α and W^β with $0 \le \alpha, \beta \le \mu$ are said to be *in conjunction* if a terminal wnode of W^α embraces or is embraced by a terminal wnode of W^β. The conjunction of W^α and W^β is a walk of rank $\max(\alpha, \beta)$, as is easily seen. It follows that ρ-wconnectedness is a transitive and indeed and an equivalence relationship for the set $\mathcal{B}$ of branches in $\mathcal{G}_w^{\mu+1}$.

A *ρ-wsection* $\mathcal{S}_w^\rho$ $(0 \le \rho \le \mu)$ of $\mathcal{G}_w^{\mu+1}$ is a subgraph of $\mathcal{G}_w^{\mu+1}$ induced by a maximal set of branches that are pairwise ρ-wconnected. Because ρ-wconnectedness is an equivalence relationship between branches, the ρ-wsections *partition* $\mathcal{G}_w^{\mu+1}$ (i.e., each branch is in exactly one ρ-wsection). In fact, if $0 \le \rho < \lambda \le \mu+1$, any λ-wsection is partitioned by the ρ-wsections within it because ρ-wconnectedness implies λ-wconnectedness. We say that an

[6] See [4, page 32] for the argument establishing this.

α-wtip ($0 \leq \alpha \leq \rho$) is *traversed by* $\mathcal{S}_w^\rho$ or *belongs to* $\mathcal{S}_w^\rho$ or is *in* $\mathcal{S}_w^\rho$ if all the branches of any representative (and, therefore, of all representatives) of that α-wtip are in $\mathcal{S}_w^\rho$.

A *bordering wnode* of a ρ-wsection $\mathcal{S}_w^\rho$ is a wnode of rank greater than ρ that embraces a wtip belonging to $\mathcal{S}_w^\rho$. A wnode of $\mathcal{S}_w^\rho$ that is not embraced by a bordering wnode of $\mathcal{S}_w^\rho$ is called an *internal wnode* of $\mathcal{S}_w^\rho$. A *boundary wnode* of $\mathcal{S}_w^\rho$ is a bordering wnode that also embraces a wtip not belonging to $\mathcal{S}_w^\rho$.

An α-walk *through* a ρ-wsection is defined as before, but now ρ may equal μ. We can now define a $(\mu+1)$-walk exactly as was a μ-walk (9) except that μ is replaced by $\mu+1$. Other definitions are so-extended, too. For instance, $\mathcal{G}_w^{\mu+1}$ is said to be $(\mu+1)$-*wconnected* if, for every two branches, there exists a walk of rank less than or equal to $\mu+1$ that terminates at 0-nodes of those branches. The $(\mu+1)$-wsections of $\mathcal{G}_w^{\mu+1}$ are the *components* of $\mathcal{G}_w^{\mu+1}$. Moreover, we have the wtip sequence of any extended one-ended $(\mu+1)$-walk, and thus the equivalence between two such $(\mu+1)$-walks as before. The resulting equivalence classes are the $(\mu+1)$-wtips, which are taken to be the infinite extremities of $\mathcal{G}_w^{\mu+1}$.

We have completed one more cycle of our recursive development of wgraphs.

6 $\vec{\omega}$-wgraphs

We now assume that there is a wgraph having wnodes of all natural-number ranks. That is, the process of establishing a μ-wgraph from ρ-wgraphs ($\rho = 0, 1, \ldots, \mu-1$) has continued unceasingly, always yielding wnodes of ever-increasing ranks. Thus, we have nonempty wnodes sets $\mathcal{X}^0, \mathcal{X}^1, \ldots, \mathcal{X}^\mu, \ldots$ for all natural-number ranks μ.

As in [4, page 4], $\vec{\omega}$ denotes the *arrow rank* that precedes the limit-ordinal rank ω and is larger than every natural-number rank μ. We now define another kind of wnode $x^{\vec{\omega}}$ of rank $\vec{\omega}$. $x^{\vec{\omega}}$ is an infinite sequence of μ_k-wnodes $x_k^{\mu_k}$ ($k = 0, 1, 2, \ldots$):

$$x^{\vec{\omega}} \;=\; \{x_0^{\mu_0}, x_1^{\mu_1}, x_2^{\mu_2}, \ldots\} \tag{14}$$

where the μ_k are increasing natural numbers: $\mu_0 < \mu_1 < \mu_2 < \ldots$, each $x_k^{\mu_k}$ is the exceptional element of $x_{k+1}^{\mu_{k+1}}$, and $x_0^{\mu_0}$ does not have an exceptional element. (In the definition of an $\vec{\omega}$-wgraph given below, it is not required that there be any $\vec{\omega}$-wnodes.) As usual, we say that $x^{\vec{\omega}}$ *embraces* itself, all its μ_k-wnodes, and all the wtips in those μ_k-wnodes. Here, too, it can be shown as an easy consequence of [4, Lemma 2.2-1] that, if two $\vec{\omega}$-wnodes embrace a common μ-wnode of natural-number rank μ, then the two $\vec{\omega}$-wnodes are the same; also, if an $\vec{\omega}$-wnode $x^{\vec{\omega}}$ and a μ-wnode x^{μ} embrace a common wnode, then $x^{\vec{\omega}}$ embraces x^{μ}. As with μ-wnodes, we speak of $x^{\vec{\omega}}$ as *shorting together* its embraced elements.

We now define an $\vec{\omega}$-wgraph $\mathcal{G}_w^{\vec{\omega}}$ of rank $\vec{\omega}$ to be the infinite set of sets:

$$\mathcal{G}_w^{\vec{\omega}} = \{\mathcal{B}, \mathcal{X}^0, \mathcal{X}^1, \ldots, \mathcal{X}^{\vec{\omega}}\} \tag{15}$$

where $\mathcal{B}$ is a set of branches, each $\mathcal{X}^{\mu}$ $(\mu = 0, 1, 2, \ldots)$ is a nonempty set of μ-wnodes, and $\mathcal{X}^{\vec{\omega}}$ is a (possibly empty) set of $\vec{\omega}$-wnodes.

Given any subset $\mathcal{B}_s$ of $\mathcal{B}$, we define the *wsubgraph of* $\mathcal{G}_w^{\vec{\omega}}$ *induced by* $\mathcal{B}_s$ exactly as was a wsubgraph defined in the preceding Section (see(13)) except that now there may be infinitely many nonempty wnodes sets $\mathcal{X}_s^{\mu}$ $(\mu = 0, 1, 2, \ldots)$ and perhaps a nonempty $\mathcal{X}_s^{\vec{\omega}}$ inserted into the right-hand side of (13).

Two branches (resp. two wnodes) are said to be *$\vec{\omega}$-wconnected* if there exists a two-ended walk of any natural-number rank that terminates at 0-nodes of those two branches (resp. at those two wnodes). $\vec{\omega}$-wconnectedness is an equivalence relationship for the branch set $\mathcal{B}$; indeed, it is clearly reflexive and symmetric—and also transitive because the conjunction of two walks of natural-number ranks is again a walk with a natural-number rank.

An *$\vec{\omega}$-wsection* is a wsubgraph of $\mathcal{G}_w^{\vec{\omega}}$ induced by a maximal set of branches that are $\vec{\omega}$-wconnected. Because $\vec{\omega}$-wconnectedness is an equivalence relationship for $\mathcal{B}$, the $\vec{\omega}$-wsections partition $\mathcal{G}_w^{\vec{\omega}}$. Presently, $\vec{\omega}$-wsections are simply *components* of $\mathcal{G}_w^{\vec{\omega}}$, that is, there is no walk of any rank connecting branches in two different $\vec{\omega}$-wsections, but, later on when we define ω-wgraphs, there may be such walks of rank ω.

Also, a ρ-wsection of $\mathcal{G}_w^{\vec{\omega}}$, where again ρ is a natural number, is defined exactly as it was in the preceding Section. With λ being any natural number with $\lambda > \rho$, any λ-wsection is partitioned by the ρ-wsections within it, and so, too, is $\mathcal{G}_w^{\vec{\omega}}$.

There is no such thing as a two-ended $\vec{\omega}$-walk, but we can define one-ended and endless $\vec{\omega}$-walks. In doing so, we wish to do it in such a fashion that a unique sequence of wtips, somewhat like that of (10), identifies a one-ended $\vec{\omega}$-walk as a representative of an infinite extremity of $\mathcal{G}_w^{\vec{\omega}}$.

A one-ended $\vec{\omega}$-walk is a one-way infinite alternating sequence of μ_m-wnodes $x_m^{\mu_m}$ and nontrivial α_m-walks $W_m^{\alpha_m}$:

$$W^{\vec{\omega}} = \{x_0^{\mu_0}, W_0^{\alpha_0}, x_1^{\mu_1}, W_1^{\alpha_1}, x_2^{\mu_2}, \ldots, W_{m-1}^{\alpha_{m-1}}, x_m^{\mu_m}, W_m^{\alpha_m}, x_{m+1}^{\mu_{m+1}}, \ldots\} \tag{16}$$

where the μ_m comprise a strictly increasing sequence of natural numbers: $\mu_0 < \mu_1 < \mu_2 < \ldots$, and $0 \leq \alpha_m < \mu_{m+1}$ for each $m = 0, 1, 2, \ldots$; moreover, $W_m^{\alpha_m}$ reaches $x_m^{\mu_m}$ and $x_{m+1}^{\mu_{m+1}}$ through its terminal wtips, and at least one of $W_{m-1}^{\alpha_{m-1}}$ and $W_m^{\alpha_m}$ reaches $x_m^{\mu_m}$ through a $(\mu_m - 1)$-wtip.

An endless $\vec{\omega}$-walk is the *conjunction* of two one-ended $\vec{\omega}$-walks in the sense that the terminal wnode of one one-ended $\vec{\omega}$-walk embraces or is embraced by the terminal wnode of the other one.

There are many ways of representing a given one-ended $\vec{\omega}$-walk $W^{\vec{\omega}}$ as in (16) because the $x_m^{\mu_m}$ and $W_m^{\alpha_m}$ can be chosen in different ways. In order to get a unique sequence of wtips characterizing $W^{\vec{\omega}}$ as stated above, we proceed as follows. First of all, since $x_0^{\mu_0}$ need not be a maximal wnode, we can write (16) is such a fashion that $W_0^{\alpha_0}$ reaches $x_0^{\mu_0}$ through a σ_0-wtip $s_0^{\sigma_0}$ where $\mu_0 = \sigma_0 + 1$. Similarly, no $x_m^{\mu_m}$ need be maximal; therefore, we can let $W_{m-1}^{\alpha_{m-1}}$ reach $x_m^{\mu_m}$ through a τ_{m-1}-wtip $t_{m-1}^{\tau_{m-1}}$ and let $W_m^{\alpha_m}$ reach $x_m^{\mu_m}$ through a σ_m-wtip $s_m^{\sigma_m}$ where $\mu_m = \max(\tau_{m-1}, \sigma_m) + 1$. Under this condition, we can continue to assume that $\{\mu_m\}_{m=0}^{\infty}$ is a strictly increasing sequence. Furthermore, we can let $x_1^{\mu_1}$ be the first wnode after $x_0^{\mu_0}$ of rank μ_1 greater than μ_0 that $W^{\vec{\omega}}$ meets in accordance with its representation (16). In general, for each m, we can let $x_{m+1}^{\mu_{m+1}}$ be the first wnode after $x_m^{\mu_m}$ of rank μ_{m+1} greater than μ_m that $W^{\vec{\omega}}$ meets in accordance with its representation (16). Under these additional conditions, we refer to $W^{\vec{\omega}}$ as a *canonical* $\vec{\omega}$-walk. In this way, every $\vec{\omega}$-walk has a unique canonical form, and that form has a unique set of wtips:

$$\{s_0^{\sigma_0}, t_0^{\tau_0}, \ldots, t_{m-1}^{\tau_{m-1}}, s_m^{\sigma_m}, t_m^{\tau_m}, \ldots\} \tag{17}$$

such that, for each $m \geq 0$, $\mu_m = \max(\tau_{m-1}, \sigma_m) + 1$, as above. The sequence (17) characterizes the way $W^{\vec{\omega}}$ leaves and enters wsections of increasing ranks. Indeed, there will be a nested sequence of wsections $\mathcal{S}_w^{\mu_m}$ of increasing ranks:

$$\mathcal{S}_w^{\mu_1} \subset \mathcal{S}_w^{\mu_2} \subset \mathcal{S}_w^{\mu_3} \subset \ldots \subset \mathcal{S}_w^{\mu_m} \subset \ldots \tag{18}$$

with $x_m^{\mu_m}$ being a boundary wnode of $\mathcal{S}_w^{\mu_m}$. Moreover, the truncation of (16) at $x_m^{\mu_m}$:

$$\{x_0^{\mu_0}, W_0^{\alpha_0}, x_1^{\mu_1}, W_1^{\mu_1}, \ldots, x_m^{\mu_m}\}$$

lies in $\mathcal{S}_w^{\mu_m}$, and $W^{\vec{\omega}}$ leaves $\mathcal{S}_w^{\mu_m}$ through $x_m^{\mu_m}$ along the wtips $t_{m-1}^{\tau_{m-1}}$ and $s_m^{\sigma_m}$. We shall refer to (17) as the *wtip sequence* of a canonical $\vec{\omega}$-walk, or simply as a *canonical wtip sequence*.

Let us now assume that $\mathcal{G}_w^{\vec{\omega}}$ contains at least one canonical $\vec{\omega}$-walk. Two such canonical $\vec{\omega}$-walks will be considered *equivalent* if their canonical wtip sequences are eventually identical. Thus, those two $\vec{\omega}$-walks "approach infinity" eventually along the same sequence of wnodes of strictly increasing ranks, eventually passing through those wnodes via the same wtips. This truly is an equivalence relationship for the set of canonical $\vec{\omega}$-walks in $\mathcal{G}_w^{\vec{\omega}}$, and the resulting equivalence classes will be called the *$\vec{\omega}$-wtips* of $\mathcal{G}_w^{\vec{\omega}}$. Those $\vec{\omega}$-wtips will be viewed as the "infinite extremities" of $\mathcal{G}_w^{\vec{\omega}}$.

A one-ended $\vec{\omega}$-walk is said to *traverse* an $\vec{\omega}$-wtip if the $\vec{\omega}$-walk is canonical and is a member of the $\vec{\omega}$-wtip. In the same way, an endless $\vec{\omega}$-walk can traverse two $\vec{\omega}$-wtips—or possibly the same $\vec{\omega}$-wtip.

7 ω-wgraphs

With μ-wgraphs (μ a natural number) and $\vec{\omega}$-wgraphs in hand, we can define ω-wgraphs as follows. Assume that an $\vec{\omega}$-wgraph $\mathcal{G}_w^{\vec{\omega}}$ has a nonempty set $\mathcal{P}^{\vec{\omega}}$ of $\vec{\omega}$-wtips. Partition $\mathcal{P}^{\vec{\omega}}$ in any fashion to obtain $\mathcal{P}^{\vec{\omega}} = \cup_{i \in I_{\vec{\omega}}} \mathcal{P}_i^{\vec{\omega}}$, where, as before, $I_{\vec{\omega}}$ is the index set for the partition, $\mathcal{P}_i^{\vec{\omega}}$ is nonempty for all $i \in I_{\vec{\omega}}$, and $\mathcal{P}_i^{\vec{\omega}} \cap \mathcal{P}_k^{\vec{\omega}} = \emptyset$ if $i \neq k$. Also, for each i, let $\mathcal{N}_i^{\vec{\omega}}$ be either the empty set or a singleton whose only member is either a μ-wnode or an $\vec{\omega}$-wnode. We also require that, if $\mathcal{N}_i^{\vec{\omega}}$ is not empty, its sole member is not the member of any other $\mathcal{N}_k^{\vec{\omega}}$ ($k \neq i$). For each $i \in I_{\vec{\omega}}$, the set

$$x_i^{\omega} \;=\; \mathcal{P}_i^{\vec{\omega}} \cup \mathcal{N}_i^{\vec{\omega}} \tag{19}$$

is called an *ω-wnode*. As usual, if $\mathcal{N}_i^{\vec{\omega}}$ is not empty, its wnode is called the *exceptional element* of x_i^{ω}. Also, as before, we say that x_i^{ω} *embraces* itself, all its elements, and all elements embraced by its exceptional element. We say that x_i^ω *shorts together* all its embraced elements. When x_i^ω is a singleton, its one and only $\vec{\omega}$-wtip is said to be *open*.

The following facts can be proven: If an α-wnode $(0 \leq \alpha \leq \vec{\omega})$ and an ω-wnode embrace a common wnode, then the ω-wnode embraces the α-wnode; moreover, if $\alpha = \omega$, then the α-wnode and the ω-wnode are the same wnode. (This is a generalization of Lemma 2.4-1 of [4].)

A one-ended α-walk W^α $(-1 \leq \alpha \leq \vec{\omega})$ is said to *reach* an ω-wnode if the α-walk traverses an α-wtip embraced by the ω-wnode, in which case we say that W^α does so *through* that α-wtip.

Let $\mathcal{X}^\omega$ be the set of all ω-wnodes. An ω-*wgraph* $\mathcal{G}_w^\omega$ is defined to be

$$\mathcal{G}_w^\omega = \{\mathcal{B}, \mathcal{X}^0, \mathcal{X}^1, \ldots, \mathcal{X}^{\vec{\omega}}, \mathcal{X}^\omega\}. \tag{20}$$

$\mathcal{X}^{\vec{\omega}}$ is the set of $\vec{\omega}$-wnodes, which may be empty. All other $\mathcal{X}^\nu$ $(\nu = 0, 1, \ldots, \omega; \nu \neq \vec{\omega})$ have to be nonempty in order for $\mathcal{G}_w^\omega$ to exist.

A wnode in $\mathcal{G}_w^\omega$ is called *maximal* if it is not embraced by a wnode of higher rank.

A *wsubgraph* $\mathcal{G}_s$ of $\mathcal{G}_w^\omega$ induced by a subset $\mathcal{B}_s$ of the branch set $\mathcal{B}$ is defined exactly as before. For instance, with μ being a natural number as always, a μ-wnode x^μ in $\mathcal{G}_w^\omega$ is also a μ-wnode in $\mathcal{G}_s$ if x^μ contains a $(\mu - 1)$-wtip with a representative all of whose branches are in $\mathcal{B}_s$. Similarly, an ω-wnode x^ω in $\mathcal{G}_w^\omega$ is an ω-wnode in $\mathcal{G}_s$ if x^ω contains an $\vec{\omega}$-wtip with a representative all of whose branches are in $\mathcal{B}_s$. On the other hand, if infinitely many of the wnodes in an $\vec{\omega}$-wnode are in $\mathcal{G}_s$, then the set of those wnodes is an $\vec{\omega}$-wnode in $\mathcal{G}_s$.

For $\mathcal{G}_w^\omega$, μ-*wconnectedness* and $\vec{\omega}$-*wconnectedness* are defined exactly as before, as are μ-*wsections* and $\vec{\omega}$-*wsections*, too. Such wsections of a given rank *partition* every wsection of higher rank and $\mathcal{G}_w^\omega$, too. A *bordering wnode* of an $\vec{\omega}$-wsection $\mathcal{S}_w^{\vec{\omega}}$ is an ω-wnode[7] that embraces a wtip traversed by $\mathcal{S}_w^{\vec{\omega}}$. (As before, by *traversed* we mean that the wtip has a representative whose branches are all in $\mathcal{S}_w^{\vec{\omega}}$.) A wnode of $\mathcal{S}_w^{\vec{\omega}}$ that is not embraced by a bordering wnode of $\mathcal{S}_w^{\vec{\omega}}$ is called an *internal* wnode of $\mathcal{S}_w^{\vec{\omega}}$; thus, the rank of an internal wnode of $\mathcal{S}_w^{\vec{\omega}}$ is no greater than $\vec{\omega}$. A *boundary wnode* of an $\vec{\omega}$-wsection $\mathcal{S}_w^{\vec{\omega}}$ is a bordering

[7] In wgraphs of ranks higher than ω, such a bordering wnode of a $\vec{\omega}$-wsection can have a rank higher than ω.

wnode of $\mathcal{S}_w^{\vec{\omega}}$ that also embraces a wtip not traversed by $\mathcal{S}_w^{\vec{\omega}}$.

A *nontrivial ω-walk* W^ω is an alternating sequence of ω-wnodes x_m^ω and nontrivial α_m-walks ($0 \leq \alpha_m \leq \vec{\omega}$) through α_m-wsections:

$$W^\omega \;=\; \{\ldots, x_{m-1}^\omega, W_{m-1}^{\alpha_{m-1}}, x_m^\omega, W_m^{\alpha_m}, x_{m+1}^\omega, W_{m+1}^{\alpha_{m+1}}, x_{m+2}^\omega, \ldots\} \tag{21}$$

such that, for each m, x_m^ω and x_{m+1}^ω are bordering wnodes of the $\vec{\omega}$-wsection through which $W_m^{\alpha_m}$ passes, $W_m^{\alpha_m}$ reaches x_m^ω and x_{m+1}^ω, and either $W_{m-1}^{\alpha_{m-1}}$ or $W_m^{\alpha_m}$ (perhaps both) reaches x_m^ω through an $\vec{\omega}$-wtip; it is also required that, if the sequence (21) terminates on either side, it does so at a wnode x of rank ω or less.. The wnode x is called a *terminal wnode* of W^ω, and the wtip with which the adjacent walk in (21) reaches x is called a *terminal wtip* of W^ω. Here, too, we say that W^ω *terminates* at a wnode y if it terminates at a wnode x that embraces or is embraced by y. We define *two-ended*, *one-ended*, and *endless* ω-walks in the usual way.

A *trivial ω-walk* is a singleton whose only member is an ω-wnode.

Two branches (resp. two wnodes) are said to be *ω-wconnected* if there exists a two-ended β-walk ($0 \leq \beta \leq \omega, \beta \neq \vec{\omega}$) that terminates at 0-nodes of those two branches (resp. at those two wnodes). A *branch-induced wsubgraph* has the usual definition (see (13) but replace $\mu + 1$ by ω). Also, an ω-wsection is a subgraph induced by a maximal set of branches that are ω-wconnected. Without wgraphs of ranks higher than ω being defined, ω-wsections are simply the components of $\mathcal{G}_w^\omega$; that is, there is no walk of any rank terminating at the 0-nodes of two branches in different ω-wsections.

A one-ended ω-walk is called *extended* if its ω-wnodes are eventually pairwise distinct. Corresponding to each $W_m^{\alpha_m}$ in (21), we have two wtips $s_m^{\sigma_m}$ and $t_m^{\tau_m}$ with which $W_m^{\alpha_m}$ reaches x_m^ω and x_{m+1}^ω, respectively. Thus, $-1 \leq \sigma_m, \tau_m \leq \vec{\omega}$ and, for each $m \geq 1$, $\max(\tau_{m-1}, \sigma_m) = \vec{\omega}$. Hence, corresponding to (21), we have a unique sequence of wtips just like (8) except for the revised conditions on the σ_m and τ_m:

$$\{s_0^{\sigma_0}, t_0^{\tau_0}, s_1^{\sigma_1}, t_1^{\tau_1}, s_2^{\sigma_2}, t_2^{\tau_2}, \ldots\} \tag{22}$$

Two one-ended ω-walks will be called *equivalent* if their sequences of wtips, as in (22), are eventually identical. As before, this partitions the set of all one-ended ω-walks into equivalence classes. We refer to these equivalence classes as ω-*wtips* and view them as the "infinite extremities" of $\mathcal{G}_w^\omega$.

We have now completed one more cycle of our recursive development of transfinite wgraphs. We could continue as in Sec. 5 to construct wgraphs of ranks $\omega+1, \omega+2, \ldots$, then $(\omega\vec{\cdot}2)$-wgraphs, (where $\omega\vec{\cdot}2 = \omega + \vec{\omega}$) as in Sec. 6, then $(\omega \cdot 2)$-wgraphs as in this section, and so on through still higher ranks of wgraphs.

8 Branch-Count Distances in Walk-Based Transfinite Graphs

Under the path-based theory of distances in transfinite graphs given in [7], not all pairs of nodes can have a distance between them because certain pairs of transfinitely distant nodes have no transfinite paths connecting them even though they have transfinite walks connecting them. Examples of this are presented below. To circumvent this difficulty, the distance function had to be restricted to subsets of the node set. Such a subset was called a "metrizable node set," and in general a transfinite graph would have many different metrizable node sets.

On the other hand, in the walk-based theory presented herein, a distance function can now be defined on all pairs of nodes. We can also state more general results concerning nodal eccentricities, radii and diameters of walk-based graphs. These quantities take their values in the well-ordered set of transfinite ranks [4, page 4], [6, page 4]. As before, this is the set of all countable ordinals (including the natural numbers) augmented with "arrow ranks." Each arrow rank $\vec{\lambda}$ precedes a limit ordinal λ and is larger than all the ordinals less than λ.[8]

9 More Examples

Example 9.1. Consider again the 1-graph of Fig. 1, which we redraw in Fig. 5 in order to indicate other 0-tips arising from paths that switch infinitely often between the a_k and b_k branches. There are uncountably many such paths and 0-tips; in fact, the cardinality of the set of 0-tips is the cardinality of the continuum. Indeed, at each 0-node encountered in

[8] The first arrow rank $\vec{0}$ is never used in the following, and, when speaking of an arrow rank, we tacitly mean that it is not $\vec{0}$.

a traversal toward the right, one makes a binary decision concerning the next branch a_k or b_k to traverse; thus, each path starting at n^0 can be designated by a binary representation of the real numbers between 0 and 1.

As before, let x^1 (resp y^1) denote the 1-node corresponding to the path proceeding only along the a_k branches (resp. b_k branches). There is a path connecting n^0 to x^1 and another connecting n^0 to y^1, but there is no path connecting x^1 to y^1 because any tracing between x^1 and y^1 must repeat 0-nodes. Thus, no distance can be assigned between x^1 and y^1 if we define distances based upon paths as in [7]. However, there is a walk connecting x^1 and y^1 (in fact, infinitely many of them), and a distance based upon walks can and will be assigned between x^1 and y^1 in Example 11.2. □

Example 9.2. The graph of Fig. 5 appears in modified forms as subgraphs of many other 1-graphs, and thus those 1-graphs may also fail to have distances based upon paths for certain pairs of 1-nodes. Such is the 1-graph of Fig. 6, consisting of a ladder graph with uncountably many singleton 1-nodes, each containing a 0-tip. Here, too, there are uncountably many 0-paths starting at, say, the 0-node x^0 and proceeding infinitely toward the right. Let x_a^1 (resp. x_b^1) denote the 1-node containing the 0-tip of the path proceeding through all the a_k branches (resp. b_k branches with c_0, too). Each of the other 1-nodes corresponds to a path that passes infinitely often through some c_k branches. There is no path but there is a walk connecting any one of those latter 1-nodes to any other 1-node, and so we must resort to distances based upon walks in order to encompass all these 1-nodes in a theory of distances for transfinite graphs. Note, however, that there are paths connecting x_a^1 and x_b^1. Since a path is a special case of a walk, the distance between x_a^1 and x_b^1 as defined below will be the same as that defined in [7]. □

Example 9.3. We redraw Fig. 3 in Fig. 7 to show a new kind of transfinite node of rank 2, which we have called a 2-wnode. There is no path that passes through all the 1-nodes of Fig. 7, but there is a one-ended 1-walk that starts at the 1-node x_1^1 and passes through all the other 1-nodes $x_2^1, x_3^1, x_4^1, \ldots$ in sequence. The infinite extremity for that one-ended 1-walk (called a 1-wtip) was defined in Sec. 4. Furthermore, that 1-wtip is encompassed within the 2-wnode x^2. Once again, a distance based upon walks can be defined between x^2

and any 1-node or 0-node of Fig. 7, something that is impossible if we base our definition of distance upon paths. That distance will be ω^2, according to the distance definition (27) given below.

Now, just as infinitely many replications of the graph of Fig. 1 were connected in series to get the graph of Fig. 3, we can connect in series infinitely many two-way infinite replicates of the graph of Fig. 7 at their 2-wnodes. The resulting graph will have an infinite extremity of still higher rank (called a 2-wtip), which in turn can be encompassed within a transfinite node x^3 of rank 3 (called a 3-wnode). The distance between x^3 and any of the other nodes will be defined to be ω^3.

This construction can be continued on to still higher ranks. □

10 Lengths of Walks

Given any walk W of any rank, $|W|$ will denote its length. A trivial walk of any rank is assigned the length 0. All the walks considered below are understood to be nontrivial unless triviality is explicitly stated.

0-walks: A 0-walk W^0 is simply a walk as defined in conventional graph theory. If W^0 is two-ended (i.e., finite), $|W^0|$ is the number τ_0 of branch traversals in it. In other words, it is a count of the branches in W^0 with each branch counted according to the number of times the branch appears in W^0.[9] We do not assign a length to any arbitrary infinite 0-walk, but, if the 0-walk W^0 is one-ended and extended, we set $|W^0| = \omega$. On the other hand, if W^0 is endless and extended in both directions, we set $|W^0| = \omega \cdot 2$.

1-walks: These are defined by (2) and its associated conditions. We assign a length $|W^1|$ to W^1 by counting tips as follows. If W^1 is one-ended and extended, we set $|W^1| = \omega^2$. If W^1 is endless and extended in both directions, we set $|W^1| = \omega^2 \cdot 2$. If, however, W^1 is two-ended, we set

$$|W^1| = \sum_m |W_m^0|, \tag{23}$$

[9]This why a trivial 0-walk is assigned the length 0, and a similar reasoning is applied to trivial walks of higher ranks.

where the sum is over the finitely many 0-walks W_m^0 in (2); thus, in this case, $|W^1| = \omega \cdot \tau_1 + \tau_0$, where τ_1 is the number of 0-tips W^1 traverses,[10] and τ_0 is the number of branch traversals in all the two-ended 0-walks in (2). In fact, $\sum_m |W^0|$ is the natural sum of ordinals yielding a normal expansion of an ordinal [1, pages 354-355]. τ_1 is not 0 because W^1 is a nontrivial 1-walk; τ_0 may be 0.

2-walks: These are defined by (7) and its associated conditions. If the 2-walk W^2 is one-ended and extended, we set $|W^2| = \omega^3$. If W^2 is endless and extended in both directions, we set $|W^2| = \omega^3 \cdot 2$. On the other hand, if W^2 is two-ended, we set

$$|W^2| \;=\; \sum_m |W_m^{\alpha_m}|, \tag{24}$$

where the summation is over the finitely many α_m-walks in (7). Since each α_m satisfies $0 \leq \alpha_m \leq 1$, each $|W_m^{\alpha_m}|$ is defined as above; here, too, the summation in (24) denotes a normal expansion of an ordinal obtained through a natural sum of ordinals [1, pages 354-355]. So, $|W^2| = \omega^2 \cdot \tau_2 + \omega \cdot \tau_1 + \tau_0$, where the τ_k are natural numbers. τ_2 is the number of 1-wtip traversals for W^2 (counting each 1-wtip by the number of times it is traversed); τ_2 is not 0 because W^2 is two-ended and nontrivial. Also, for $k = 0, 1$, we obtain τ_k by adding ordinals in accordance with the natural summation of ordinals; τ_k can be 0.

μ-walks: We can continue recursively to define in this way the length of a μ-walk, where μ is a natural number. A μ-walk W^μ is defined by (9) and its associated conditions. If W^μ is one-ended and extended, we set $|W^\mu| = \omega^{\mu+1}$. If W^μ is endless and extended in both directions, we set $|W^2| = \omega^{\mu+1} \cdot 2$. If W^μ is two-ended, we recursively apply natural summations to get

$$|W^\mu| \;=\; \sum_m |W^{\alpha_m}| \;=\; \omega^\mu \cdot \tau_\mu + \omega^{\mu-1} \cdot \tau_{\mu-1} + \cdots + \omega \cdot \tau_1 + \tau_0 \tag{25}$$

Here, too, the τ_k ($k = 0, \ldots, \mu$) are natural numbers. Also, τ_μ is not 0, but the τ_k can be 0 if $k < \mu$.

$\vec{\omega}$-walks: As was indicated in Sec. 6, $\vec{\omega}$-walks always have canonical forms, and they are always extended. Moreover, they are either one-ended or endless—never two-ended. The

[10] Each traversal of a given 0-tip adds 1 to the count for τ_1.

length of a canonical one-ended (resp. endless) $\vec{\omega}$-walk is by definition $|W^{\vec{\omega}}| = \omega^\omega$ (resp. $|W^{\vec{\omega}}| = \omega^\omega \cdot 2$).

ω-walks: Finally, we consider an ω-walk W^ω defined by (21) and its associated conditions. If W^ω is one-ended (resp. endless) and extended (resp. extended in both directions), we set $|W^\omega| = \omega^{\omega+1}$ (resp. $|W^\omega| = \omega^{\omega+1} \cdot 2$). If W^ω is two-ended, we recursively apply natural summation to get the normal expansion of the ordinal length $|W^\omega|$:

$$|W^\omega| \;=\; \sum_m |W_m^{\alpha_m}| \;=\; \omega^\omega \cdot \tau_\omega + \sum_{k=0}^{\infty} \omega^k \cdot \tau_k \tag{26}$$

where τ_ω and the τ_k are natural numbers. Here, too, $\tau_\omega \neq 0$. Also, only finitely many (perhaps none) of the τ_k are nonzero because W^ω is two-sided and therefore has only finitely many $W_m^{\alpha_m}$ terms.

11 Distances between Wnodes

Henceforth, we restrict the rank of the wgraph $\mathcal{G}_w^\nu$ to be no larger than ω (i.e., $\nu \leq \omega$), and we assume that $\mathcal{G}_w^\nu$ is wconnected, that is, given any two wnodes in $\mathcal{G}_w^\nu$ there is a two-ended γ-walk ($0 \leq \gamma \leq \omega, \gamma \neq \vec{\omega}$) that terminates at those two wnodes. Such a walk will have an ordinal length in accordance with the preceding Section. Consequently, we can define the distance $d_w(x_a^\alpha, x_b^\beta)$ between any two maximal wnodes x_a^α and x_b^β in $\mathcal{G}_w^\nu$ as follows:

$$d_w(x_a^\alpha, x_b^\beta) \;=\; \min\{|W_{a,b}|\} \tag{27}$$

where the minimum is taken over all two-ended walks $W_{a,b}$ terminating at x_a^α and x_b^β. The minimum exists because the ordinals comprise a well-ordered set. If, however, $x_a^\alpha = x_b^\beta$, we get $d_w(x_a^\alpha, x_b^\beta) = 0$ from the trivial walk at x_a^α. Thus, we have a function mapping pairs of maximal wnodes into the set of nonnegative real numbers. If x_a^α and/or x_b^β are not maximal, we define the distance between them as being the same as that between the maximal wnodes embracing x_a^α and/or x_b^β. So, unless something else is explicitly stated, we will henceforth confine our attention to the maximal wnodes in $\mathcal{G}_w^\nu$.

Clearly, if $x_a^\alpha \neq x_b^\beta$, then $d_w(x_a^\alpha, x_b^\beta) > 0$; moreover, $d_w(x_a^\alpha, x_b^\beta) = d_w(x_b^\beta, x_a^\alpha)$. Furthermore, the triangle inequality

$$d_w(x_a^\alpha, x_b^\beta) \;\leq\; d_w(x_a^\alpha, x_c^\gamma) \,+\, d_w(x_c^\gamma, x_b^\beta) \tag{28}$$

holds for any three wnodes x_a^α, x_b^β, and x_c^γ; it is understood here that we are using the "natural summation" of ordinals in order to get the right-hand side of (28) as the "normal expansion" of an ordinal [1, pages 354-355]. That (28) is true follows directly from the fact that the conjunction of two walks is again a walk. Thus, a walk from x_a^α to x_c^γ followed by a walk from x_c^γ to x_b^β is a walk from x_a^α to x_b^β. So, by taking minimums appropriately, we get (28). Thus, we have

Proposition 11.1. *The distance function d_w satisfies the metric axioms.*

Example 11.2. In the 1-graph of Fig. 5, the distance (27) between any 0-node and x^1 (or y^1) is ω, and the distance between x^1 and y^1 is $\omega \cdot 2$. For distances defined only by paths [7, Equation (7)], the distance between any 0-node and x^1 (or y^1) is ω, but there is no distance defined between x^1 and y^1. □

Example 11.3. In the 1-graph of Fig. 6, the distance between any 0-node and any 1-node is ω. Also, the distance between any two 1-nodes is $\omega \cdot 2$; this stands in contrast to the path-based definition of distance for which the distance between those 1-nodes could only be defined for the pair x_a^1 and x_b^1. □

Example 11.4. In the 2-graph of Fig. 7, the distance $d_w(x_1^1, x_2^1)$ is $\omega \cdot 2$, according to our definition (27). Similarly, $d_w(x_i^1, x_k^1) = \omega \cdot (2|i-k|)$. On the other hand, for every i, $d_w(x^2, x_i^1) = \omega^2$. □

A special case is worth noting here. Since 1-wnodes and 1-nodes are the same thing and since a two-ended 1-path is a special case of a two-ended 1-walk, the distance between two 1-nodes in a 1-graph as defined by (27) will be the same as the distance defined by [7, Equation (7)] so long as a two-ended 0-path or 1-path exists between those 1-nodes. However, this statement need not be true for wnodes of ranks higher than 1 because there may be a two-ended walk terminating at two such nodes that is shorter than any path terminating at them.

12 Eccentricities and Related Ideas

The ideas of eccentricities of wnodes, radii, diameters, centers, and peripheries for wgraphs can be defined just as they are in [7, Sec. 6], using now the distance function d_w of (27) instead of the distance function d of [7, Equation (7)]. The only difference is that the restriction to a metrizable set of nodes [7, Sec. 3] is no longer needed. All the wnodes of a wgraph constitute a metrizable set under the distance function (27). So, let us present just one example illustrating this generality.

Example 12.1. In order to get a result distinguishable from those relating to path-based graphs, we shall use the 1-graph of Fig. 5 in place of a branch. Thus, in Fig. 8, a bold line between 1-nodes (shown as solid dots) will denote that 1-graph, where it is understood that connections to it are only made at the two 1-nodes x^1 and y^1. Also, a one-way infinite series connection of such 1-graphs will be denoted by three bold dashes.

With this notation, consider the 2-wgraph of Fig. 8. Let $e(v)$ denote the eccentricity of any wnode v. The eccentricities of the wnodes of Fig. 8 are as follows:[11] $e(x_k^1) = \omega^2 \cdot 2 + k$ for $k = 1, 2, 3, \ldots$; $e(y^2) = \omega^2 \cdot 2$; $e(z_k^1) = \vec{\omega^2} \cdot 2$ for $k = \ldots, -1, 0, 1, \ldots$; and $e(\omega^2) = \vec{\omega^2} \cdot 3$. Consequently, the radius of this wgraph is $\vec{\omega^2} \cdot 2$, its diameter is $\vec{\omega^2} \cdot 3$, its center is $\{z_k^1 : k = \ldots, -1, 0, 1, \ldots\}$, and its periphery is $\{w^2\}$. □

Three theorems concerning eccentricities, radii, diameters, and centers of transfinite graphs were established in [7]. Those theorems can be extended readily to wgraphs. In fact, the proofs for wgraphs are simpler because of the fact that the conjunction of two walks is again a walk. Moreover, these results for wgraphs are stronger because they hold for all wnodes—not just for nodes in some metrizable set. So, let us simply state the versions of those theorems that hold for wgraphs. In the following, it is understood that every wnode discussed is a maximal wnode. The eccentricity of any nonmaximal wnode x is the same as that of the maximal wnode embracing x.

Theorem 12.1. *Let $\mathcal{S}_w^\rho$ be any ρ-wsection in the ν-wgraph $\mathcal{G}_w^\nu$, where $0 \leq \rho < \nu \leq \omega$. Assume that all the bordering wnodes of $\mathcal{S}_w^\rho$ are incident to $\mathcal{S}_w^\rho$ only through ρ-wtips. Then, all the internal wnodes of $\mathcal{S}_w^\rho$ have the same eccentricity.*

The radius (resp. diameter) of a wgraph is denoted by rad (resp. diam). If rad is an arrow rank, then rad$^+$ will denote the next higher limit ordinal rank.

Theorem 12.2. *For any given wgraph of radius* rad *and diameter* diam, *the following*

[11] As always, an arrow above a symbol λ (i.e., $\vec{\lambda}$) denotes the arrow rank $\vec{\lambda}$ just preceding the limit ordinal λ.

hold:

(i) *If* rad *is an ordinal, then* rad $\leq$ diam $\leq$ rad $\cdot\, 2$.

(ii) *If* rad *is an arrow rank, then* rad $\leq$ diam $\leq$ rad$^{+} \cdot 2$.

Theorem 12.3. *The ν-wnodes of any wgraph $\mathcal{G}_w^\nu$ $(0 \leq \nu \leq \omega, \nu \neq \vec{\omega})$ comprise the center of some ν-graph $\mathcal{H}^\nu$.*

13 Walk-Based Transfinite Electrical Networks

Our objective in the rest of this work is to extend electrical network theory to networks whose graphs are wgraphs. We refer to such networks as "wnetworks," just as we did for other entities relating to wgraphs. As should be expected, the solution spaces for the current regimes of wnetworks are larger than those for the networks considered in [3], [4], and [6], and the solutions we now obtain can have stranger configurations. Examples of this are given in [5, Secs. VC and VD]; that paper only employed the first rank of transfiniteness for walks and therefore encountered only the 0-graphs and 1-graphs considered previously in the path-based theory—not wgraphs. Herein, we examine the voltage-current regimes of wnetworks at higher ranks. A notable result is that, in contrast to path-based networks, wnetworks possess unique node voltages with respect to a given ground node whenever node voltages exist.

We continue to assume that we have in hand a ν-wgraph $\mathcal{G}_w^\nu$ where $0 \leq \nu \leq \omega$.

14 Tours and Tour Currents

Let γ be an ordinal such that $0 \leq \gamma \leq \omega$ and $\gamma \neq \vec{\omega}$. We define a *closed* γ-walk as a two-ended γ-walk such that its first wnode embraces or is embraced by its last wnode. (Thus, a γ-loop is a special case of a closed γ-walk.) Also, when $\gamma = 1$, a closed γ-walk will pass through any branch only finitely many times because each 0-walk between consecutive 1-nodes in it does so.[12] However, for $\gamma > 1$, a γ-tour might pass through a branch infinitely often; this can result in infinite power dissipation for current regimes based on closed walks.

[12] In fact, that 0-walk can be decomposed into three 0-paths.

We wish to disallow this because our fundamental Theorem 15.2, given below, is based upon finite-power regimes. For this reason, we restrict the allowable closed walks still further: A *γ-tour* is defined to be a closed γ-walk that passes through each branch at most finitely many times (possibly never for some branches). As with any walk, every γ-tour is assigned an orientation (i.e., a direction for tracing it). A *tour* is a γ-tour of some unspecified rank γ.

The next step is to assign an electrical structure to the ν-wgraph $\mathcal{G}_w^\nu$ at hand. As was stated above, we restrict ν to $0 \leq \nu \leq \omega$; in addition, we now require that $\gamma \leq \nu$. Let J be the set of indices j assigned to the branches b_j. J may be uncountable. All entities relating to the branch b_j will also carry the same index j. We take it that every branch b_j of $\mathcal{G}_w^\nu$ is in the Thevenin form of a positive resistor r_j in series with a pure voltage source e_j. Thus, r_j is a positive real number, and e_j is any real number—possibly 0. Also, every branch b_j is assigned an orientation. The current i_j and voltage v_j on b_j are measured with respect to that branch orientation. By Ohm's law, these quantities are related by

$$v_j \;=\; i_j r_j - e_j. \tag{29}$$

Thus, we are taking v_j to a voltage "drop" and e_j to be a voltage "rise" with respect to b_j's orientation, but these quantities can be either positive, negative, or zero. $i = \{i_j\}_{j\in J}$, $v = \{v_j\}_{j\in J}$, and $e = \{e_j\}_{j\in J}$ will denote the *branch-current vector*, the *branch-voltage vector*, and the *branch-voltage-source vector*, respectively. On the other hand, the mapping $R\colon i \mapsto r_j i_j$ will denote *the branch resistance operator*. R maps i into a voltage vector Ri. Thus, Ohm's law can be written in this notation as follows:

$$v \;=\; Ri \,-\, e \tag{30}$$

We refer to $\mathcal{G}_w^\nu$ with these assigned branch parameters as a *ν-wnetwork* and denote it by $\mathbf{N}_w^\nu$. This, too, is assumed given and fixed in the following.

Next, a *tour current* is a constant flow f of current passing along the tour in the direction of the tour's orientation; f is any real number. That flow f produces branch currents as follows: If the tour does not pass through a branch b_j, then the branch current i_j equals 0. If a tour passes through b_j just once, then $i_j = \pm f$, with the $+$ sign (resp. $-$ sign) used if the b_j's orientation and the tour's orientation agree (resp. disagree). If, however, the tour passes through a branch more than once, the corresponding branch current i_j is a multiple of f obtained by adding and/or subtracting f for each passage of the tour through the branch, addition (resp. subtraction) being used when the orientation of the tour and branch agree (resp. disagree); that branch current may be 0 because of cancellation.

15 The Solution Space $\mathcal{T}$

In the following, the summation symbol $\sum$ will denote a sum over all branch indices j unless something else is explicitly indicated. $\mathcal{I}$ will denote the space of current vectors that dissipate only a finite amount of power in the resistors:

$$\mathcal{I} \;=\; \{i\colon \sum i_j^2 r_j\} \;<\; \infty \tag{31}$$

We assign to $\mathcal{I}$ the inner product $(i,s) = \sum i_j s_j r_j$, where $i, s \in \mathcal{I}$. In the standard way, $\mathcal{I}$ can be shown to be complete under the corresponding norm $\|i\| = (\sum i_j^2 r_j)^{1/2}$, and thus $\mathcal{I}$ is a Hilbert space. Each $i \in \mathcal{I}$ will be called *finite-powered*. A branch-current vector corresponding to a tour current may or may not be finite-powered in this sense.

No requirement concerning Kirchhoff's current law is being imposed on the members of $\mathcal{I}$. Nonetheless, we do wish to satisfy that law whenever possible—certainly at 0-nodes of finite degree. To this end, we construct a solution space $\mathcal{T}$ that will be searched for a current vector i such that i and its corresponding voltage vector v (as determined by Ohm's law (30)) satisfy Kirchhoff's laws whenever possible. Since each tour passes through any branch only finitely many times, we can define linear combinations of finitely many tour currents by taking linear combinations of the currents in each branch. It follows from Schwarz's inequality that a linear combination of finite-powered branch-current vectors is again finite powered. Thus, the span $\mathcal{T}^o$ of all finite-powered tour currents will be a subspace of $\mathcal{I}$. Each member of $\mathcal{T}^o$ will satisfy Kirchhoff's current law at every 0-node.

Finally, we let $\mathcal{T}$ be the closure of $\mathcal{T}^o$ in $\mathcal{I}$. Consequently, $\mathcal{T}$ is a Hilbert subspace of $\mathcal{I}$ with the same inner product (31). $\mathcal{T}$ will be the solution space that we will search for a unique branch-current vector satisfying Tellegen's equation, given below.

16 The Fundamental Theorem

Next, we confine the branch-voltage-source vector e to be of *finite total isolated power* by requiring that

$$\sum e_j^2 g_j \;<\; \infty \tag{32}$$

where $g_j = 1/r_j$. It follows from Ohm's law (29) and Schwarz's inequality that, for $i, s \in \mathcal{I}$ and e restricted by (32), $\sum |v_j s_j|$ is finite. Indeed,

$$\begin{aligned} \sum |v_j s_j| &= \sum r_j |i_j s_j|, + \sum |e_j s_j| \\ &= \sum |i_j| \sqrt{r_j} |s_j \sqrt{r_j} + \sum |e_j| \sqrt{g_j} |s_j| \sqrt{r_j} \\ &\leq \left(\sum i_j^2 r_j \sum s_j^2 r_j\right)^{1/2} + \left(\sum e_j^2 g_j \sum s_j^2 r_j\right)^{1/2} \;<\; \infty. \end{aligned}$$

We let $\langle w, s\rangle = \sum w_j s_j$ denote the coupling between any voltage vector w and any current vector s whenever the sum exists. Then, Tellegen's equation can be written as

$$\langle v, s\rangle \;=\; 0 \tag{33}$$

where in the following we will have $v = Ri - e$, $i \in \mathcal{T}$, $s \in \mathcal{T}$, and e restricted by (33).

Lemma 16.1. *If e satisfies (32), then $e : s \mapsto \langle e, s\rangle$ is a continuous linear functional on $\mathcal{I}$ and therefore on $\mathcal{T}$, too.*

The proof of this lemma is the same as that for [4, Lemma 5.2-5].

We are now ready to state the fundamental theorem for a unique voltage-current regime in $\mathbf{N}_w^\nu$.

Theorem 16.2. *If e satisfies (32), then there exists a unique branch-current vector $i \in \mathcal{T}$ such that the corresponding unique branch-voltage vector $v = Ri - e$ satisfies Tellegen's equation (33) for every $s \in \mathcal{T}$.*

The proof of this theorem is the same as that for [4, Theorem 5.2-8]. It is an easy consequence of Lemma 16.1 and the Riesz representation theorem.

17 A Larger Solution Space $\mathcal{S}$

It may happen that, for the given ν-wnetwork $\mathbf{N}_w^\nu$, there are certain finite-powered branch-current vectors that are not members of $\mathcal{T}$. The examples illustrated by [4, Figs. 9 and

10] are two instances of this for the case of a 1-network. We can enlarge $\mathcal{T}$ by allowing a thinning out of the current vector as it spreads out toward infinite extremities.[13] To do so, we define a ***splayed*** branch-current vector in just the same way as a "basic current" is defined in [4, page 127-128] in the case of path-based current vectors. (Just replace "path" by "walk" and "loop" by "closed walk.") Then, we let $\mathcal{S}^o$ be the span of all finite-powered splayed-current vectors, and let $\mathcal{S}$ be the closure of $\mathcal{S}^o$ in $\mathcal{I}$. It follows that $\mathcal{T} \subset \mathcal{S}$, with strict inclusion holding for some ν-wnetworks. The arguments of Sec. 16 can be applied once again to establish Theorem 16.2 with $\mathcal{T}$ replaced by $\mathcal{S}$. This expands the scope of that theorem.

18 Uniqueness of Wnode Voltages

A short-coming of electrical network theory for path-based ν-networks is that node voltages need not be unique even though they exist. That is, the node voltage determined by summing branch voltages along a path from a node to fixed ground node may depend upn the path chosen. As a result, a special condition must be imposed to assure the uniqueness of node voltages; see Condition 5.4-1 and Theorem 5.5-4 of [4].

This difficulty does not exist for walk-based ν-wnetworks. Indeed, having chosen arbitrarily any wnode as the ground wnode x_g, we assign a wnode voltage to another wnode x if a "permissive" walk exists between those two wnodes. A walk is called *permissive* if the branch resistances sum to a finite amount along the walk, where each resistance is added every time the walk passes along its branch. Then, the *wnode voltage* at x is the sum $\sum_W \pm v_j$, where $\sum_W$ denotes the sum along the branches of the walk from x to x_g. Each time a branch b_j is traced by the walk, a term $\pm v_j$ is added with the $+$ sign (resp. $-$ sign) used if the branch orientation agrees (resp. disagrees) with the orientation of the walk through that branch. Then, two different permissive walks from x to x_g will yield the same wnode voltage at x. The proof of this result is the same as that for [5, Theorem 6.1] except that now we are dealing with wnetworks of ranks greater than 1 as well. In other words, Kirchhoff's voltage law is always satisfied around closed walks. However, this

[13]Such a construction is given explicitly for a 1-network in [5, Sec V-D].

advantage might be paid for in certain cases by the collapse of Kirchhoff's current law at certain wnodes. The 1-networks of Figs 9 and 10 in [5] provide examples of these results.

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Figure Captions

There are only figure numbers—no captions.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7

Fig. 8

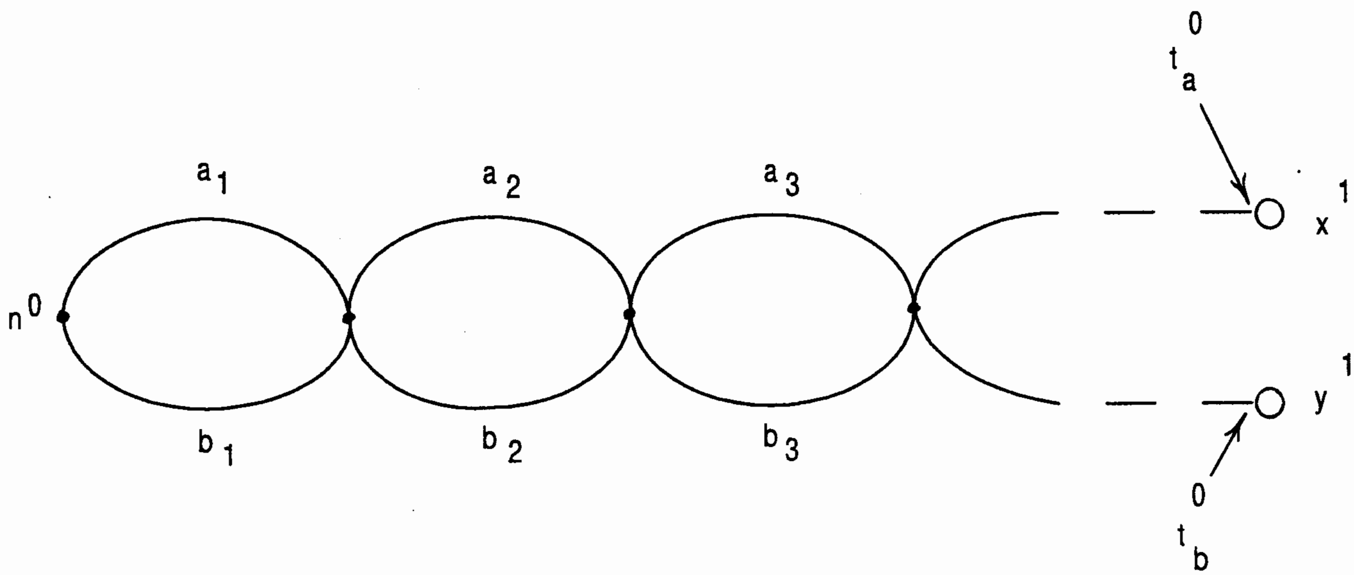

Fig. 1

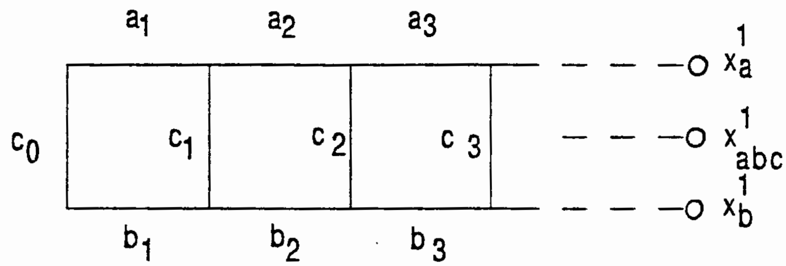

Fig. 2

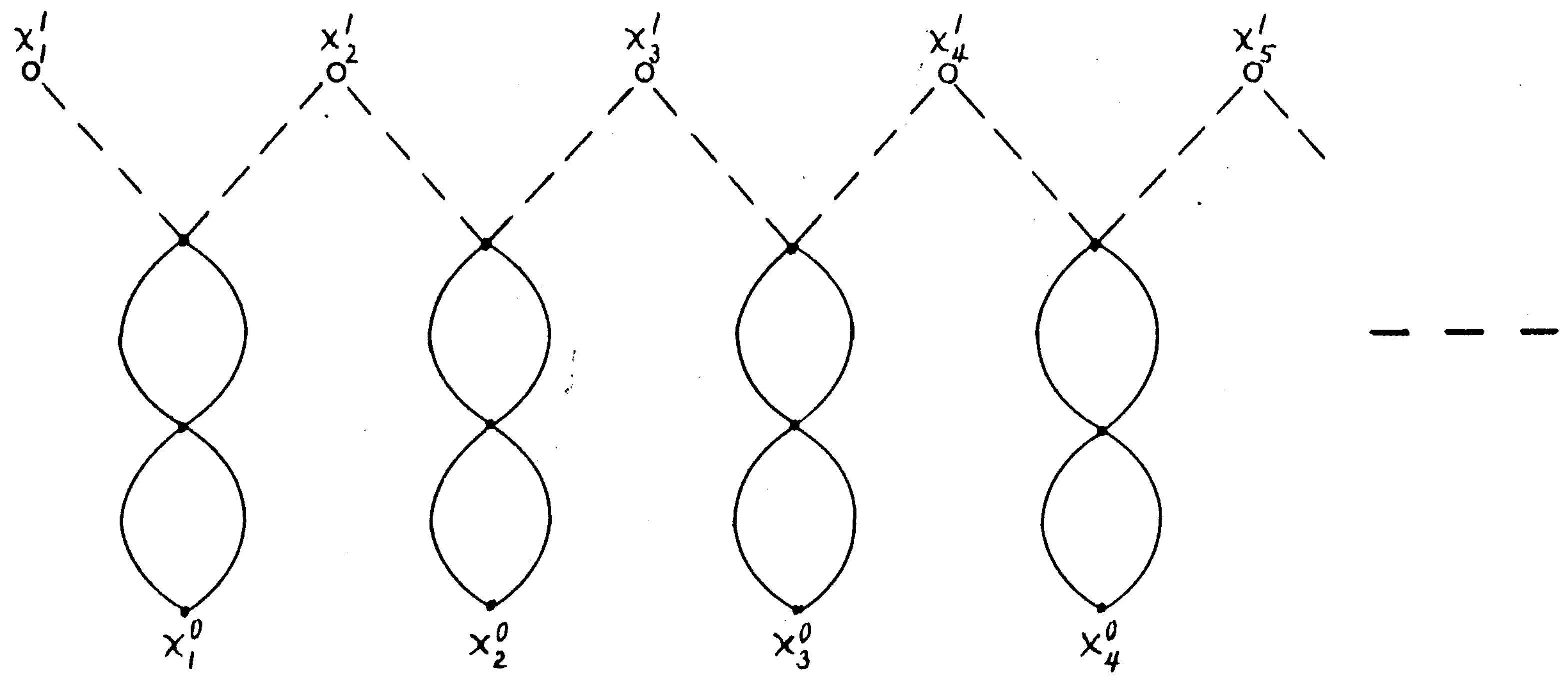

Fig. 3

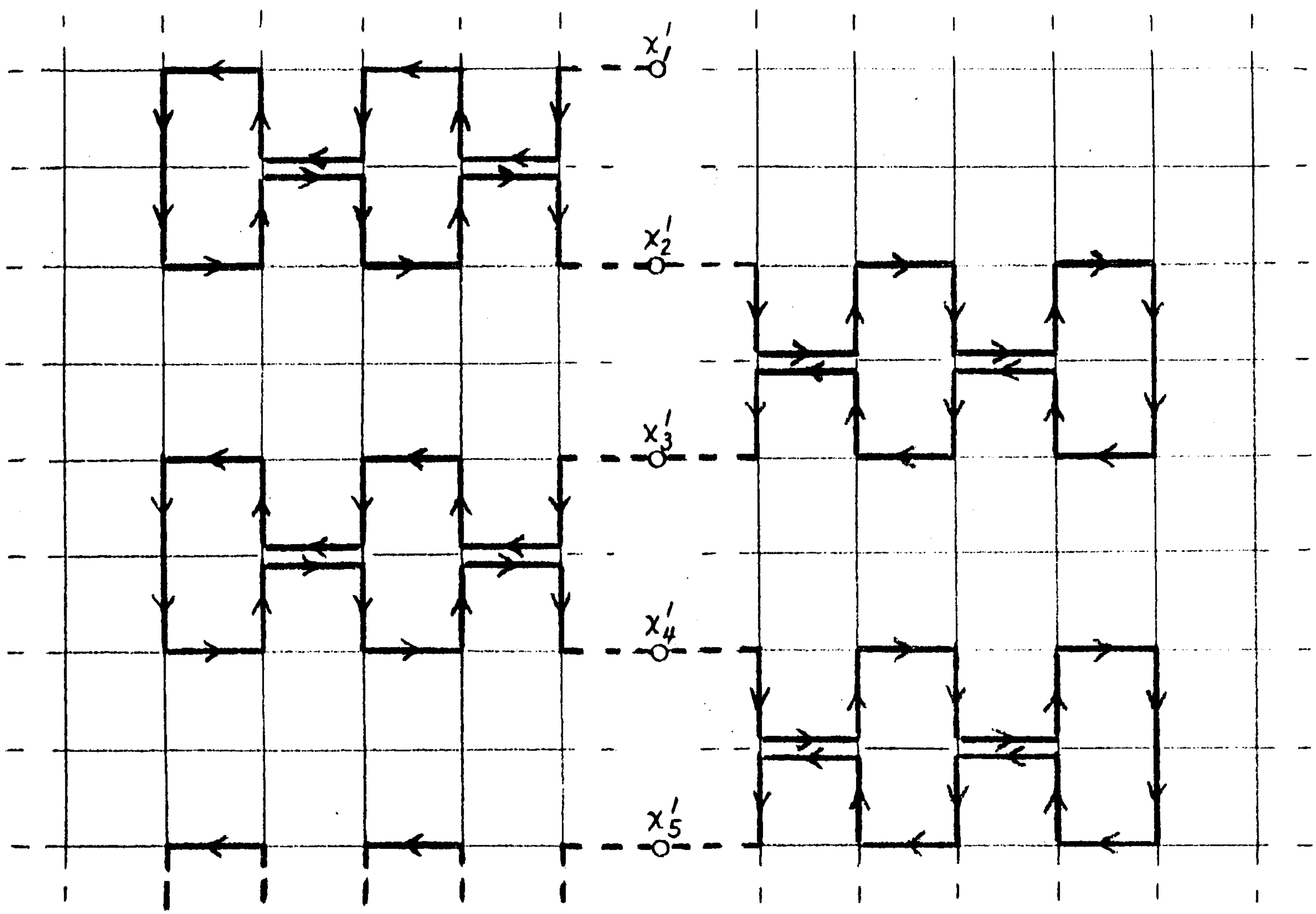

Fig. 4

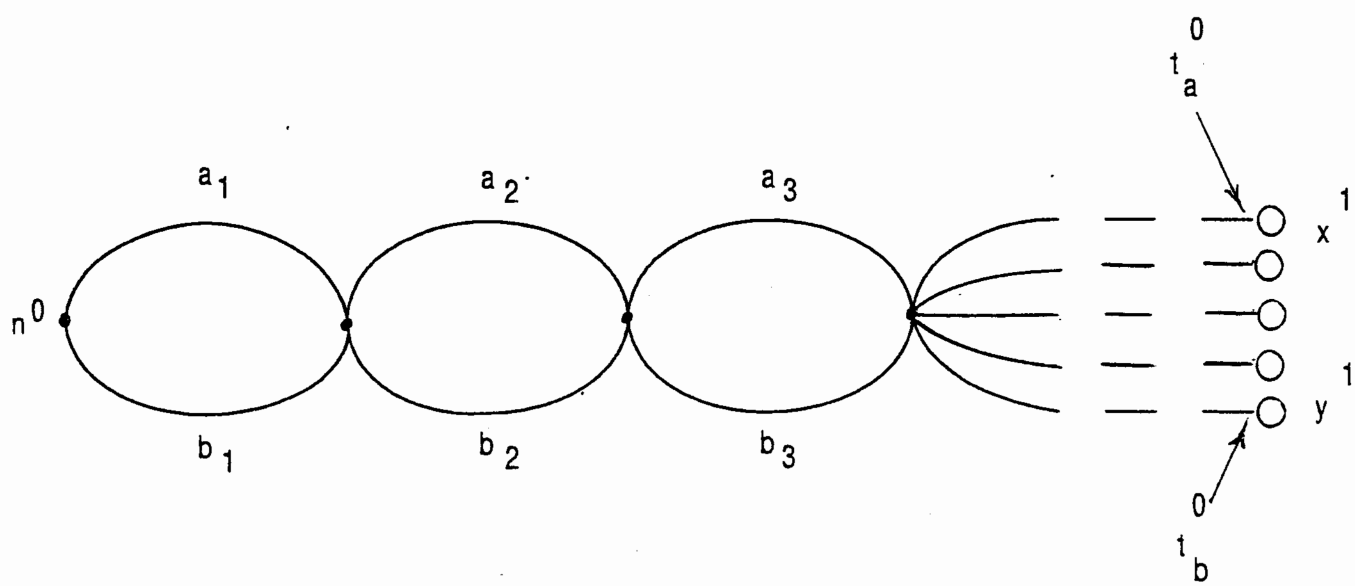

Fig. 5

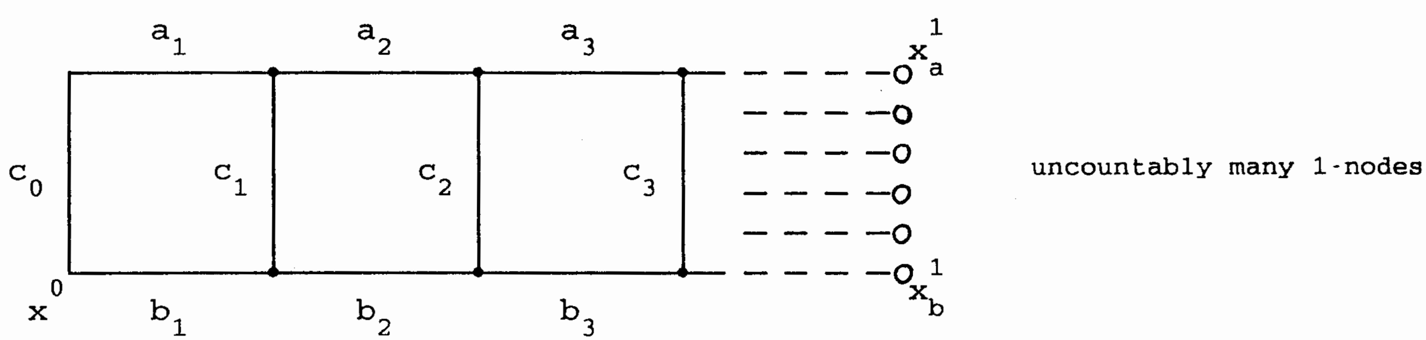

Fig. 6

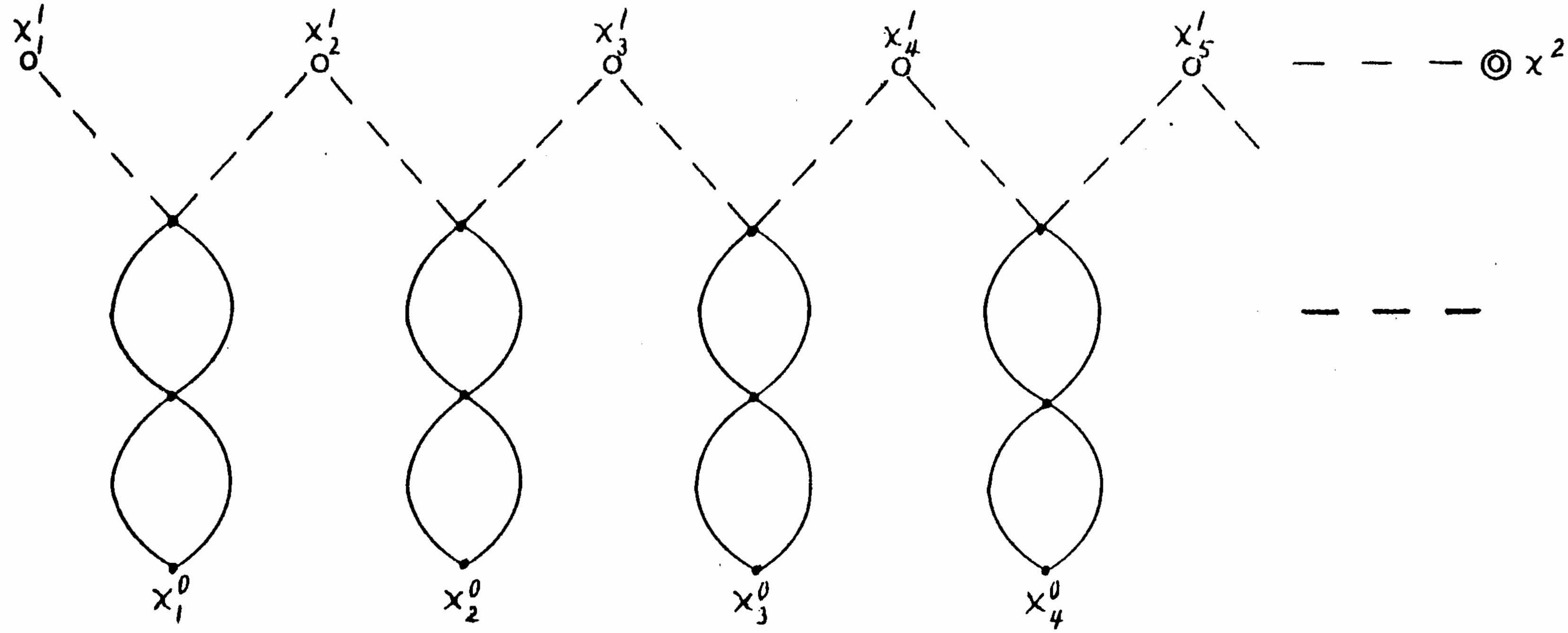

Fig. 7

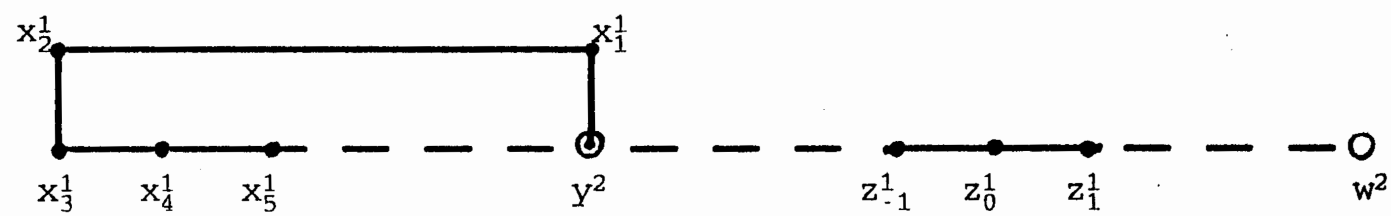

Fig. 8