UNIVERSITY AT STONY BROOK

CEAS Technical Report 791

ORDINAL DISTANCES IN PRISTINE GRAPHS

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May 2, 2001

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Abstract — Ordinal distances were defined between certain nodes of transfinite graphs and a variety of results concerning nodal eccentricities were established in a prior work. When, however, a transfinite graph is restricted to being pristine (i.e., no node embraces a node of lower rank), a number of simplifications and extensions of those prior results accrue. They are presented in this short paper.

Key Words: Distances in graphs, transfinite pristine graphs, transfinite ordinal distances, nodal eccentricities.

1 Introduction

Distances in transfinite graphs can be represented by transfinite ordinals. Thus was proposed and examined a prior work [3], and generalizations of several standard results concerning distances in finite graphs were achieved as well as some other results having no counterparts for finite graphs. That prior work considered transfinite graphs [1] in general. Pristine graphs [2], being a special case of transfinite graphs, have additional properties, some of which represent simplifications of the prior general results. In this short paper, we present several theorems of this nature. The terminologies we use herein are defined in [2], which has a thorough index. One definition of importance for this work is the following. The set of ranks for transfinite graphs and the paths within them is the set of countable ordinals along with the insertion of an *arrow rank* $\vec{\lambda}$ just before each limit ordinal λ [1, page 4], [2, page 4].

A pristine ν-graph is a transfinite graph of rank ν in which no node embraces a node of lower rank; as a result, all nodes are maximal, that is, no node is embraced by a node of higher rank. In such a graph, each μ-node consists only of $(\mu-1)$-tips. Also, no node can be of rank $\vec{\omega}$, in contrast to transfinite graphs in general, which can have $\vec{\omega}$-nodes. Nonetheless, a pristine graph can be of rank $\vec{\omega}$ by containing a one-ended $\vec{\omega}$-path but no ω-node. We always assume in the following that the ν-graph $\mathcal{G}^\nu$ is pristine and is ν-connected. Also, we restrict the rank ν to $0 \leq \nu \leq \omega$. All our results can be extended to many ranks larger than ν with virtually no change in the arguments [2, Sec. 2.4]. Furthermore, the following is also assumed throughout.

Condition 1.1. *If two tips are nondisconnectable* [2, page 31], *they are either shorted together(i.e., are of the same rank and belong to the same node) or at least one of them is open (i.e., is the sole member of a singleton node).*

This is the pristine version of Condition 3.2 of [2]. It ensures that connectedness is a transitive binary relation for nonsingleton nodes [2, Theorem 3.1-6]. It also ensures that between any two nonsingleton nodes there exists at least one path terminating at them. As a result, ordinal-valued distances can be defined between all pairs of nonsingleton nodes and in fact between all pairs of nodes in any metrizable set $\mathcal{M}$ of nodes [3, Proposition 4.1]. $\mathcal{M}$ contains all the nonsingleton nodes in $\mathcal{G}^\nu$ as well as some singleton nodes. It will be understood that a metrizable set $\mathcal{M}$ of nodes has been selected and fixed in the following. One possibility is that $\mathcal{M}$ is just the set of nonsingleton nodes in $\mathcal{G}^\nu$.

2 ν-Graphs Having One-Ended ν-Paths

A one-ended ν-path ($\nu \neq \vec{\omega}$) is one that starts at some node of rank ν or less and passes through infinitely many ν-nodes. When it appears in a ν-graph, its ν-tip is not embraced by any node of rank ν or greater. In effect, such a ν-graph extends infinitely without being a subgraph of a graph of higher rank. A similar result holds when $\nu = \vec{\omega}$, except that now the starting node is of rank less than $\vec{\omega}$ because there are no $\vec{\omega}$-nodes in a pristine graph, and moreover the path passes through nodes of all ranks less than $\vec{\omega}$. Because of this, the eccentricities of all the nodes are all the same arrow rank, in contrast to the eccentricities of nonpristine transfinite graphs whose eccentricities can be ordinal ranks as well as arrow ranks and can differ. More precisely, we have the following theorem.

Theorem 2.1. *Assume that the ν-graph $\mathcal{G}^\nu$ contains a one-ended ν-path P^ν. Then, every node of $\mathcal{M}$ has an eccentricity of $\omega^{\vec{\nu}+1}$.*

Note. When $\nu = \vec{\omega}$, $\omega^{\vec{\nu}+1}$ denotes $\omega^{\vec{\omega}}$.

Proof. Let x be any node in $\mathcal{M}$, and let y be any nonsingleton node in P^ν (and therefore in $\mathcal{M}$ as well). Then, there is a two-ended path P_{xy} terminating at x and y. It follows from [2, Theorem 3.1-6] that there is a one-ended ν-path Q^ν in $P_{xy} \cup P^\nu$ terminating at x. The length of Q^ν is $\omega^{\vec{\nu}+1}$, and there is no path in $\mathcal{G}^\nu$ having a greater length. Thus, the eccentricity of x is $\omega^{\vec{\nu}+1}$. □

The next theorem states conditions ensuring the presence of a one-ended ν-path in $\mathcal{G}^\nu$. The critical properties are a transfinite generalization of local-finiteness along with the assumed ν-connectedness of $\mathcal{G}^\nu$.

Theorem 2.2

(i) *With $\nu \neq \vec{\omega}$, assume that there are infinitely many nonsingleton ν-nodes in $\mathcal{G}^\nu$ and that, for each rank $\rho = 0, 1, \ldots, \nu$ (possibly, $\nu = \omega$), every nonsingleton ρ-node is ρ-adjacent [2, page39] to only finitely many nonsingleton ρ-nodes. Then, for any node $x \in \mathcal{M}$, there exists at least one one-ended ν-path in $\mathcal{G}^\nu$ starting at x. Thus, the eccentricity of every node in $\mathcal{G}^\nu$ is $\omega^{\vec{\nu}+1}$.*

(ii) *For the $\vec{\omega}$-graph $\mathcal{G}^{\vec{\omega}}$, assume that, for each natural number $\rho = 0, 1, 2, \ldots$, every nonsingleton ρ-node is ρ-adjacent to only finitely many nonsingleton ρ-nodes. Then, for any node $x \in \mathcal{M}$, there exists at least one one-ended $\vec{\omega}$-path in $\mathcal{G}^{\vec{\omega}}$ starting at x. Moreover, the eccentricity of every node in $\mathcal{G}^{\vec{\omega}}$ is $\omega^{\vec{\omega}}$.*

Proof. These conclusions follow from transfinite versions of König's lemma given by Corollary 3.3-6 and Theorem 3.3-7 of [2] with $\nu \neq \vec{\omega}$ and $\nu = \vec{\omega}$ respectively. □

Let us note in passing that König-type theorems hold for nonpristine graphs as well, but under some strong assumptions [1, Corollary 4.2-5 and Theorem 4.2-7]. This ensures the existence of one-ended ν-paths, and consequently the conclusions of Theorem 2.2 hold once again.

3 ν-Graphs With Only Finitely Many Nonsingleton ν-Nodes

Very different results arise when $\mathcal{M}$ has at least one and no more than finitely many ν-nodes, which means in particular that there are only finitely many nonsingleton ν-nodes in $\mathcal{G}^\nu$. Since there are no $\vec{\omega}$-nodes in a pristine graph, we now have the restriction that $\nu \neq \vec{\omega}$, which is imposed throughout this section. We will now find that all eccentricities are ordinals (never arrow ranks).

In this section, $\nu \neq \vec{\omega}$ because there are no $\vec{\omega}$-nodes in a pristine graph.

Theorem 3.1. *Let $\mathcal{M}$ have at least one and no more than finitely many ν-nodes. Then, the eccentricities of all the nodes in $\mathcal{M}$ take their values in a finite set of ordinals having the form*

$$\{\omega^\nu \cdot p \colon 1 \leq p \leq 2m\} \tag{1}$$

where m and p are positive natural numbers and m is the number of ν-nodes in $\mathcal{M}$.

Proof. Let P be any path in $\mathcal{G}^\nu$ terminating at two nodes of $\mathcal{M}$. All the other nodes of P are nonsingletons and therefore in $\mathcal{M}$, too. P can embrace no more than m ν-nodes and therefore can traverse no more than $2m$ $\nu - 1$-tips. Thus, the length of P is no larger than $\omega^\nu \cdot 2m$. By Theorem 7.1 of [3], the nodes of any $(\nu - 1)$-section $\mathcal{S}^{\nu-1}$ all have the same rank. Also, the length of any one-ended $(\nu - 1)$-path in $\mathcal{S}^{\nu-1}$ connecting a node of $\mathcal{S}^{\nu-1}$ to a bordering node of $\mathcal{S}^{\nu-1}$ is exactly ω^ν. So, if x and y are nodes of $\mathcal{M}$ in different $(\nu-1)$-sections, any two-ended path terminating at them has a length equal to $\omega^\nu \cdot p$, where the natural number p satisfies $1 \leq p \leq 2m$. This is also true if either (or both of) x and y is a bordering node in $\mathcal{M}$ of a $(\nu - 1)$-section. (That bordering node will have rank ν.) So, the set of lengths of all paths terminating at nodes of $\mathcal{M}$ is a subset of (1) (possibly all of (1)). Consequently, the distance between any such nodes will take their values in that finite set. It follows that the eccentricity of any node in $\mathcal{M}$ will also be a value in that finite set. □

In the next result, rad and diam denote the radius and diameter of a transfinite graph as defined in [3].

Corollary 3.2. *Under the hypothesis of Theorem 3.1,* rad *and* diam *are ordinals of the form $\omega^\nu \cdot p$ and $\omega^\nu \cdot q$, where p and q are positive natural numbers no larger than $2m$.*

Moreover, rad $\leq$ diam $\leq$ rad$\cdot$2.

Proof. Because (1) is a finite set, rad and diam are respectively the minimum and maximum of 1). Whence the first conclusion. The second conclusion follows from Theorem 7.2(i) of [3]. □

Corollary 3.3. *Assume again the hypothesis of Theorem 3.1. Let x^ν be a bordering node of a $(\nu-1)$-section $\mathcal{S}^{\nu-1}$, and let y^α be an (internal) node of $\mathcal{S}^{\nu-1}$. With $\omega^\nu \cdot k$ and $\omega^\nu \cdot p$ denoting the eccentricities of x^ν and y^α respectively, the following is true: $|k-p| \leq 1$.*

Proof. This follows from the fact that any $(\nu-1)$-path in $\mathcal{S}^{\nu-1}$ that terminates at y^α and reaches x^ν has the length ω^ν. □

That $k-p$ can equal 0 in the last corollary is verified by the next example.

Example 3.4. Consider the pristine 1-graph of Fig. 1 consisting of a one-ended path of 0-nodes w_k^0 and an endless path of 0-nodes y_k^0 connected to two 1-nodes x^1 and z^1, as shown. The eccentricities are $e(w_k^0) = \omega \cdot 3$ for $k = 1, 2, 3, \ldots$, $e(x^1) = \omega \cdot 2$, $e(y_k^0) = \omega \cdot 2$ for $k = \ldots, -1, 0, 1, \ldots$, and $e(z^1) = \omega \cdot 3$. Thus, $e(x^1) - e(y_k^0) = 0$, as asserted. Note also that rad $= \omega \cdot 2$, diam $= \omega \cdot 3$, the center consists of x^1 and all the y_k^0, and the periphery consists of z^1 and all the w_k^0. □

An immediate consequence of Corollary 3.3 is the following.

Corollary 3.5. *Under the hypothesis of Theorem 3.1, the set of eccentricities for $\mathcal{M}$ form a consecutive set of values in (1).*

4 ν-Graphs With Infinitely Many ν-Nodes But No One-Ended ν-Paths

There are ν-graphs that do not satisfy the hypotheses in the prior sections. There is, however, another circumstance wherein the eccentricities are all the same arrow rank. Specifically, first consider the case where $\nu \neq \vec{\omega}$, and assume that there is no one-ended ν-path in $\mathcal{G}^\nu$. Assume furthermore that for each node in $\mathcal{M}$ there are two-ended ν-paths of all possible lengths $\omega^\nu \cdot k$ ($k = 1, 2, 3, \ldots$). (Such is the case, for example, for a star graph, whose hub is a 0-node x^0 of infinite degree and whose spokes are two-ended paths starting at x^0 and of lengths $\omega^\nu \cdot k$ ($k = 1, 2, 3, \ldots$).) In this case, the eccentricity of each node in

$\mathcal{M}$ is $\omega^{\vec{\nu}+1}$.

Similarly, when $\nu = \vec{\omega}$, assume again that there is no one-ended $\vec{\omega}$-path in $\mathcal{G}^\nu$, but for each node in $\mathcal{M}$ there are two-ended μ-paths of all natural-number ranks $\mu = 1, 2, 3, \ldots$ starting at every node. (Again, the star graph with hub x^0 and infinite many spokes of such μ-paths serves as an example.) In this case, the eccentricity of each node in $\mathcal{M}$ is $\omega^{\vec{\omega}}$.

References

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