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TRANSFINITE ELECTRICAL NETWORKS

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Abstract — This is a tutorial/survey paper that provides an explanation of "transfinite electrical networks." The theory of such networks has been developing only over the past decade, and undoubtedly much is yet to be discovered. It is quite abstract, and its mathematics is complicated. In this work we try to facilitate an understanding of its main ideas without encumbering comprehension with too much mathematical jargon. Instead, definitions and arguments are heuristically presented, and many examples are given. Transfiniteness for electrical networks is a radical generalization of conventional network theory, which opens up an entirely new area of research. It may be of interest to mathematical circuit theorists but will disappoint those looking for practical applications.

1 Introduction

This paper is an explanation of a recently introduced idea for a new kind of electrical network, the "transfinite network." The graph of such a network is not a graph in any prior sense but is instead an extension roughly analogous to Cantor's extension of the natural numbers to the transfinite ordinals. Cantor introduced the transfinite numbers over 100 years ago and profoundly altered thereby mathematical ideas, as for example the concurrent invention of sets and the consequent examination of the foundations of mathematics [3], [4]. In contrast to number theory, graph theory, which is another fundamental subject with many applications in mathematics, science, and engineering, remained on "this side of infinity" until a decade ago. An initial embryonic idea [9] concerning "connections at infinity" was introduced in 1975, but it was only after 1987 [11] that transfinite graphs and

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networks were investigated on a continuing basis. This has enriched the theories of graphs and networks with radically new constructs and research problems. In general, many solved problems of conventional graphs and networks have now reopened and comprise a largely unexplored research area, and in addition there are transfinite-network problems having no counterparts in conventional theories.

Let us be more indicative about the difference between conventionally infinite graphs and transfinite graphs. In a conventionally infinite graph two nodes are either connected through a finite path or are not connected at all. On the other hand, in a transfinite graph two nodes may also be connected by an infinite path or more generally by a union of infinite paths — possibly infinitely many infinite paths, and this may even be the only way those two nodes are connected. Moreover, there may be two nodes in a transfinite graph that are connected by a walk (i.e., a tracing in the graph wherein nodes repeat) but not by any path (i.e., a tracing wherein nodes do not repeat); consequently, we now have to distinguish between "walk-connectedness" and "path-connectedness," something that is unnecessary for conventionally infinite graphs.

The basic idea for constructing a transfinite graph is the following. Any conventional (finite or infinite) graph will henceforth be called a 0-*graph*, and its nodes will be called 0-*nodes*. As is explicated in Section 3, the infinite extremities of an infinite 0-graph can be defined in a certain precise fashion, and these extremities will called 0-*tips*. As a heuristic example, consider the infinite checkerboard graph of Fig. 1; some of its 0-tips are the infinite extremities of the horizontal and vertical lines therein, one 0-tip to each half-line, but there are other 0-tips as well, such as the infinite extremity of a path that spirals outward infinitely.

Now, a new kind of node, the "1-node," can be defined as a set consisting of some of those 0-tips; we can interpret this as the shorting together of those 0-tips. 1-nodes may connect many 0-graphs together at their 0-tips; the result will be called a "1-graph" or alternatively a "graph of rank 1." This process can be repeated. Infinitely many 0-graphs can be connected through 1-nodes, and the result may have infinite extremities of a higher rank; the latter are called "1-tips." A "2-node" can then be defined as a set of 1-tips, and these can be used to connect together 1-graphs to obtain a "2-graph," synonymously, a "graph of rank 2." Moreover, this process can be repeated any finite number of times to get a μ-graph for any natural number μ.

Continuing our heuristic example, let us consider infinitely many checkerboard graphs laid out in a checkerboard fashion with adjacent 0-tips connected through 1-nodes. The result is the checkerboard of checkerboards, indicated in Fig. 2. That 1-graph has 1-tips; some of them are the infinite extremities of horizontal and vertical lines that pass through infinitely many of the checkerboards. We can repeat this procedure using those latter 1-tips and 2-nodes that short together adjacent 1-tips to get a checkerboard of checkerboards of checkerboards. This can be repeated finitely many times to get a checkerboard of checkerboards of ... of checkerboards.

We can go even further. Imagine that this process has been continued through increasing ranks without ceasing, and then jump in your imagination to the result. Call the result an "$\vec{\omega}$-graph." ω denotes Cantor's first transfinite ordinal, and the arrow above it is indicative of the never-ceasing process of construction through all the natural-number ranks. Continuing our example, we now have a checkerboard of checkerboards of ..., unceasingly. Now, a $\vec{\omega}$-graph may have infinite extremities, which we now call "$\vec{\omega}$-tips," as for example the infinite extremity of a line that starts at a 0-node and passes horizontally through our unending checkerboard-like $\vec{\omega}$-graph. Then, ω-nodes, which are sets of $\vec{\omega}$-tips can be used to create an "ω-graph," as for example a checkerboard of our unending checkerboard-like $\vec{\omega}$-graphs.

This process still need not end. Indeed, we can define ω-tips and then short them together through $(\omega + 1)$-nodes to get an $(\omega + 1)$-graph, and so on through succeeding transfinite-ordinal ranks $\omega + 2$, $\omega + 3$, ... , $\omega \cdot 2$, $\omega \cdot 2 + 1$, ... We have hereby a hierarchy of transfinite graphs of natural-number ranks and then transfinite-ordinal ranks. We can convert these transfinite graphs into transfinite electrical networks by inserting electrical elements into the branches. Does such a transfinite network have a voltage-current regime, that is, an operating point in the resistive case? Yes. We shall explain how in Section 4.

But, what good is all this? Why should one bother with such strange infinite networks? Actually, the first critical step is the generalization from finite networks to conventionally infinite networks. Once infinite concepts are introduced into any analytical description of a physical phenomenon, that description becomes perforce a mathematical abstraction that can both simplify and complicate the original description. For example, a physical resistor consists of a complicated configuration of molecules, atoms, electrons, nuclei, and so on. It is far simpler to represent it as a continuous medium described by a resistivity constant and the spatial dimensions of the medium. However, such a continuous medium is itself an infinite concept; indeed, the resistance between two connections to that medium, having say irregular boundaries, can only be approximately computed using our customary theory of finite networks by representing the medium as finite grid of lumped resistors. In general, infinities may intrude when smoothing out physical descriptions, and finiteness can only be restored by accepting still further approximations.

Once conventionally infinite concepts have been accepted, it is no longer a radical leap to continue on to transfinite ideas so far as mathematical analysis is concerned. To put this more specifically, finite sets are well-behaved; infinite sets require a leap in conceptualization (and lead at times to antinomies, that is, paradoxes that cannot be resolved [1, pages 611-635],[3, Chapter 11]). Nonetheless, we use infinite sets (while circumventing their antinomies) because of the power and conveniences that infinite sets provide. Transfiniteness is basically no more than a classification of infinite sets.

The unfamiliarity of transfinite electrical networks may provoke uneasiness, but hopefully not its rejection. In fact, in some circumstances transfinite ideas must be introduced to resolve questions regarding conventionally infinite ones, as we shall see in the next section. But, let us not beg the question, "What good are transfinite networks?" It is presently the case that there are no practical applications of transfinite models other than some models at the borderline between the conventionally infinite and the transfinite [13], [14, Example 1.7-1]. The motivation for research into transfinite electrical networks has to be purely mathematical, at least for now. Electrical network theory, which is so important to our vast electrical industry, is also part of mathematics, and it behooves at least a few of us to develop its theory in the various ways that theory may point, whatever be the abstractions encountered.

In the rest of this paper, we shall restrict our discussions to the first rank of transfiniteness, that is, to 1-graphs and 1-networks. Most of the ideas and many of the peculiarities occurring in transfinite network theory can be explained at this first level of transfiniteness. This may be enough for a tutorial/survey paper.[1]

2 A Need for Transfiniteness

Let us now ask a simple question about the conventionally infinite ladder shown in Fig. 3. What is its input resistance $R_{in} = v_1/i_1$? When the ladder is uniform (that is, when all the r_k have the same value r and all the g_k are the same value g), it is customarily asserted that R_{in} is the characteristic resistance R_{ch}. Its value can be determined by solving the circuit of Fig. 4; that is, R_{ch} is the positive solution of the quadratic equation

$$R_{ch} \;=\; \frac{1}{g + \frac{1}{r+R_{ch}}}$$

Thus,

$$R_{ch} \;=\; -\frac{r}{2} + \left(\frac{r^2}{4} + \frac{r}{g}\right)^{1/2}. \tag{1}$$

Wait. We may be missing something. Let us examine this matter a little more closely. Can we agree that, given any network, any voltage-current regime (i.e., any set of branch voltages and branch currents) that satisfies Ohm's law on every branch, Kirchhoff's current law at every node, and Kirchhoff's voltage law around every loop is a solution of the network? If so, we can then assert that the input resistance R_{in} of any infinite ladder, whether uniform or nonuniform, is any arbitrarily chosen real number (positive, negative, or zero). Indeed, to compute R_{in}, we can set $v_1 = 1$ and compute i_1 to get $R_{in} = 1/i_1$. Now, choose i_1 to be any real number. Then, a solution can be obtained by computing recursively according to

$$i_{k+1} \;=\; i_k - g_k v_k \tag{2}$$

$$v_{k+1} \;=\; v_k - r_k i_{k+1} \tag{3}$$

[1] We should also point out that two prior tutorial/survey papers on infinite electrical networks have appeared; the first one [10] did not discuss transfinite networks, and the second one [12] hardly touched upon them.

for $k = 1, 2, 3 \ldots$. This recursion can be continued indefinitely, and any branch voltage or current can thereby be determined.[2] It turns out however that, for the uniform ladder, there is one and only one choice of i_1 for which the i_k and v_k tend to 0 as $k \to \infty$; it is $i_1 = 1/R_{ch}$. Every other choice of i_1 yields currents i_k and voltages v_k whose absolute values tend to ∞ as $k \to \infty$. A regime of this latter kind dissipates infinite power and requires two power sources, one injecting the finite power $i_1 v_1$ at the input and another injecting infinite power at the infinite extremity of the ladder. Thus, it appears that we can distinguish R_{ch} from all other input resistances by stipulating one more condition, namely, the total power dissipated throughout the ladder is to be finite. This insures that the only power injected into the network comes from the input.

We seem to have resolved the problem of determining a "proper" input resistance for an infinite ladder: Just impose Ohm's law, Kirchhoff's laws, and a finite power condition. Let us apply this conclusion to a nonuniform ladder. For the sake of specificity, let it be the ladder of Fig. 3, where now $g_k = 2^{-k}\,\mho$ and $r_k = 2^{-k}\,\Omega$ for all k. Some computation will show that any choice of i_1 will lead to a finite-power regime. Indeed, whatever be i_1, i_k will approach a constant value as $k \to \infty$, and so too will v_k. The fact that g_k and r_k tend exponentially to 0 then insures that the power dissipated is finite.

Perhaps, we should try another way of determining a "proper" input resistance. Let us truncate the ladder by opening the nth resistance pair (i.e., set $r_n = \infty$), then compute the corresponding input resistance R_n, and finally take $\lim_{n\to\infty} R_n$ as the "proper" value of R_{in}. We get $R_{in} = 1.1245\,\Omega$. Just to check this result, let us now truncate the ladder by placing a short across the nth conductance g_k, compute the resulting input resistance, and then take the limit again as $n \to \infty$. We now get $R_{in} = .6384\,\Omega$. We are forced to conclude that R_{in} depends on whether we send an open or a short out to infinity, and we are led heuristically to the idea that R_{in} depends upon what is connected to the infinite extremity of the ladder — in these two cases, an open or a short.[3] Indeed, we can move any resistance R_L out to infinity by truncating the ladder either just after g_n or just after the $r_n/2$ pair

[2] A recursive procedure like this can be performed on any conventionally infinite network [14, Chapter 2].

[3] This is equivalent to writing R_{in} as an infinite continued fraction using the rules for series and parallel combinations of the r_k and g_k and then noting that the continued fraction does not converge [15, Section 1.4].

and using R_L as a load resistance at that truncated output. Upon sending $n \to \infty$, we get a value of R_{in} depending upon the value of R_L. We might interpret this as the input resistance R_{in} when "R_L is connected to the ladder at infinity." For example, $R_{in} = .8380\,\Omega$ when $R_L = 1\,\Omega$. In fact, this even works for negative values of R_L; for instance, $R_{in} = .5\,\Omega$ when $R_L = -3.1800\,\Omega$, and $R_{in} = -.5\,\Omega$ when $R_L = -1.0058\,\Omega$.

Let us review our investigation. We started by asking an ostensibly local question: What is the input resistance R_{in} of the ladder of Fig. 3? For the uniform ladder, the answer $R_{in} = R_{ch}$ was obtained through some local reasoning so long as a finite-power condition was also assumed. However, for the nonuniform ladder where $r_k = g_k = 2^{-k}$, local reasoning and the finite power assumption did not provide a unique value for R_{in} — even approximately. We were forced to go transfinite, that is, we had to declare what is connected to the ladder at its infinite extremity in order to determine a unique value for R_{in}. The essential difference between the uniform ladder and our nonuniform one is that "infinity is imperceptible" in the first case and "infinity is perceptible" in the second case. Indeed, in the uniform case there is infinite series resistance $r_1 + r_2 + \ldots$ and infinite shunting conductance $g_1 + g_2 + \ldots$ between the input and the infinity extremity of the ladder. Consequently, any finite-power source at the input is unable to send power to infinity; instead, voltages and currents decay to zero along the ladder. As a result, what is connected to the ladder at infinity is of no consequence; it cannot be perceived at the input. On the other hand, in our nonuniform case the total series resistance and total shunting conductance are finite, and power can therefore be sent to infinity. This forces us to specify just what the load at infinity is if we wish to determine what power the input voltage source produces, or equivalently what R_{in} is. In short, going transfinite in our nonuniform case is not generalization just for the sake of generalization. Going transfinite is forced upon us if we wish to answer a simple natural question. It is remarkable that this question was not posed and resolved transfinitely during the hundred or so years that engineers have been contemplating infinite transmission lines.

There is however a difficulty with this transfinite resolution of our problem. How does one connect the load resistor R_L to the infinite extremity of the ladder? There is no last pair of nodes to which R_L can be connected. What is needed is the invention of a new kind of node, the "1-node," which enables us combine the nodes of R_L with the infinite extremities of the ladder.

3 Transfinite Graphs

By a 0-*graph* we mean a conventional (finite or infinite) graph. Let us now explain more specifically just what is a transfinite graph of the first rank, namely, a 1-graph. First of all, we need to define precisely what are the infinite extremities of an infinite 0-graph. By a 0-*path* we will mean a path in the conventional sense; it is an alternating sequence of 0-nodes and branches

$$\{\ldots, n_a^0, b_1, n_b^0, b_2, n_c^0, \ldots\}$$

wherein each branch is incident in the graph to the two 0-nodes adjacent to it in the sequence and no 0-node (and therefore no branch) appears more than once. A 0-path may be *finite* because it terminates on the left and right at 0-nodes, *one-ended* because it terminates only on one side at a 0-node, or *endless* because it terminates on neither side. A 0-*loop* is a finite 0-path except that the two terminal 0-nodes are the same. We take it that one approaches an infinite extremity of a given 0-graph $\mathcal{G}^0$ by following unceasingly a one-ended path in $\mathcal{G}^0$. However, the infinite extremity that this tracing is extending toward should be independent of how the one-ended path initially roams. So, let us consider two one-ended paths as being equivalent if they are eventually identical, that is, if they differ by no more than finitely many branches and nodes. Then, let us partition all the one-ended paths in a given infinite 0-graph into equivalence classes, each class consisting of all the one-ended paths that are equivalent to each other. Each such class will be called a 0-*tip*, and these we take to be precisely the infinite extremities of $\mathcal{G}^0$. Just as we can connect branches by combining their terminals within 0-nodes, so too we can connect 0-graphs by combining their 0-tips within 1-nodes. More precisely, a 1-*node* is defined as a set of 0-tips along with (possibly but not necessarily) a single 0-node.[4] Those 0-tips may come from the same 0-graph or from

[4] We do not allow two or more 0-nodes in a single 1-node so as not to alter the 0-graphs by introducing a new short between 0-nodes.

different 0-graphs. The possible presence of a 0-node in the 1-node allows a terminal of a branch to be shorted to an infinite extremity.

For example, consider the infinite ladder graph of Fig. 5(a). One of its 0-tips consists of all the one-ended paths that eventually follow the a_k branches; let us denote this by 0-tip by t_a^0. Another 0-tip t_b^0 consists of all the one-ended paths that eventually follow the b_k branches. There are still others, such as the 0-tip, one of whose one-ended paths passes through the branches a_0, c_1, b_1, c_2, a_2, c_3, b_3, c_4, a_4, $\ldots$, thus zigzagging down the ladder. How many 0-tips are there? Uncountably many of them, in fact, as many as there are real numbers because a traversal of a horizontal branch is followed by a binary choice of a vertical or horizontal branch traversal. This is equivalent to following the binary representation of a real number.[5] Let us now combine t_a^0 with a 0-node of a branch b_L to make a 1-node n_a^1, and do the same thing with t_b^0 and the other 0-node of b_L to get another 1-node n_b^1, as is shown in Fig. 5(b). The result is a 1-graph. It is transfinite because b_L is connected to any other branch only through an infinite path. In this way, we have succeeded in a precise mathematical way of connecting a branch b_L to two of the infinite extremities of the ladder. This firms up the heuristic idea of a load resistance at infinity discussed in Section 2.

Of course, we could also connect the branch b_L to any two 0-tips of the ladder. This would be another perfectly legitimate 1-graph, but a complication arises in this case.[6] The trouble comes from the fact that, except for the pair t_a^0 and t_b^0, the representative paths of any two 0-tips remain forever entangled; that is, any path for one 0-tip and any path for the other 0-tip meet infinitely often. Thus, electrical power cannot be sent through to those two 0-tips simply because there is no transfinite loop for the flow of current through them. A consequence of this, as we shall see, is that node voltages with respect to a chosen ground node need not exist uniquely in an electrical network based on a 1-graph, even though that network has a unique operating point (i.e., a unique set of branch voltages and branch currents). Since we are presently discussing graphs, not electrical networks, it is too early to examine this electrical difficulty more closely.

However, there is a related connectedness problem that can be discussed. We can strip

[5]See [15, page 21] for a more detailed argument.

[6]This should not be surprising when infinities are introduced.

the problem down to its essentials by examining the simple 1-graph of Fig. 6. We first need to define a 1-*path*. This is an alternating sequence of 1-nodes and 0-paths

$$\{\ldots, n_a^1, P_1^0, n_b^1, P_2^0, n_c^1, \ldots\} \tag{4}$$

satisfying the following condition. No 0-node within any 0-path P_k^0 or 1-node n_z^1 appears more than once within the elements of the sequence (and thus no branch repeats either). Each of the two 1-nodes adjacent to a 0-path P_k^0 in the sequence (4) contains either a 0-tip of P_k^0 or a terminal 0-node of P_k^0, which implies that P_k^0 either extends infinitely toward the 1-node or terminates at that 1-node. Furthermore, if the sequence terminates on one side, it does so either with a 1-node or a 0-node. The fact that no 0-node appears within two or more 0-paths of (4) implies that, for the two 0-paths adjacent to a 1-node n_z^1 in (4), at least one of them must reach n_z^1 with a 0-tip. Finally, a 1-path is called *two-ended* if the sequence (4) is finite, it is called *one-ended* if (4) terminates only one side, and it is called *endless* if (4) does not terminate but instead extends infinitely on both sides. A 1-*loop* is a two-ended 1-path except that one terminal node embraces or is embraced by the other terminal node.[7]

By "0-connectedness" we mean connectedness in the conventional way. Thus, two nodes are said to be 0-*connected* if there is a finite 0-path terminating at them, and a graph is said to be 0-*connected* if every two nodes in it are 0-connected. Note that a 1-graph would be 0-connected if every 1-node in it contained a 0-node and there was a finite 0-path between every pair of nodes whatever their ranks. Furthermore, two nodes are said to be 1-*connected* if there exists a finite 0-path or a two-ended 1-path that terminates at those nodes.

Consider now the 1-graph of Fig. 6. The 0-node n_c^0 is 0-connected to n_a^1 and 1-connected to all the n_k^0, where $k = 1, 2, \ldots$. However, n_c^0 is not 1-connected to either n_b^1 or n_d^0 because there is no 1-path that terminates at n_c^0 and n_b^1 and similarly for n_c^0 and n_d^0. Indeed, any tracing from n_c^0 to n_d^0 will require a repetition of infinitely many of the n_k^0. Thus, we might say that n_c^0 and n_d^0 are "connected by a 1-walk" but not by a 1-path.

There is a related peculiarity concerning "sections." A *k-section* ($k = 0, 1$) is a maximal

[7] We say that a 1-node *embraces* itself, its 0-tips, and its single 0-node if it has one. We also say that a 0-node *embraces* itself.

subgraph such that every two nodes in it are k-connected. The k-sections of a 1-graph $\mathcal{G}^1$ are said to *partition* $\mathcal{G}^1$ if every branch belongs to one and only one k-section. It is a fact that 0-sections always partition every 1-graph, but this need not be so for 1-sections. For instance, in Fig. 6 the 0-subgraph to the left of the 1-nodes n_a^1 and n_b^1 is a 0-section. Branch β_a along with its two incident 0-nodes is another 0-section, and similarly β_b with its 0-nodes is a third 0-section. These 0-sections partition that 1-graph. On the other hand, all the branches a_k, b_k ($k = 1, 2, \ldots$) along with branch β_a induce (i.e., comprise the branch set of) a 1-section, and all the branches a_k, b_k, ($k = 1, 2, \ldots$) along with branch β_b induce another 1-section. In this 1-graph the 1-sections overlap and thus do not partition the 1-graph. Nothing of this sort can be said about conventional graphs; the 0-sections are simply the components of a 0-graph.

Both of these graph-theoretic peculiarities and the electrical difficulty alluded to before arise from the fact that the binary relationship "1-connectedness between nodes" is not in general transitive. We have seen in Fig. 6, for instance, that node n_c^0 is 1-connected to n_1^0, and n_1^0 is 1-connected to n_d^0, but n_c^0 is not 1-connected to n_d^0. This problem can be avoided if we restrict a 1-graph as follows. We shall say that two 0-tips t_a^0 and t_b^0 are *nondisconnectable* if any path in t_a^0 and any path in t_b^0 meet infinitely often. When this is so, every pair of paths for t_a^0 and t_b^0 will meet infinitely often. Furthermore, we shall say that a 0-tip t_a^0 is *open* if it is the only element in the 1-node containing it; this means that nothing is connected to the 0-tip, that is, after tracing a one-ended path in t_a^0 to reach t_a^0, there is no way to leave t_a^0 through another one-ended path not in t_a^0 or through some branch.

Here is a condition making 1-connectedness well-behaved:

Condition 3.1. *If two 0-tips are nondisconnectable, then either they are shorted together (i.e., are members of the same 1-node) or at least one of them is open.*

Theorem 3.2. *If a 1-graph $\mathcal{G}^1$ satisfies Condition 3.1, then 1-connectedness is transitive in $\mathcal{G}^1$.*

For example, in the networks of Fig. 5 there is only one pair of nondisconnectable 0-tips, the 0-tip t_a^0 induced by the a_k branches and the 0-tip t_b^0 induced by the b_k branches. All other pairs of 0-tips have representative paths meeting infinitely often. So, if we assume that all 0-tips other than t_a^0 and t_b^0 are open, then 1-connectedness becomes well-behaved so far as connections to the infinite extremities of the ladder are concerned.

Two consequences of Theorem 3.2 are the following:

Corollary 3.3. *Assume that the 1-graph $\mathcal{G}^1$ satisfies Condition 3.1. Then, the following are true.*

(i) *The 1-sections of $\mathcal{G}^1$ partition $\mathcal{G}^1$ (i.e., the 1-sections do not overlap).*

(ii) *Let P_a^1 and P_b^1 be two-ended 1-paths in $\mathcal{G}^1$ that meet at one or more nodes (0-nodes and/or 1-nodes). Let $\{n_k\}_{k\in K}$ denote the set of nodes where they meet. Totally order $\{n_k\}_{k\in K}$ according to an orientation for P_a^1. Then, $\{n_k\}_{k\in K}$ has a first node and a last node.*

Theorem 3.2 is easy to state but hard to prove (see [15, Chapter 3]). There is another condition that ensures the transitivity of 1-connectedness (see [15, Theorem 3.2-2]). It is harder to state but easier to prove.

4 Transfinite Electrical Networks

As was mentioned above, any transfinite graph can be converted into an electrical network by specifying electrical parameters for the branches. Much has been accomplished for transfinite networks of resistors and sources, but there are hardly any results available when inductors and capacitors are allowed. So, let us restrict our attention to the case where each branch has the Thevenin form of a resistor r in series with a voltage source e as shown in Fig. 7, where $r > 0$ and e may be any real number including 0. With J denoting an index set for the branches[8] and with $j \in J$, we assign the subscript j to the quantities in Fig. 7. Thus, Ohm's law becomes

$$v_j + e_j = r_j i_j \tag{5}$$

Let $v = \{v_j\}_{j\in J}$ and $i = \{i_j\}_{j\in J}$ denote the vector of branch voltages and the vector of branch currents respectively. Also, let us refer to the pair $\{v, i\}$ as a *voltage-current regime*

[8] The set of branches, and therefore J too, may be uncountably large.

— or in engineering parlance as an *operating point* — if Ohm's law and a generalization of Tellegen's equation (to be specified later) are satisfied. That generalization will encompass Kirchhoff's laws whenever those laws can hold. Unfortunately, Kirchhoff's laws do collapse for certain transfinite networks, and that is why we are forced to resort to that generalized form of Tellegen's equation. As a special case, we wish to have[9] $\sum v_j i_j = 0$. However, we now need to make sure that $\sum v_j i_j$ is a finite quantity before we can hope to set it equal to 0. We can accomplish this by adopting two more restrictions, one on i and the other on e:

$$\sum i_j^2 r_j < \infty \tag{6}$$

$$\sum e_j^2 g_j < \infty \tag{7}$$

Here $g_j = 1/r_j$. Restriction (6) asserts that every allowable current vector i dissipates only a finite amount of power in the resistors. The second one (7) concerns how much power can be extracted from all the voltage sources; the maximum such power occurs when a short is placed across each branch, in which case all the power extracted from the source of a branch is dissipated in that branch's resistor. We refer to this as the *total isolated source power*, and (7) asserts that it too is finite.

Furthermore, let $e = \{e_j\}_{j\in J}$ be the vector of so restricted voltage sources, and set $\langle x, y, \rangle = \sum x_j y_j$ for any two vectors x and y. Then, $\langle e, s\rangle = \sum e_j s_j$ denotes the power delivered by e when s is the current in the network. (As with general applications of Tellegen's equation, we are not requiring that s be the current vector generated by e.) Then, by Schwarz's inequality we have that

$$\langle e, s\rangle = \sum e_j \sqrt{g_j}\, s_j \sqrt{r_j} \le \left[\sum e_j^2 g_j \sum s_j^2 r_j\right]^{1/2},$$

and thus $\langle e, s\rangle$ is finite whenever the two summations on the right-hand side are finite, conditions we shall continue to assume. Altogether then, for any current vector $s = \{s_j\}_{j\in J}$ with $\sum s_j^2 r_j < \infty$, we have from Ohm's law (5), condition (6) on i, condition (7) on e, and Schwarz's inequality that

$$\sum |v_j s_j| = \sum r_j |i_j s_j| + \sum |e_j s_j| = \sum |i_j| \sqrt{r_j}\, |s_j| \sqrt{r_j} + \sum |e_j| \sqrt{g_j}\, |s_j| \sqrt{r_j}$$

[9]All summations will be over the branch index set J unless something else is specified. See [15, Appendix B] for a discussion of summations having uncountably many terms.

$$\leq \sum i_j^2 r_j \sum s_j^2 r_j \, + \, \sum e_j^2 g_j \sum s_j^2 r_j \, < \, \infty.$$

Thus, $\sum v_j s_j$ now has a meaning, and we can therefore impose Tellegen's condition: $\langle v, s\rangle = \sum v_j s_j = 0$.

Next, let R be the operator that assigns to each current vector $i = \{i_j\}_{j\in J}$ the vector $Ri = \{r_j i_j\}_{j\in J}$ of voltages across the branch resistors. Then, we can rewrite Tellegen's equation as

$$\langle v, s\rangle \; = \; \langle Ri - e, s\rangle \; = \; 0 \tag{8}$$

It is important to note that since Kirchhoff's laws need not hold everywhere, (8) is now being imposed, rather than being a consequence of those laws as in the case finite networks. We might say that Tellegen's equation now steps forward as a more fundamental relationship, which determines a voltage-current regime for the transfinite network, something that Kirchhoff's laws are unable to do in the case of transfinite networks.

However, these finite power conditions are not enough to generate a unique current vector i that we might accept as a sensible solution to a transfinite electrical network. We still wish to satisfy Kirchhoff's laws whenever it is possible to do so, especially for Kirchhoff's current law at finite maximal 0-nodes (that is, at each 0-node not contained in a 1-node and having only finitely many incident branches) and for Kirchhoff's voltage law around 0-loops (which perforce are finite loops). To do this, we set up a solution space $\mathcal{A}$ of allowable current vectors i that fulfill these desired properties regarding Kirchhoff's laws and in addition dissipate only a finite amount of power. $\mathcal{A}$ is the space that will be searched for a vector i (hopefully a unique one) that can be accepted as a solution to the network.

It turns out that there are at least four ways of setting up $\mathcal{A}$, as we shall explicate in the next section. Which space is preferable depends upon how much generality is desired. The greater the generality, the more likely it is that i will have an unexpected — perhaps, peculiar — distribution of values on the branches. The less the generality, the more likely it is that i will have more restricted properties, such as the inability to pass through 1-nodes.

The proof of the existence of a solution i can be informally described as follows. $\mathcal{A}$ will be chosen to be a "Hilbert space." This means that there is an "inner product" (synonymously, "scalar product" or "dot product") defined on all pairs i, s of current vectors in $\mathcal{A}$. In our case, that inner product is taken to be

$$(i, s) = \sum r_j i_j s_j. \tag{9}$$

Corresponding to this, there is a "norm" $\|i\|$ on each $i \in \mathcal{A}$ given by

$$\|i\| = \sqrt{\sum i_j^2 r_j}. \tag{10}$$

The "distance" between two current vectors i and s is $\|i - s\| = \sqrt{\sum (i_j - s_j)^2 r_j}$. Also, $\mathcal{A}$ is a "linear space," which means essentially that adding two vectors in $\mathcal{A}$ by adding their corresponding components yields a vector that is also in $\mathcal{A}$. Finally, $\mathcal{A}$ is "complete" in the following sense. If any current vector x is arbitrarily close to vectors in $\mathcal{A}$ (that is, if, given any $\epsilon > 0$, there is an $s \in \mathcal{A}$ such that $\|x - s\| < \epsilon$), then x is in $\mathcal{A}$ as well.

Now, by virtue of condition (7) on e, it turns out that a common theorem of functional analysis, called the Riesz representation theorem,[10] asserts that corresponding to the given e satisfying (7) there is a unique i such that (8) holds for all $s \in \mathcal{A}$. This is a fundamental theorem. We shall discuss four versions of it in the next section by choosing four different $\mathcal{A}$. At this point, let us present this fundamental theorem more explicitly.

Conditions 4.1. *The network* **N** *is either a 0-network or a 1-network (that is, its graph is either a 0-graph or a 1-graph), and its branches have the Thevenin form shown in Fig. 7 with positive resistors in all branches. Moreover, the total isolated source power is finite (that is, (7) holds).*

Theorem 4.2. *Under Conditions 4.1, there exists a unique branch-current vector i such that*

$$\langle e, s \rangle = \langle Ri, s \rangle \tag{11}$$

for every $s \in \mathcal{A}$. Furthermore, there is a unique branch-voltage vector v satisfying Ohm's law: $v = Ri - e$.

Here then is how Tellegen's equation (8) changes from a consequence of Kirchhoff's laws when dealing with finite networks to a more fundamental principle for transfinite networks,

[10] Also, the Riesz-Fischer theorem.

from which Kirchhoff's laws can be derived for certain nodes and loops including those mentioned above.[11]

5 Solution Spaces

Here we discuss and compare four solution spaces for the branch current vector i. The third and fourth of them are presented here for the first time.

5a. *Loop Currents:* The simplest of the solution spaces can be constructed by aping what is done with finite networks. We start with loop currents; these are currents that are restricted to 0-loops and 1-loops and are constant thereon. However, we also require that the power dissipated by each loop current be finite. This does not restrict the 0-loop currents, for they pass only through finitely many branches. On the other hand, it does restrict the 1-loop currents by requiring that the total resistance in any admissible 1-loop be finite. We let $\mathcal{L}^o$ denote the span[12] of such admissible loop currents.

Unfortunately, $\mathcal{L}^o$ is usually not large enough to encompass the current vector we seek, as we shall note in an example below. So, let us complete $\mathcal{L}^o$ by appending all current vectors that are arbitrarily close to members of $\mathcal{L}^o$ in the following sense: i will be appended to $\mathcal{L}^o$ if, for each $\epsilon > 0$, there is an $i' \in \mathcal{L}^o$ such that $\|i - i'\| < \epsilon$ (i.e., such that the power dissipated by $i - i'$ is less than ϵ^2). We let $\mathcal{L}$ denote the resulting expansion of $\mathcal{L}^o$. The fundamental Theorem 4.2 now holds with $\mathcal{A} = \mathcal{L}$. It turns out that every member of $\mathcal{L}$ will satisfy Kirchhoff's current law at every 0-node whose incident branch conductances have a finite sum and also Kirchhoff's voltage law around every loop whose branch resistances have a finite sum.

For example, consider the network of Fig. 5(a) with all 0-tips open. Let there be a voltage source in branch c_0 and no voltage sources anywhere else. Now, every member of $\mathcal{L}^o$ is a finite linear combination of 0-loop currents and therefore can be nonzero on only finitely many branches. However, Kirchhoff's laws require that any current vector that is nonzero on any one branch must be nonzero on infinitely many branches, that is, the current

[11] Because of space limitations, we do not derive Kirchhoff's laws here but instead refer the reader to [14, Section 3.4] and [15, Section 5.3].

[12] i.e., the set of all finite linear combinations of those loop currents.

distribution must spread out throughout the ladder. Thus, $\mathcal{L}^o$ cannot provide a solution to our network. However, $\mathcal{L}$ does do exactly this, as is asserted by our fundamental Theorem 4.2.

5b. *Basic Currents:* Depending upon the choice of the network **N**, $\mathcal{L}$ can in general be expanded by adding *basic currents*, which are defined to be a sum $i = \sum_{m \in M} i_m$ of current vectors i_m, where M is a countably infinite index set, each i_m is a 0-loop current or a 1-loop current, each branch or maximal 0-node meets only finitely many of the loops of the i_m, and the branches on which i is nonzero are contained in no more than finitely many 1-sections. Basic currents can be intuitively explained as an infinite superposition of loop currents that together permit a spreading and thinning out of a current regime as it flows toward infinity from some node of injection. This might allow such a current regime to dissipate finite power, even when an infinite power dissipation would result were that injected current to flow out to infinity along a single path.

An example of this is shown by the binary tree of Fig. 8, wherein a current of 1 A flows from the apex node n^0, spreads out uniformly through the tree, and then is gathered through a short at infinity and returned through a source branch to the apex node. When every branch of the binary tree is a 1 Ω resistor, the total power dissipated is finite; were that 1 A current to flow along a single path from n^0 to infinity, the power dissipation would be infinite. The current regime shown in Fig. 8 is a basic current and is the one dictated by the fundamental theorem.

For any network, the span of all finite-power basic currents is denoted by $\mathcal{K}^o$, and the completion $\mathcal{K}$ of $\mathcal{K}^o$ is obtained as before by appending all current vectors that are arbitrarily close to members of $\mathcal{K}^o$ as measured by the norm (10). Thus, every member of $\mathcal{K}$ also dissipates finite power. We always have $\mathcal{L} \subset \mathcal{K}$, and in general $\mathcal{K}$ is larger than $\mathcal{L}$, as is the case for Fig. 8. In fact, for that figure, $\mathcal{L}$ consists only of the zero vector, but $\mathcal{K}$ is much larger. The corresponding fundamental theorems yield $\mathbf{i} = \mathbf{0}$ under $\mathcal{L}$ and $\mathbf{i} \neq \mathbf{0}$ under $\mathcal{K}$. Moreover, Kirchhoff's voltage law is violated around every transfinite loop in Fig. 8 under $\mathcal{L}$ but is fulfilled under $\mathcal{K}$.

5c. *Tour Currents:* Let us define a *track* as a finite sequence $\{P_1, P_2, \ldots, P_m\}$ of oriented finite 0-paths or two-ended 1-paths such that, for each $k = 1, \ldots, m-1$, the last node of P_k embraces or is embraced by the first node of P_{k+1}.[13] It is not required that the these paths be disjoint. Thus, a track may pass through a node or branch several times — but at most finitely many times. Next, let us define a *tour* to be a track such that the last node of P_m embraces or is embraced by the first node of P_1. Thus, a tour generalizes a loop. Finally, we take a *tour current* to be a constant flow f of current passing along a tour. Thus, if a tour repeats a branch, the corresponding branch current is a multiple of f obtained by adding and/or subtracting f for each passage of the tour through the oriented branch, addition being used when the flow f and branch orientation agree, subtraction otherwise. In fact, the branch current may be 0 by cancellation. Note that a loop current is a special case of a tour current. Fig. 9 illustrates a tour current along a transfinite tour. This is the simplest network that must sustain a nonloop tour current if a certain branch (in this case, b_L) is to have a nonzero current, and it can often occur as a subnetwork in many more complicated networks.

Let $\mathcal{T}^o$ be the span of all finite-power tour currents,[14] and let $\mathcal{T}$ be the completion of $\mathcal{T}^o$, as before. Thus, every member of $\mathcal{T}$ dissipates finite power. $\mathcal{A}$ is now $\mathcal{T}$ in Theorem 4.2. Moreover, we have that $\mathcal{L} \subset \mathcal{T}$. A branch may have a nonzero solution current in $\mathcal{T}$ but a zero solution current in $\mathcal{L}$ and in $\mathcal{K}$. In fact, this is the case for the example of Fig. 9, (for whose network $\mathcal{L} = \mathcal{K}$). Indeed, the branch current in b_L dictated by Theorem 4.2 can be nonzero under $\mathcal{T}$ but must be zero under $\mathcal{L} = \mathcal{K}$ whatever be the choices of the branch resistances and voltage sources. This is because there is no loop passing through b_L, but there are tours doing so. When $\mathcal{A} = \mathcal{T}$, when there is a source in b_L, when no other branch has a source, and when the resistances in the branches to the left of the 1-nodes have a finite sum, there will be a nonzero current in b_L but zero current in all other branches; this is an apparent violation of Kirchhoff's current law. However, Kirchhoff's voltage law is satisfied around tours so long as resistances decrease rapidly enough as stated in Fig. 9.

5d. *Splayed Currents:* $\mathcal{T}$ can be expanded into a generally larger space $\mathcal{S}$ in much the

[13]See footnote 7 for "embrace."

[14]Obviously, a tour current is of finite power if and only if the resistances in the tour sum to a finite amount.

same way as $\mathcal{L}$ is expanded into $\mathcal{K}$. For certain networks this will permit a thinning out of currents as they flow toward infinity, thereby enabling a finite-power current distribution that may not be available in $\mathcal{T}$. A "splayed current" is specified as is a basic current except that loops are replaced by tours. Specifically, we define a *splayed current* to be a sum $\mathbf{i} = \sum_{m \in M} \mathbf{i}_m$, where M is a countably infinite index set and each $\mathbf{i}_m$ is a tour current, such that each maximal 0-node and each branch meets no more than finitely many tours of the $\mathbf{i}_m$ and moreover the branches where i is nonzero are contained in no more than finitely many 1-sections. Thus, a tour current is a special case of a splayed current. For $\mathbf{i}$ to dissipate finite power, it is not required that any of the $\mathbf{i}_m$ do so; in fact, it is possible for every $\mathbf{i}_m$ to dissipate infinite power, whereas $\mathbf{i}$ dissipates finite power nonetheless. We now let $\mathcal{S}^o$ be the span of all finite-power splayed currents, and let $\mathcal{S}$ be the completion of $\mathcal{S}^o$ as before. Thus, the members of $\mathcal{S}$ also dissipate finite power. This time, we have $\mathcal{K} \subset \mathcal{S}$. Once again, the fundamental Theorem 4.2 holds for the still more general case where $\mathcal{A} = \mathcal{S}$. Moreover, Kirchhoff's voltage law now holds around permissive tours, that is, around tours whose resistances sum to a finite amount.

Fig. 10 illustrates a particular splayed current $\mathbf{i} = \sum_{m=1}^{\infty} \mathbf{i}_m$ on a quarter-plane square grid along with other branches connected to certain of the grid's extremities (i.e., 0-tips). All branch resistances are 1 Ω. The top three parts of that figure indicate $\mathbf{i}_m$ for $m = 1, 2, 3, 4, 5$. The small circle labeled n_a (resp. n_b) is a 1-node that connects to the 0-tip determined by the lowest (resp. the next lowest) path of horizontal branches. The small circles labeled w_m denote 1-nodes that connect to 0-tips determined by paths that wiggle rectangularly, passing along vertical and horizontal branches. All of these 0-tips and their corresponding 1-nodes w_m are different from each other. For m even (resp. odd), there is a branch at infinity connecting n_a (resp. n_b) to w_m, which allows $\mathbf{i}_m$ to close on itself at infinity. The indicated pattern repeats itself for $m = 6, 7, 8, \ldots$ with the vertical wiggles expanding upward and with the first (i.e., leftmost) wiggle shifting to the right as m increases. The bottom part of Fig. 10 shows the total splayed current $\mathbf{i}$ so far as the currents within the grid are concerned. All the branches at infinity carry nonzero currents too, but are not shown in this part. All the tour currents $\mathbf{i}_m$ in $\mathbf{i}$ cancel to zero within the grid except on the squares shown. Note that every $\mathbf{i}_m$ by itself dissipates infinite power. On the other hand, $\mathbf{i} = \sum \mathbf{i}_m$ dissipates finite power. This provides an example of how an admissible splayed current can be a superposition of inadmissible tour currents. Note also that with respect to $\mathbf{i}$ each current at infinity passes through its branch but goes no further. This is yet another apparent violation of Kirchhoff's current law.

5e. *Which Solution Space Should We Use?*

As was indicated above, $\mathcal{L} \subset \mathcal{K} \subset \mathcal{S}$ and $\mathcal{L} \subset \mathcal{T} \subset \mathcal{S}$, with strict inclusion occurring for certain networks. So, which one of these solution spaces is preferable?

The smallest space $\mathcal{L}$ is the simplest to comprehend, but it may yield only the trivial solution $i = 0$ and lead to a violation of Kirchhoff's voltage law around some transfinite loops, as we have noted in Subsection 5b. In contrast to $\mathcal{L}$, the space $\mathcal{K}$ may provide a nontrivial solution i and may ensure a satisfaction of Kirchhoff's voltage law, at least in special cases where $\mathcal{L}$ is unable to do so. This is the case for the network of Fig. 8. As will be noted at the end of the next section, the nonsatisfaction of Kirchhoff's voltage law is reflected in the loss of uniqueness for node voltages with respect to a chosen ground even when node voltages exist throughout the network. On the other hand, the spaces $\mathcal{T}$ and $\mathcal{S}$ may provide nontrivial solutions i in cases where $\mathcal{L}$ and $\mathcal{K}$ are both unable to do so. In fact, $\mathcal{T}$ and $\mathcal{S}$ always provide unique node voltages with respect to a chosen ground so long as the node voltages exist; this is because Kirchhoff's voltage law around permissive tours holds. But then, apparent contradictions of Kirchhoff's current law may arise, as was noted in Subsections 5c and 5d. It seems that the restoration of unique node voltages is paid for by possible violations of Kirchhoff's current law. Furthermore, the spaces $\mathcal{T}$ and $\mathcal{S}$ seem unnatural — at least with regard to our conventional thinking about electrical currents in finite networks as being superpositions of loop currents.

Compromising between these features, we feel that the space $\mathcal{K}$ is the most natural one to use. The theories presented in [14] and [15] have been based upon $\mathcal{K}$.

6 Node voltages

Another possible peculiarity of a transfinite network is that node voltages need not exist throughout the network whatever be the choice of solution space. An example of this is given in [15, pages 151-153]. This is a consequence of the facts that there are pairs of nodes in a transfinite network that are connected only through transfinite paths or tracks and that the sum of branch voltages along such a path or track may diverge.

Still another peculiarity can occur even when sums of voltages along paths or tracks all converge and a ground node has been selected: The node voltages need not be uniquely determined despite the fact that they exist. This is a shortcoming of the use of $\mathcal{L}$ and $\mathcal{K}$, but not of $\mathcal{T}$ or $\mathcal{S}$. To see this, we must first define "node voltages" for transfinite networks.

A track (and, as a special case, a path) is called *permissive* if the resistances in the track sum to a finite amount. Let n_g and n_0 be any two totally disjoint nodes of the transfinite network $\mathbf{N}$ with n_g designated as ground, and let T be a permissive track starting at n_0 and stopping at n_g. We define the *node voltage* v_0 at n_0 with respect to T to be the sum $\sum_{(T)} \pm v_j$, where $\sum_{(T)}$ denotes a sum along the branches of T with an additional term for each occurrence of a branch as T is traced from n_0 to n_g and where the $+$ (resp. $-$) sign is used with v_j if the orientation of the branch b_j agrees (resp. disagrees) with that tracing of T for the considered occurrence of b_j. Note that a node voltage will be assigned to n_0 if and only if there exists a permissive track between n_0 and n_g.

Theorem 6.1. *Let* $\mathbf{N}$ *be a transfinite network satisfying Conditions 4.1. Let the current regime in* $\mathbf{N}$ *be that dictated by the fundamental theorem based on the solution space* $\mathcal{T}$ *or* $\mathcal{S}$. *Let* n_g *be a chosen ground node and let* n_0 *be any other node. Let* T_1 *and* T_2 *be two permissive tracks in* $\mathbf{N}$ *starting at* n_0 *and stopping at* n_g. *Then, the node voltage assigned to* n_0 *along* T_1 *is the same as that along* T_2. *(That is, node voltages will be unique whenever they exist.)*

Proof. Let $-T_1$ denote the track T_1 with a reversed orientation — that is, $-T_1$ starts at n_g and stops at n_0. Let $(-T_1) \cup T_2$ denote the tour consisting of $-T_1$ followed by T_2. By the fundamental Theorem 4.2 based upon $\mathcal{T}$, $\sum v_j s_j = 0$ for any $\mathbf{s} \in \mathcal{T}$. Upon choosing $\mathbf{s}$ as a tour current along $(-T_1) \cup T_2$, we find that $\mathbf{s} \in \mathcal{T}$ because of the permissivities of T_1

and T_2, and from this we obtain $\sum_{(T_1)} \pm v_j = \sum_{(T_2)} \pm v_j$. The same argument holds when $\mathcal{T}$ is replaced by $\mathcal{S}$. Q.E.D.

Corollary 6.2. *Let* N *be as in Theorem 6.1. Let there be a permissive track from a chosen ground node n_g to every node of* N. *Then, there is a unique set of node voltages throughout* N *when the solution of* N *is based upon either $\mathcal{T}$ or $\mathcal{S}$.*

As an example, consider again the network of Fig. 9, where now the branches to the left of the 1-nodes have no sources, their resistances have a finite sum, and the branch b_L is a series circuit of a 1 Ω resistor and a 1 V source oriented upwards. Then, as was noted in Subsection 5c, under $\mathcal{L}$ or $\mathcal{K}$, currents must be zero in all branches including b_L because the network does not have any 1-loops coupling b_L to the other branches. However, there are tours passing through b_L. Consequently, under $\mathcal{T}$ or $\mathcal{S}$, a nontrivial current distribution will occur; specifically, the current in b_L is 1 A and the currents in all other branches is 0 A. Then, with any 0-node to the left of the 1-nodes chosen as ground, we have that, with regard to the solution found in $\mathcal{L}$ or $\mathcal{K}$, the node voltage at the upper 1-node is either 0 V or 1 V depending on whether we choose a path passing along the upper branches or a path passing along the lower branches and then through b_L. On the other hand, with regard to the solution found in $\mathcal{T}$ or $\mathcal{S}$, that upper 1-node voltage along either path is 0 V because the 1 A current in b_L produces a zero branch voltage for b_L. However, this restoration of a unique voltage is paid for by an apparent violation of Kirchhoff's current law.

7 Simplifications

It is remarkable how easily the existence of an operating point for a transfinite network, as expressed by Theorem 4.2, can be obtained. The proof of that theorem is quite short [14, pages 77-78], [15, pages 132-134]. However, not much more can be developed if we maintain the generality of Conditions 4.1 because that generality allows quite complicated structures for transfinite graphs. We have to simplify those graphs if major progress is to be made. In this section we list a variety of such restrictions and point out what they lead to. Moreover, much of the strange behavior of transfinite networks can be tamed in this way. All this enables richer analyses of the simplified networks.

By describing possible simplifications, we can provide an indication of how complicated transfinite networks can be in general.

7a. *Nondisconnectable Permissive 0-Tips Shorted Together:* We noted in Section 3 that 1-connectedness becomes transitive if Condition 3.1 is satisfied by nondisconnectable 0-tips. A similar but nonetheless different condition insures the uniqueness of node voltages under the solution spaces $\mathcal{L}$ and $\mathcal{K}$. We shall say that a 0-tip is *permissive* (resp. *nonpermissive*) if it has a representative one-ended 0-path whose resistances have a finite sum (resp. infinite sum). Every representative one-ended 0-path will then have that same property.

Condition 7.1. *If two 0-tips are permissive and nondisconnectable, then they are shorted together.*

Theorem 7.2. *If a network satisfies Conditions 4.1 and 7.1, then under the solution spaces $\mathcal{L}$ and $\mathcal{K}$ its node voltages (with respect to a chosen ground node) are unique whenever they exist.*

See [15, Section 5.5] for a proof. This result extends to certain nonlinear networks as well [16]. Note also that, according to Corollary 6.2, node voltages are unique under the solution spaces $\mathcal{T}$ and $\mathcal{S}$, whether or not Condition 7.1 holds.

7b. *If Every 1-Node Contains No 0-Node, Sections and Subsections Coincide:* The fact that a 1-node can contain a 0-node leads to another kind of complication. In particular, the following distinction must be made. In contrast to a 0-section, which was defined in Section 3, a 0-*subsection* is a maximal subgraph whose nodes are connected by 0-paths that do not meet 1-nodes. For example, in Fig. 6 there are three 0-sections, one consisting of the branches to the left of the 1-nodes, another having branch β_a alone, and the last having branch β_b alone. On the other hand, there are three 0-subsections, one identical to the first 0-section and the other two being degenerate graphs consisting of only the 0-nodes n_c^0 and n_d^0 separately. However, if we add another branch incident to the 0-nodes n_1^0 and n_a^0, then the first two 0-sections coalesce into a single 0-section, but the 0-subsections remain unaffected. In this latter case, we can "pass through infinity" (i.e., we can pass through the 1-node n_a^1) via a finite number of branches. Thus, 0-sections no longer serve as a means of distinguishing conventionally infinite subnetworks within a transfinite network; instead, we must resort to 0-subsections for this purpose. This leads to complications when trying to establish a maximum principle for node voltages or when analyzing random walks on transfinite networks [15, Chapters 6 and 7]. Much simplification occurs when we simply assume that no 1-node contains a 0-node. When this is so, 0-sections and 0-subsections coincide.

7c. *Transfinite Ends:* A useful concept for conventionally infinite graphs is that of an *end* [7, page 40]. This is defined as an equivalence class of one-ended paths, where two such paths are equivalent if there is a third one which meets each of the first two infinitely often. More heuristically and particularly, one can think of a 0-graph as having finitely many ends if it is shaped like an octopus with a central finite body and finitely many infinitely long arms. If every 0-section in a 1-graph has such a structure and no 1-node contains a 0-node, then 1-nodes can only connect 0-sections at the extremities of their arms, that is, at their ends. Such a structure is illustrated in Fig. 11. In this case, 1-nodes can be isolated from each other by severing arms.[15] When every 0-section has only finitely many ends, each severing of an arm can be implemented by a finite set of branches, which serve as a cutset for the considered 0-section. This too is advantageous when trying to extend Kirchhoff's current law to 1-nodes.

7d. *Terminals:* As was noted in Section 2, the infinite extremities of a uniform ladder network cannot be perceived by a source at its input. Thus, so far as electrical behavior is concerned, there is no point to defining connections at those extremities. We should therefore only do so when electrical power can be transmitted through those extremities, but we need some way of determining this. Such can be done by converting the set of 0-nodes of each 0-section into a metric space as follows.

Let m and n be two 0-nodes in the same 0-section $\mathbf{S}^0$, and let no 0-node be a member of a 1-node (i.e., let 0-sections and 0-subsections coincide). Let $\mathcal{P}(m,n)$ be the set of all 0-paths in $\mathcal{S}^0$ that terminate at m and n. For each path P in $\mathcal{P}(m,n)$, let $|P|$ be its resistive

[15] This structure can be extended to 1-graphs by letting 0-sections take the role of branches to obtain thereby "1-ends," which are then connected together by 2-nodes — and so on for higher ranks of transfiniteness.

length, that is, the sum of resistance values for the branches in P. Set

$$d(m,n) = \inf\{|P|: P \in \mathcal{P}(m,n)\}.$$

It can be shown that d is a metric on the set $\mathcal{N}_{\mathbf{S}^0}$ of all 0-nodes in $\mathbf{S}^0$. $\mathcal{N}_{\mathbf{S}^0}$ then becomes a discrete[16] metric space, but it is not in general complete. Upon taking the completion of $\mathcal{N}_{\mathbf{S}^0}$, we obtain limit points, and these turn out to be the extremities of $\mathbf{S}^0$ through which electrical power can be sent [2]. We call these limit points *terminals.* Heuristically speaking, we append terminals to each 0-section at places where the 0-nodes of the 0-section crowd up infinitely as measured by the metric d.

For example, if every resistor of the ladder of Fig. 3 is 1 Ω, there are no terminals, and no connection at infinite is warranted. If $r_k = 2^{-k}$ Ω and $g_k = 2^k$ $\mho$ for all k, there is exactly one terminal, but with just one terminal there is no point in connecting anything at infinity. If however $r_k = 2^{-k}$ Ω and $g_k = 2^{-k}$ $\mho$ for all k, then there are two terminals. They coincide with the 0-tips t_a^0 and t_b^0 of Fig. 5(b), and thus an electrical connection at infinity now makes sense.

7e. *Permissive and Nonpermissive 1-Nodes:* Each terminal T of a 0-section can be identified with a set of permissive 0-tips of S^0 as follows. If a 0-tip t^0 of $\mathbf{S}^0$ has a representative 0-path whose 0-nodes converge under the metric d to the terminal T of $\mathcal{N}_{\mathbf{S}^0}$, then t^0 is a permissive 0-tip and is a member of the set of 0-tips with which T is identified. In fact, the terminals of $\mathbf{S}^0$ partition the set of permissive 0-tips of $\mathbf{S}^0$. A helpful simplification occurs if we use the terminals to create the connections between 0-sections. In particular, we define a *permissive* 1-node to consist of all the 0-tips (perforce permissive) in a set of terminals chosen from among all the terminals of all the 0-sections of the network $\mathbf{N}$. A permissive 1-node can be viewed as a short among terminals. Every nonpermissive 0-tip can be effectively discarded so far as electrical behavior is concerned by making it the sole member of a singleton 1-node, called a *nonpermissive 1-node.* In other words, we open all nonpermissive 0-tips.

These are strong assumptions, for they remove the distinction between the solution spaces $\mathcal{L}$ and $\mathcal{K}$, and also between $\mathcal{T}$ and $\mathcal{S}$. The thinning out of current flows among

[16] There is a positive distance between every two 0-nodes.

nonpermissive 0-tips is what made $\mathcal{K}$ larger than $\mathcal{L}$, and $\mathcal{S}$ larger than $\mathcal{T}$. By opening the nonpermissive 0-tips, we prevent such thinning out. However, these simplifications may be worth the restriction.

7f. *Finite Incidences Between 0-Sections and 1-Nodes:* A further simplification occurs when we assume that each 0-section has only finitely many terminals and therefore only finitely many incident 1-nodes and conversely assume that each permissive 1-node has only finitely many terminals and therefore is incident to only finitely many 0-sections. This is advantageous when discussing current flows through 0-sections and when trying to extend Kirchhoff's current law to 1-nodes. The 0-sections incident to a 1-node play a role analogous to that of the branches incident to a 0-node. "Local finiteness," which is the condition of finitely many branches incident to each 0-node, plays a strongly simplifying role in the theory of conventionally infinite graphs. Finite incidences between 0-sections and 1-nodes is an extension of local finiteness to transfinite graphs.

7g. *Coincidences between Ends and Terminals:* Still another simplification occurs if terminals and ends coincide (i.e., if there is a bijection between terminals and ends). When this is so, the extremity of each arm of a 0-section can allow power to be transmitted through itself and into other such extremities. Thus, a source in one 0-section may then send currents through 1-nodes and thereby into other 0-sections.

7h. *Consequences:* Various combinations of the above simplifications serve to tame transfinite electrical networks sufficiently to allow some rich theories about them to be constructed.[17] For example, we now have a potential theory for node voltages wherein a generalized maximum principle holds [15, Chapter 6]. Also, a theory for random walks on transfinite networks now exists [15, Chapter 7]; a random walker may now "wander through infinity." Furthermore, nonstandard analysis [6] has been extended to certain transfinite networks so that Kirchhoff's laws have been reestablished for those networks using infinitesimal and infinite hyperreal numbers [17]. Finally and most recently, the classical theory for nonlinear finite networks by Minty [5] has now been extended to certain transfinite networks whose branch characteristics are maximal monotone and may have

[17]Some further technicalities, such as the assumption of only finitely many permissive 1-nodes, are also imposed for particular results, but the above list covers the principle ideas.

restricted domains or ranges; the first rank of transfiniteness is discussed in [2] and higher ranks of transfinitenenss are covered in [18]. These two work also extend Wolaver's "no-gain" property [8] to Minty-type transfinite networks. Furthermore, for the kinds of transfinite networks considered, Kirchhoff's laws now work to establish an operating point, in contrast to the general theory of unrestricted transfinite networks.

References

[1] D.M. Burton, *The History of Mathematics*, Boston: Allyn and Bacon, 1985.

[2] B.D. Calvert and A.H. Zemanian, Operating points in infinite nonlinear networks approximated by finite networks, *Trans. Amer. Math. Soc.*, in press.

[3] J.W. Dauben, *Georg Cantor*, Princeton, N.J.: Princeton University Press, 1979.

[4] K. Devlin, *The Joy of Sets*, New York: Springer-Verlag, 1993.

[5] G. Minty, Monotone networks, *Proc. Royal Soc. London*, vol. 257, pp. 194-212, 1960.

[6] A. Robinson, *Non-Standard Analysis*, Amsterdam: North-Holland Publishing Co., 1970.

[7] W. Woess, Random walks on infinite graphs and groups — a survey on selected topics, *Bull. London Math. Soc.*, vol. 26, pp. 1-60, 1994.

[8] D.H. Wolaver, Proof in graph theory of the "no-gain" property of resistor networks, *IEEE Trans. Circuits and Systems*, vol. CT-17, pp. 436-437, 1970.

[9] A.H. Zemanian, Connections at infinity of a countable resistive network, *Circuit Theory and Applications*, vol. 3, pp. 333-337, 1975.

[10] A.H. Zemanian, Infinite electrical networks, *Proc. IEEE*, vol. 64, pp. 6-17, 1976.

[11] A.H. Zemanian, Infinite electrical networks with finite sources at infinity, *IEEE Trans. Circuits and Systems*, vol. CAS-34, pp. 1518-1534, 1987.

[12] A.H. Zemanian, Infinite electrical networks: A reprise, *IEEE Trans. Circuits and Systems*, vol. 35, pp. 1346-1358, November, 1988.

[13] A.H. Zemanian, An electrical gridlike structure excited at infinity, *Mathematics of Control, Signals, and Systems*, vol. 4, pp. 217-231, 1991.

[14] A.H. Zemanian, Infinite Electrical Networks, New York: Cambridge University Press, 1991.

[15] A.H. Zemanian, *Transfiniteness for Graphs, Electrical Networks, and Random Walks*, Boston: Birkhauser, 1996.

[16] A.H. Zemanian, The existence and uniqueness of node voltages in a non-linear resistive transfinite electrical network, *Internat. J. Circuit Theory and Appl.* vol. 24, pp. 635-644, 1996.

[17] A.H.Zemanian, Nonstandard electrical networks and the resurrection of Kirchhoff's laws, *IEEE Trans. Circuits and Systems — I: Fundamental Theory and Appl.*, vol. 44, pp. 221-233, March, 1997.

[18] A.H. Zemanian, *End-generated transfinite monotone networks*, CEAS Technical Report 752, University at Stony Brook, Stony Brook, N.Y., January, 1998.

Figure Captions

Fig. 1. An infinite square grid of branches and nodes forming an "infinite checkerboard" pattern.

Fig. 2. A 1-graph consisting of an infinite checkerboard pattern of infinite square grids, that is, an "infinite checkerboard of infinite checkerboards." The small circles represent 1-nodes that connect the square grids at their horizontal and vertical infinite extremities.

Fig. 3. A conventionally infinite ladder. The g_k are conductances, and the r_k are resistances; g_k need not be equal to $1/r_k$.

Fig. 4. When the ladder of Fig. 3 is uniform, its characteristic resistance R_{ch} can be obtained by solving this circuit for R_{ch}.

Fig. 5. (a) A conventionally infinite ladder graph. n^0 is a 0-node. The a_k, b_k, and c_k $(k = 0, 1, 2, \ldots)$ are branches. Each dashed line on the right represents a 0-tip.

(b) A branch b_L connected to the two 0-tips t_a^0 and t_b^0 of the ladder through the 1-nodes n_a^1 and n_b^1. Each of the other 0-tips is the sole member of a singleton 1-node and is not shown in this diagram.

Fig. 6. A 1-graph. The a_k, b_k, β_a, and β_b are branches. The solid dots denote 0-nodes, and the two small circles denote 1-nodes, n_a^1 and n_b^1. Each of those 1-nodes contains a 0-node, n_a^0 and n_b^0 respectively.

Fig. 7. Every branch has this Thevenin form.

Fig. 8. An infinite, purely resistive, binary tree fed by a 2 V source branch that gathers current through a short at infinity (i.e., through a 1-node) and feeds it back to the apex node n^0. The vertical dashed lines denote the uncountably many 0-tips that are shorted together by the 1-node. Every branch has a 1 Ω resistor. The numbers near arrows indicate branch currents.

Fig. 9. The transfinite network of Fig. 6 modified to have only a single branch b_L connected to the two 0-tips induced by the upper and lower branches through 1-nodes. The arrows indicate a tour current that passes through every branch once and only once. When the resistances in the loops are 2^{-k} Ω, where the index $k = 1, 2, 3, \ldots$ counts loops from the left, we can have a finite-power tour current as shown.

Fig. 10. The splayed current $i = \sum_{m \in M} i_m$ discussed in Section 5d. The quarter-plane grid is shown four times to display i_m for $m = 1, 2, 3, 4, 5$ and also i.

Fig. 11. A 1-graph illustrating the conditions in Subsections 7c and 7d. The cross-hatched areas represent six 0-sections. The small circles denote six 1-nodes. There are 16 ends.

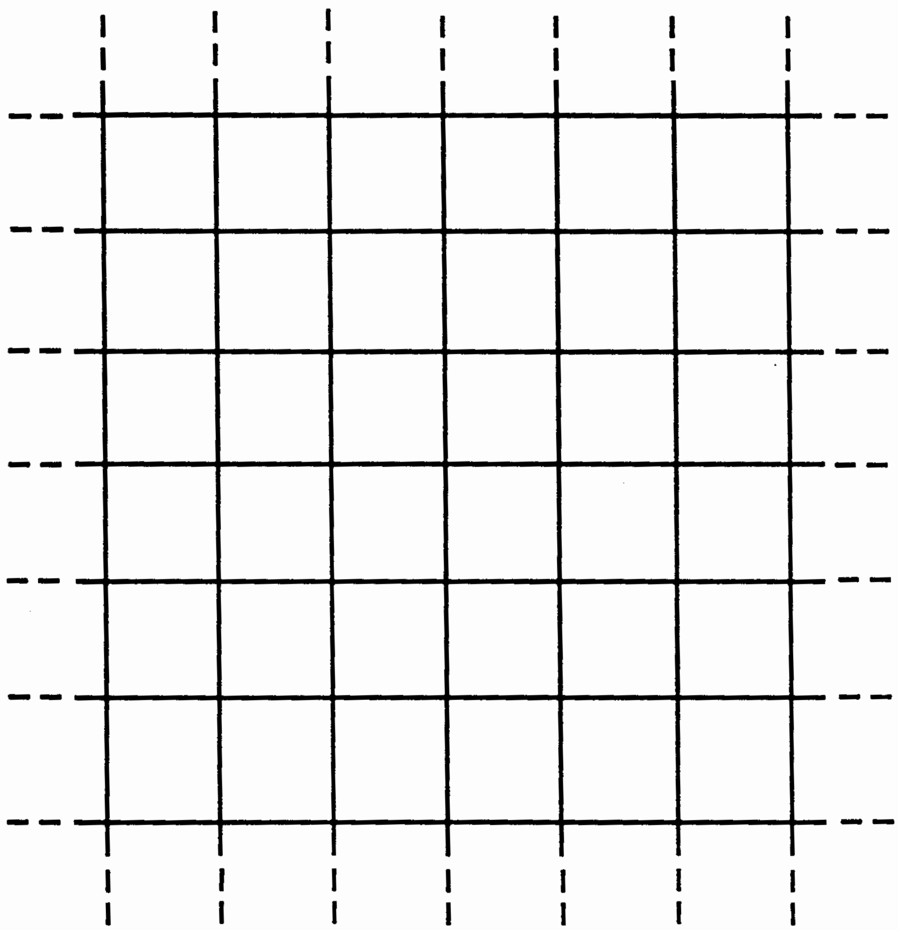


Fig. 1

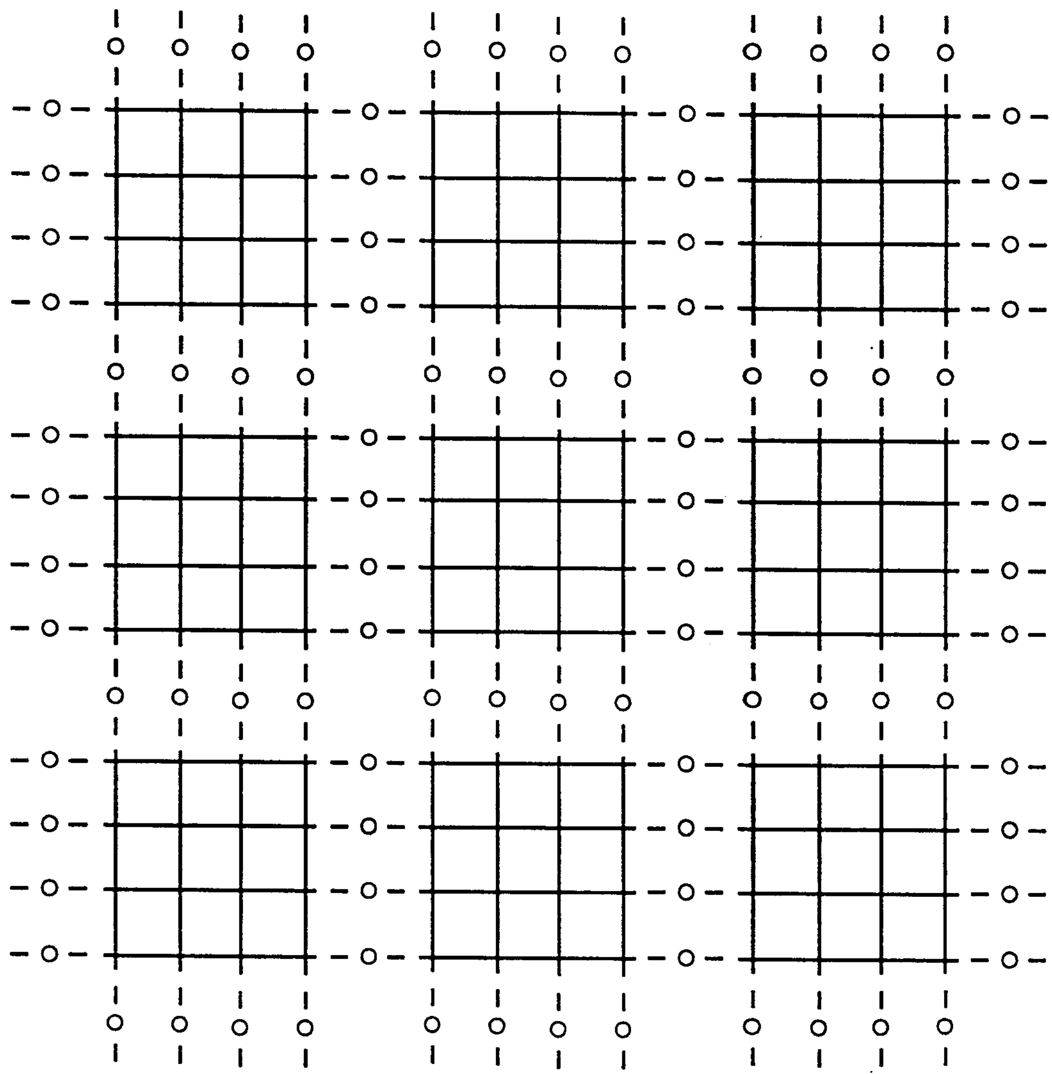


Fig. 2

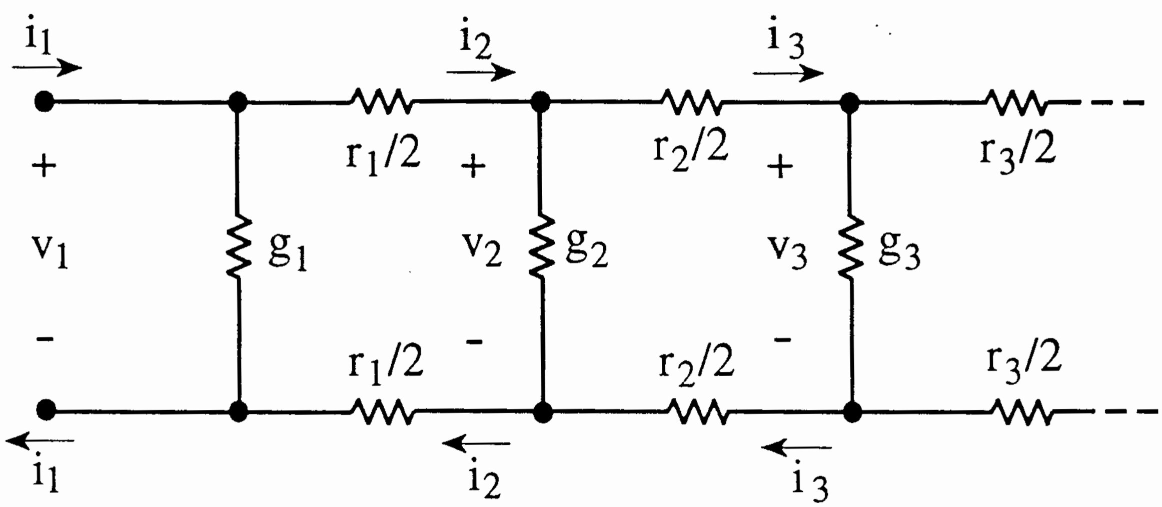


Fig. 3

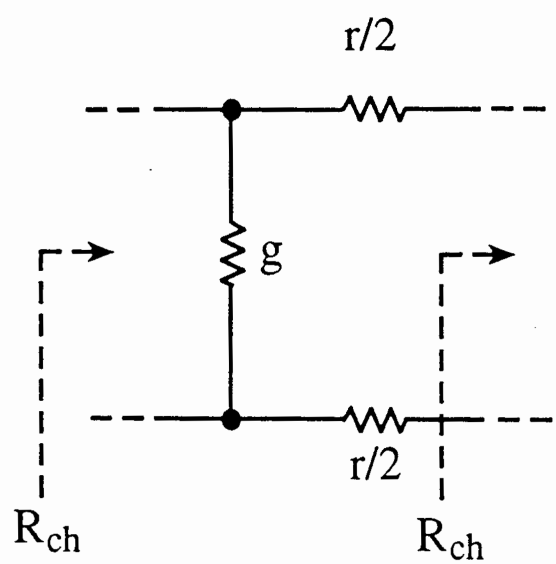
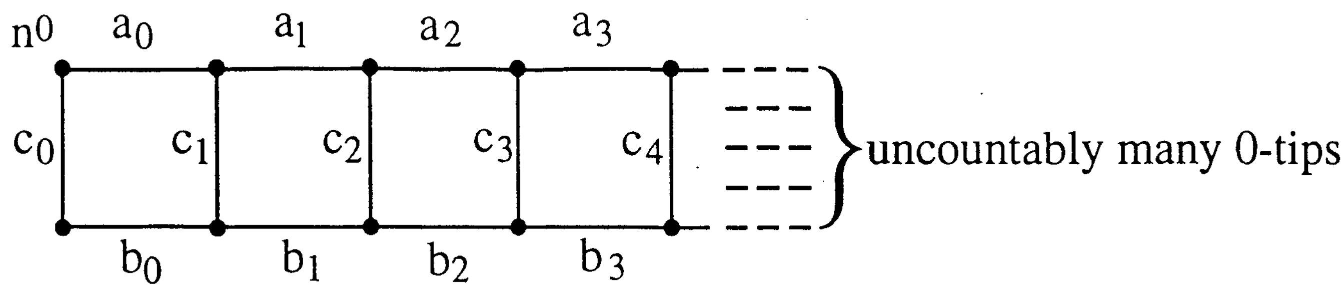
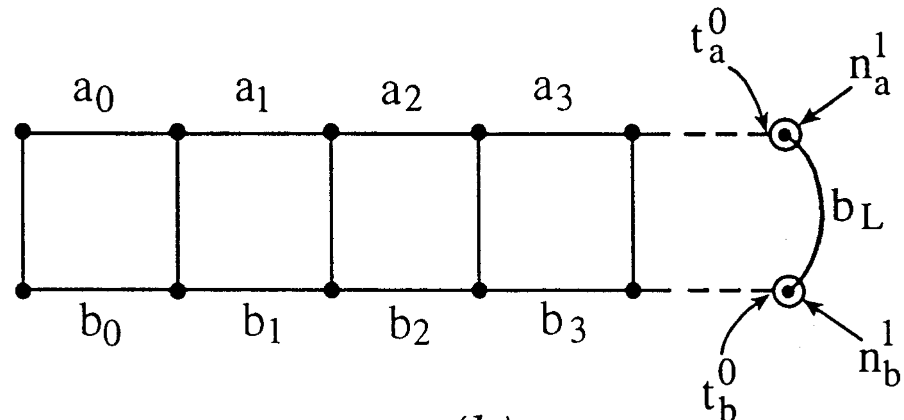


Fig. 4



n^0
a_0
a_1
a_2
a_3
c_0
c_1
c_2
c_3
c_4
uncountably many 0-tips
b_0
b_1
b_2
b_3

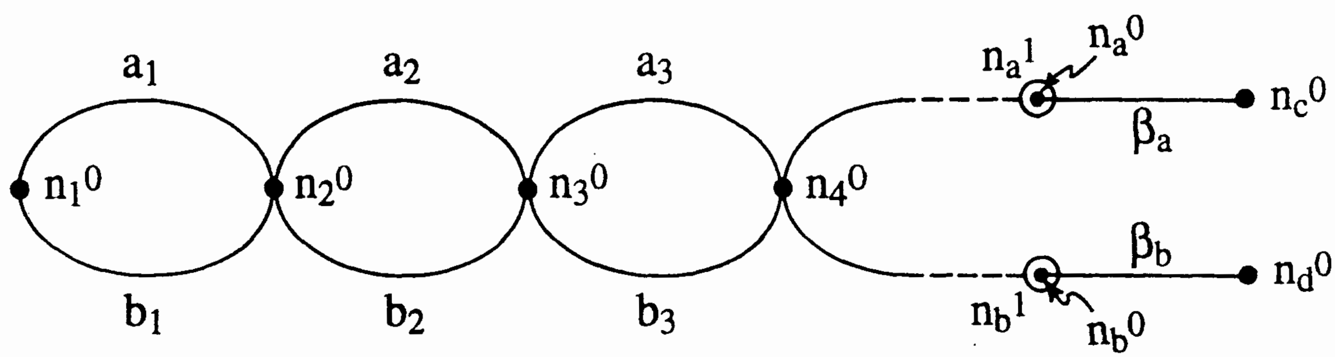
(a)



t_a^0
n_a^1
a_0
a_1
a_2
a_3
b_L
b_0
b_1
b_2
b_3
n_b^1
t_b^0

(b)

Fig. 5 a/b



a_1
a_2
a_3
n_a^1
n_a^0
n_c^0
β_a
n_1^0
n_2^0
n_3^0
n_4^0
β_b
n_d^0
b_1
b_2
b_3
n_b^1
n_b^0

Fig. 6

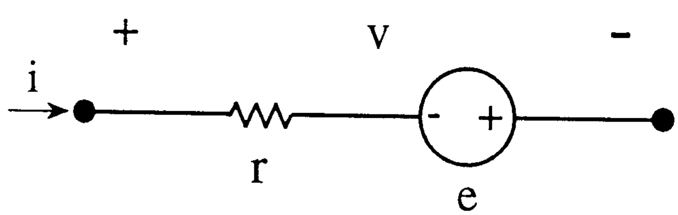


Fig. 7

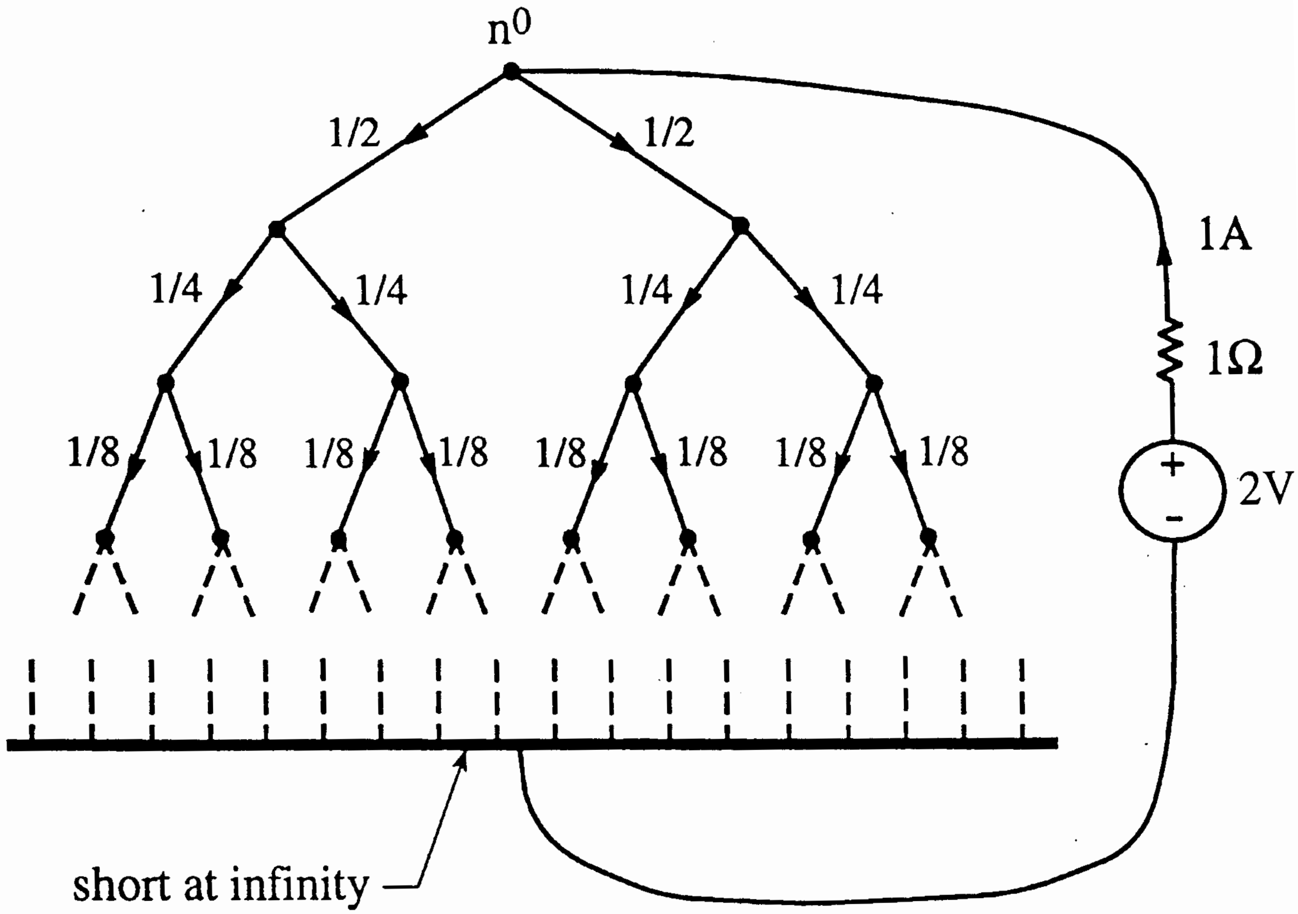

n0
1/2
1/2
1/4
1/4
1/4
1/4
1/8
1/8
1/8
1/8
1/8
1/8
1/8
1/8
1A
1Ω
+
-
2V
short at infinity

Fig. 8

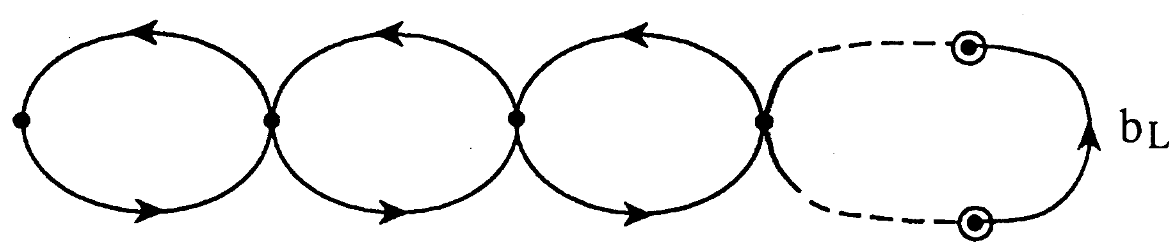
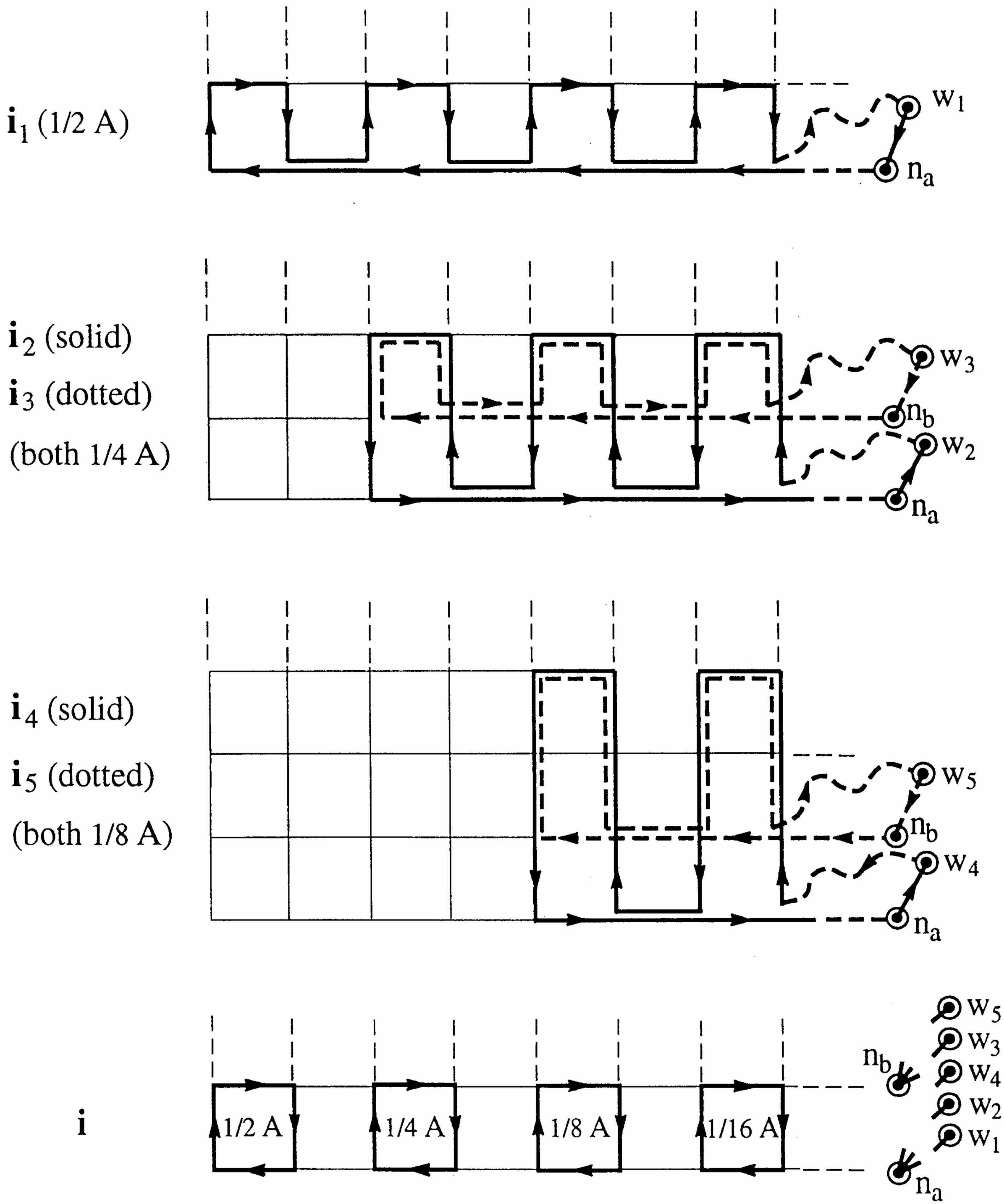


Fig. 9



i_1 (1/2 A)
w_1
n_a
i_2 (solid)
i_3 (dotted)
(both 1/4 A)
w_3
n_b
w_2
n_a
i_4 (solid)
i_5 (dotted)
(both 1/8 A)
w_5
n_b
w_4
n_a
i
1/2 A
1/4 A
1/8 A
1/16 A
w_5
w_3
n_b
w_4
w_2
w_1
n_a

Fig. 10

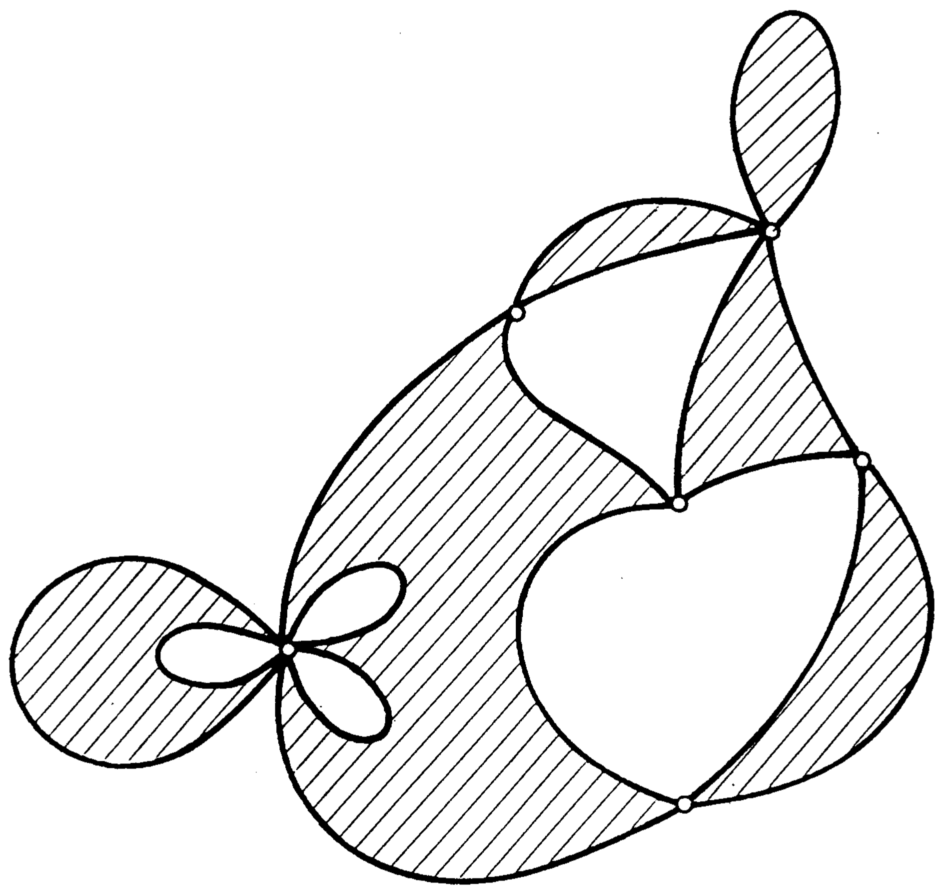


Fig. 11