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BUOYANCY AND SURFACE-TENSION INDUCED INSTABILITIES OF FLUID IN OPEN AND CLOSED VERTICAL CYLINDRICAL CONTAINERS

by

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Abstract

Existing attempts to extend two dimensional treatments of stability problems to the three dimensional cases under consideration are shown to be erroneous. These mistakes stem from a lack of clarity in the accepted formulation of the governing differential system. The author restates it in a form that can fully account for the presence of the surrounding solid and free surfaces as well as for surface-tension variations in the latter. Approximate solutions for circular cylindrical geometry are then found to be physically plausible. Thus, an increase in the conductivity of the solid surface has a stabilizing effect. An increase in either the conductivity of the top surface or the slenderness of the container retard the Marangoni type of instability.

1. Introduction

This work is concerned with the instability of fluid which is either completely confined in a closed vessel or partially confined in a container which is open on top. The instability is either of the Ragleigh or the Marangoni type or due to both effects combined. This work is therefore similar to Pellew and Southwell's[1], Yih's[2], Nield's[3] and many other treatments[4,5,6]. It differs from these in that it fully accounts for the presence of laterally confining vertical walls. Thus, unlike previous ones this work is concerned with a bona fide three dimensional problem. In most of the works cited a horizontal layer of fluid is considered and pertinent variables are assumed to satisfy the Helmholtz equation in the horizontal co-ordinates (x_1, x_2). Periodicity in the vertical direction x_3 is assumed when very high columns are considered[2]. Consequently only in one direction the variations of the dependent variables are a priori unknown. The critical Rayleigh and Marangoni numbers R_a and β are therefore characteristic values of ordinary homogeneous differential systems. Treatments of this type are inapplicable to the problems under consideration. Once the x_i dependence of the variables is assumed to have a particular form, it is impossible to prescribe arbitrary conditions at x_i = const. An appropriate alternative approach is proposed here and various particular cases are solved by way of example.

The need for an altogether new method has not been recognized, probably because Pellew and Southwell appear to have outlined a procedure by which some of the problems under discussion could be solved. They assume the above-mentioned horizontal variations not only for unbounded layer but also in the case of latterally confined fluid (§ 13). This assumption gives rise to cylindrical surfaces $C(x_1, x_2) = 0$ on which the vertical component of velocity u_3 vanishes. They state the conditions that should be satisfied by the other variable there, assuming that this surface is solid (§ 14). The variable u_3 and Ra are solved for simultaneously. It is proposed to complete the solution for the horizontal components of velocity (u_1, u_2) and the temperature, when u_3 and Ra are already known (§ 12). It is noteworthy that according to this procedure the conditions imposed at $C(x_1, x_2) = 0$ on all variables other than u_3 play no role in the solution for Ra. This mathematical scheme, therefore, does not reflect the full physical effect of the latteral wall on the confined fluid. Nevertheless they try this method (§§ 37, 38). Understandably, they fail to satisfy one of the conditions imposed at $C(x_1, x_2) = 0$, namely, that of zero heat flux. They conclude that "the case of rigid non conducting boundary is intractible" but fail to mention that their approach is, to say the least, unpromising. Anyway, they do not rule out the possibility that the case of rigid conducting boundary (which they believe to be unrealistic) could be analyzed by their method. It is shown here that subsequent attempts [7, 8] to do so produced wrong results. Indeed close examination with attention on its determinacy shows that the sixth order differential system proposed by Pellew and Southwell governs only the instability in a latterally unconfined layer.

The formulation of the appropriate differential system is presented in the next section. This formulation and ensuing analysis is based on the assumption that marginally stable modes are time-independent. This has been proved for the cases of completely confined fluid [2]. For the other cases this widely acceptable assumption appears to be justified in view of the existence of a non trivial time-independent solution and the plausibility of the results obtained.

Particular solutions are obtained for the marginally unstable modes in circular cylindrical containers. This is done by extending Jeffrey's [9] Goldstein's [10] method so as to account for the three dimensional nature of the problem. Dependent variables are expanded in terms of space-dependent functions which (like the Fourier Series in the one dimensional cases) form a complete series. The coefficients in the expansions are governed by infinitely many homogeneous algebraic equations. These are satisfied when Ra and β attain their critical values. Approximate results are obtained here by trunkation. These are restricted to cases in which the containers' radius to height ratio is not too large. For open containers these results are in a qualitative agreement with those obtained by Nield. For very slender containers the critical values of Ra are found to be within a few per cent from the values obtained by Yih.

2. Mathematical Formulation of the Problem

The equations considered are

$$(\partial/\partial t - \nu\nabla^2)u_i = -\rho^{-1}\partial p/\partial x_i + (0, 0, g\alpha T), \qquad (1) - (3)$$

$$\partial u_j/\partial x_j = 0, \qquad (4)$$

$$(\partial/\partial t - \kappa\nabla^2)T = \beta u_3 \qquad (5)$$

Here p and T are the deviation of the pressure and temperature respectively from their undisturbed state. The expensivity of the fluid, its kinematic viscosity and thermal diffusivity are designated by α, ν and κ, respectively. The symbols β and g denote the imposed temperature gradient and gravitational acceleration. The symbol ∇^2 represents the three-dimensional Laplace differential operator. Repeated indices denote summation and t is time.

The boundary conditions imposed on the temperature can be quite generally expressed by

$$k(\partial T/\partial \eta) + hT = 0 \qquad (6)$$

In this relationship k is the thermal conductivity of the liquid, h is the appropriate coefficient of heat transfer and η is the direction normal to the bounding surface. In addition to (6) it is necessary to prescribe boundary conditions on either u_i or the three stress components. The conditions

$$u_i = 0 \qquad (i = 1, 2, 3) \qquad (7) - (9)$$

hold at the solid surface. Therefore equations (1) - (9) form a determinant system which governs the case of completely confined fluid.

The cases in which the top surface is exposed appear to be somewhat more complicated because that surface deviates from its undisturbed plane by an unknown amount $\zeta(x_1, x_2, t)$. From continuity one has

$$\frac{\partial \zeta}{\partial t} + u_1 \frac{\partial \zeta}{\partial x_1} + u_2 \frac{\partial \zeta}{\partial x_2} = u_3, \qquad (10)$$

where the non-linear terms are neglected within the frame-work of this study of stability. Similarly, linearization of the equilibrium equation in the x_3 direction yields

$$-\rho g \zeta + \left(\frac{\partial}{\partial x_1} S \frac{\partial \zeta}{\partial x_1} + \frac{\partial}{\partial x_2} S \frac{\partial \zeta}{\partial x_2} \right) = 2 \rho \nu \frac{\partial u_3}{\partial x_3} - p, \qquad (11)$$

where the right hand side is evaluated at the undisturbed position $x_3 = const$, rather than at $x_3 = const. + \zeta$. In the last relationship the normal stress is equated to the sum of pressure variation due to gravity and membrane-type of force, produced by the surface-tension S. Equilibrium in the x_1 and x_2 directions is maintained provided the following holds

$$\left(\frac{\partial S}{\partial T}\right) \frac{\partial T}{\partial x_i} = (\rho \nu) \left[\frac{\partial u_i}{\partial x_3} + \frac{\partial u_3}{\partial x_i} \right], \qquad (i = 1, 2). \quad (12) - (13)$$

Here $(\partial S/\partial T)$ is the rate of change of surface tension with the temperature, which gives rise to the Marangoni effect. Thus, the assumption of freely distorting surface requires introducing one more dependent variable ζ and prescribing four conditions (10) - (13) rather than the three (7) - (9). It is then noted that since boundary-condition (11) is in itself an elliptic partial differential equation, the system is underdeterminant unless ζ is made to satisfy appropriate condition at the intersection of the cylindrical surface $C(x_1, x_2) = 0$ and $x_3 = \text{const}$. It is assumed that vertical capillary forces are constant so that this condition is

$$S(\partial \zeta / \partial \eta) = 0 , \qquad \text{on} \quad C(x_1, x_2) = 0 \qquad (14)$$

where η is the normal to the cylindrical surface.

Fortunately, under the assumption that time variations vanish, equations (11) and (14) are uncoupled from the differential system which has to be solved. Under this assumption the linearized equation (10) reduces to

$$u_3 = 0 . \qquad (10')$$

In such case equations (10'), (12) and (13) are the three kinematic boundary conditions and these do not contain ζ. A determinant system is therefore obtained when equation (6) together with either these three or equations (7) - (9) are prescribed at the boundaries and equations (1) - (5) are made to hold throughout the cylindrical domain. The uncoupling of equations (11) and (14) implies that as long as the relative strengths of surface-tension, viscosity and gravity keep ζ infinitesimally small, these surface effects do not influence the stability of the fluid below. The variable $\rho^{-1}p$ plays a somewhat similar role to that of ζ. It does not appear in any of the boundary conditions (6) - (9), (10′) (12) and (13) and may therefore be eliminated (by cross differentiation) from the governing equations. Once all the other physical variables are solved for $\rho^{-1}p$ (and ζ when the top surface is exposed) can be obtained by integrating equations (1) - (3) (together with (11) and (14)).

While in this work only $\rho^{-1}p$ (and ζ) are eliminated, Pellew and Southwell eliminate also u_1, u_2 and T. Their derivation is therefore believed to be permissable under some but not all circumstances. Such step is legitimate whenever the conditions which are imposed on the eliminated variables are expressed in terms of the remaining one, which is u_3 in their analysis. They indeed show that the vanishing of u_1 and u_2 on a horizontal solid boundary together with equation (4) yields the condition

$$\partial u_3 / \partial x_3 = 0, \qquad \text{on} \qquad x_3 = \text{const.}$$

Conditions imposed on T are similarly expressed in terms of u_3. Their derivation is therefore believed to be legitimate in the case of an unbounded layer of fluid. However, they propose and try to solve the latterally bounded case without reducing the conditions of vanishing u_1 and u_2 on $C(x_1, x_2) = 0$ to a condition which is expressed in terms of u_3. This loss of information is reflected in Osrach and Pnueli's[7] and Pnueli's[8] restatement of the underdeterminant problem. There, a sixth order partial differential equation governing u_3 is assumed integrable when three conditions are prescribed everywhere on the boundary. Two of these

$$u_3 = 0 \qquad \partial u_3 / \partial x_3 = 0$$

are mutually independent only on a horizontal boundary.

In eliminating ρ'/ρ from the governing equations it is convenient to make use of the functions φ and ψ which are defined by

$$u_1 = -\partial\varphi/\partial x_1 - \partial\psi/\partial x_2 \qquad u_2 = -\partial\varphi/\partial x_2 + \partial\psi/\partial x_1 \tag{15}$$

Cross differentiation of equations (1) and (2) then yields

$$(\partial/\partial t - \nu\nabla^2)\bar{\nabla}_1^2\psi = 0 \tag{16}$$

where $\bar{\nabla}_1^2$ is the Laplace operator in x_1 and x_2.
Cross differentiation of the other two possible pairs of momentum equations yields

$$\left.\begin{aligned}
\frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial t} - \nu\bar{\nabla}^2\right)\left(\frac{\partial\varphi}{\partial x_3} + u_3\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial}{\partial t} - \nu\bar{\nabla}^2\right)\frac{\partial\psi}{\partial x_3} &= g\alpha\frac{\partial T}{\partial x_1}, \\
\frac{\partial}{\partial x_2}\left(\frac{\partial}{\partial t} - \nu\bar{\nabla}^2\right)\left(\frac{\partial\varphi}{\partial x_3} + u_3\right) - \frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial t} - \nu\bar{\nabla}^2\right)\frac{\partial\psi}{\partial x_3} &= g\alpha\frac{\partial T}{\partial x_2}.
\end{aligned}\right\} \tag{17}$$

Continuity then reduces to

$$\bar{\nabla}_1^2 \varphi = \partial u_3 / \partial x_3 \,, \tag{18}$$

so that equations (1) - (4) can be substituted by (16) (18) and any one of (17).

Evidently equations (15) - (18) remain unaltered when one adds to $(\varphi + i\psi)$ the product of an arbitrary analytic function of $(x_1 + i x_2)$ times an arbitrary function of x_3. A certain flexibility is therefore permissable in prescribing the boundary conditions on φ and ψ, provided the conditions on u_1 and u_2 hold. The following relationships

$$\varphi = \psi = 0 \qquad\qquad \text{on} \quad x_3 = \text{const.} \,, \tag{19}$$

$$(\rho\nu)\, \partial\varphi/\partial x_3 = -(\partial S/\partial T)\, T, \qquad \partial\psi/\partial x_3 = 0 \qquad \text{on} \quad x_3 = \text{const.}, \tag{20}$$

are thus chosen as the boundary conditions for the solid and free horizontal boundaries, respectively. On the cylindrical surface both of the conditions

$$\varphi = 0, \qquad \partial\psi/\partial\eta = 0, \qquad -\partial\psi/\partial\xi = \partial\varphi/\partial\eta, \tag{21}$$

$$\partial\varphi/\partial\eta = 0, \qquad \psi = 0, \qquad \partial\psi/\partial\eta = \partial\varphi/\partial\xi. \tag{22}$$

make u_1 and u_2 vanish when η and ξ form an orthogonal right handed coordinate system. It is convenient to impose equations (21) and (22) in the treatments of Sections 3 and 4 respectively.

In view of the geometry of the particular cases under discussion use is made of circular cylindrical co-ordinates

$$[r,\ \theta,\ z] \equiv [((x_1/d)^2 + (x_2/d)^2)^{1/2},\ \tan^{-1}(x_2/x_1),\ x_3/d]$$

where d is the height of the container. As mentioned, time-independent solution is sought. Dependent variables are therefore assumed to have the following form

$$u_3 = (\kappa/d)\,W(r,z)\cos(n\theta) \qquad \phi = \kappa\,\Phi(r,z)\cos(n\theta)$$

$$\psi = -\kappa\,\Psi(r,z)\sin(n\theta) \qquad T = (\beta d)\,\Theta(r,z)\cos(n\theta) \tag{23}$$

where W, Φ, Ψ and Θ are dimensionless. In view of the Fourier Theorem there is no loss of generality by having the assumed dependence on θ, provided n is an integer. With the assumed form of solution equations (5) (16) (17) and (18) reduce to

$$\nabla^2\Theta = -W, \tag{5'}$$

$$\nabla^2\nabla^2\Psi = 0, \tag{16'}$$

$$\frac{\partial}{\partial r}\nabla^2\left(W + \frac{\partial\Phi}{\partial z}\right) - \frac{n}{r}\nabla^2\frac{\partial\Psi}{\partial z} = -R_a\frac{\partial\Theta}{\partial r}, \tag{17'}$$

and

$$\nabla_1^2 \Phi = \partial W / \partial z , \qquad (18')$$

where

$$R_a = g \alpha \beta d^4 \nu^{-1} \kappa^{-1} \qquad \nabla_1^2 \equiv r^{-1}(\partial/\partial r) r (\partial/\partial r) - n^2 r^{-2} \qquad \nabla^2 \equiv \nabla_1^2 + \partial^2/\partial z^2$$

These together with the boundary conditions are first solved for cases in which $n \neq 1$. Axially symmetric cases, which are more involved are considered in Section 4.

3. Non Symmetric Modes of Instability

Consider fluid which is completely confined in a container. Its temperature is taken to be uniform over the top and bottom surfaces, $z = \pm 1/2$. Its cylindrical surface $r = c$ is assumed to be either insulated or maintained at the linearly decreasing temperature of the undisturbed fluid. The last boundary condition holds when this surface is made of highly conductive material. In the light of previous studies, here too both modes in which W is symmetric and antisymmetric with respect to $z = 0$, are expected to be possible. For reason which will be explained later attention is focused on the symmetric case. The variable W is thus assumed to be expandable in the following form

$$W = \sum_{m=1}^{\infty} \sum_{i=0}^{\infty} A_i^m J_n(\alpha_m^n r) \cos[(2i+1)\pi z] . \qquad (24)$$

In this expression A_i^m are constants, J_n is Bessel Function of the order n. Throughout this section n is assumed to be a positive non-zero integer. The number α_m^n are the roots of the transcendental equation

$$J_n(\alpha_m^n c) = 0 .$$

The assumed expression for W satisfies the boundary conditions which must hold on the solid boundaries. Furthermore, in view of the completeness of the Fourier series and the series of Bessel Functions, this expression is quite general. In terms of the unknowns A_i^m the solution of equation (5'), for insulated and highly conductive cylindrical surface, is

$$\Theta = \sum_{m=1}^{\infty}\sum_{i=0}^{\infty} \frac{A_i^m}{(\alpha_m^n)^2 + (2i+1)^2\pi^2} J_n(\alpha_m^n r)\cos[(2i+1)\pi z] \; +$$

$$\varepsilon\left\{-\sum_{m=1}^{\infty}\sum_{i=0}^{\infty} \frac{A_i^m \, \alpha_m^n \, J_n'(\alpha_m^n c)}{(\alpha_m^n)^2 + (2i+1)^2\pi^2} \cdot \frac{I_n((2i+1)\pi r)}{(2i+1)\pi \, I_n'((2i+1)\pi c)} \cos[(2i+1)\pi z]\right\}, \qquad (25)$$

where

$$\varepsilon = 0 \quad \text{if} \quad \Theta(c,z) = 0 \qquad\qquad \varepsilon = 1 \quad \text{if} \quad \frac{\partial\Theta}{\partial r}(c,z) = 0 .$$

The symbol I_n designates modified Bessel Function of the first kind of order n .

In view of equation (18') and the symmetry of W, Φ must be antisymmetric with respect to z = 0. This function is therefore assumed to be expandable in the form

$$\Phi = \sum_{m=1}^{\infty}\sum_{j=1}^{\infty} B_j^m \, J_n(\alpha_m^n r)\sin(2j\pi z) \qquad (26)$$

in which B_j^m are unknown constants. Like the expression in equation (24), this form is quite general and satisfies the conditions which are imposed on φ, namely equations (19) and (21). In terms of the constants B_j^m, the solution for which satisfies (16') (19) and (21) is

$$\Psi = \sum_{m=1}^{\infty}\sum_{j=1}^{\infty} B_j^m \alpha_m^n J_n'(\alpha_m^n c)\frac{I_n'(2j\pi c)}{I_{n+1}(2j\pi c)}\left[\frac{r^n}{n c^{n-1}} - \frac{I_n(2j\pi r)}{(2j\pi) I_n'(2j\pi c)}\right] \sin(2j\pi z) \quad (27)$$

So far two doubly-infinite sets of constants, A_i^m and B_j^m, were introduced. All the boundary condition and two of the four governing equations were satisfied. The solution will be completed by reducing the two remaining governing equations to two sets of doubly-infinite algebraic equations, which are linear in A_i^m and B_j^m. The non-trivial solutions of the latter are associated with critical values of R_a, i.e. values of R_a for which the confined fluid is marginally stable. One set of equations is obtained by substituting from (24) and (26) into (18'). Termwise differentiation of the two expansions, which will be justified and discussed later, yields

$$\sum_{m=1}^{\infty}\sum_{j=1}^{\infty} B_j^m (\alpha_m^n)^2 J_n(\alpha_m^n r)\sin(2j\pi z) = \sum_{m=1}^{\infty}\sum_{i=0}^{\infty} A_i^m [(2i+1)\pi] J_n(\alpha_m^n r)\sin[(2i+1)\pi z]$$

This relationship is multiplied by $r J_n(\alpha_m^n r)\sin[(2i+1)z]$ and integrated with respect to r and z throughout the ranges $0 < r < c$, $-1/2 < z < 1/2$, to yield

$$B_j^m (\alpha_m^n)^2 \frac{1}{2} = \sum_{i=0}^{\infty} A_i^m [(2i+1)\pi] \frac{2}{\pi} \cdot \frac{(2j)(-1)^{i+j}}{(2i+1)^2 - (2j)^2} \tag{$28_{m,j}$}$$

The other set of algebraic equations is obtained by substituting from equations (24) - (27) into (17') and integrating the latter with respect to r from o to r. By processing the resulting relationship in a similar manner the following is obtained:

$$\sum_{i=0}^{\infty} A_i^m [(\alpha_m^n)^2 + (2i+1)^2\pi^2] \frac{2}{\pi} \frac{(2i+1)(-1)^{i+j}}{(2i+1)^2 - (2j)^2} + B_j^m [(\alpha_m^n)^2 + (2j\pi)^2](2j\pi) \frac{1}{2}$$

$$- \frac{\gamma_m^n}{n c^{n-1}} \sum_{p=1}^{\infty} B_j^p \alpha_p^n J_n'(\alpha_p^n c) \frac{I_n'(2j\pi c)}{I_{n+1}(2j\pi c)} \frac{1}{2} (2j\pi)^3 =$$

$$= Ra \left\{ \sum_{i=0}^{\infty} \frac{A_i^m}{(2i+1)^2\pi^2 + (\alpha_m^n)^2} \cdot \left(\frac{2}{\pi}\right) \cdot \frac{(2i+1)(-1)^{i+j}}{(2i+1)^2 - (2j)^2} \right.$$

$$\left. - \varepsilon \sum_{i=0}^{\infty} \left(\sum_{p=1}^{\infty} \frac{A_i^p (\alpha_p^n) J_n'(\alpha_p^n c)}{(\alpha_p^n)^2 + (2i+1)^2\pi^2} \right) \frac{\chi_{mi}^n}{[(2i+1)\pi] I_n'((2i+1)\pi c)} \cdot \left(\frac{2}{\pi}\right) \frac{(2i+1)(-1)^{i+j}}{(2i+1)^2 - (2j)^2} \right\}, \tag{$29_{m,j}$}$$

where

$$\gamma_m^n \equiv \frac{\int_0^c r^{n+1} J_n(\alpha_m^n r)\,dr}{\int_0^c r J_n^2(\alpha_m^n r)\,dr} = \frac{2 c^n J_{n+1}(\alpha_m^n c)}{(\alpha_m^n c)[J_n'(\alpha_m^n c)]^2},$$

$$\chi_{mi}^n \equiv \frac{\int_0^c r J_n(\alpha_m^n r) I_n((2i+1)\pi r)\,dr}{\int_0^c r J_n^2(\alpha_m^n r)\,dr} = \frac{-2(\alpha_m^n c) J_n'(\alpha_m^n c) I_n((2i+1)\pi c)}{[(\alpha_m^n)^2 + (2i+1)^2\pi^2] c^2}.$$

Of the non-trivial solutions of equations (28_{mj}) and (29_{mj}) the one which is associated with the lowest value of Ra is of interest. It is assumed here that this solution can be approximated by trunkating from the series expressions under consideration terms involving A_i^m and B_j^m for $m, j > 1$ and $i > 0$. Equations (28_{mj}) and (29_{mj}) then yield

$$Ra = \frac{(\alpha_1^n)^2\left[(\alpha_1^n)^2+\pi^2\right] + \left[(\alpha_1^n)^2+(2\pi)^2\right](2\pi)^2 - 2\,\dfrac{(2\pi)^4\, I_n'(2\pi c)\, J_{n+1}(\alpha_1^n c)}{n\, I_{n+1}(2\pi c)\, J_n'(\alpha_1^n c)}}{\dfrac{(\alpha_1^n)^2}{(\alpha_1^n)^2+\pi^2}\left(1 + \varepsilon\,\dfrac{2(\alpha_1^n)^2\, I_n(\pi c)}{\left[(\alpha_1^n)^2+\pi^2\right](\pi c)\, I_n'(\pi c)}\right)} \quad (30)$$

The corresponding flow pattern, namely, that which is represented by nonvanishing A_0^1 and B_1^1, is (say) upward throughout the sections $-\pi/2 < (n\theta) < \pi/2$, $3\pi/2 < n\theta < 5\pi/2$ ... and downwards $\pi/2 < n\theta < 3\pi/2, \ldots$ $0 < r < c, -1/2 < z < 1/2$. Within such sections it is radially outward for $z > 0$ and inward for $z < 0$. When c is not too large this is the expected circulation pattern for the least stable mode. More 'complicated' patterns in which u_3 changes its sign across an internal cylindrical surface $r = r^* < c$ are bound to produce higher viscous dissipation and consequently be more stable. In order to verify this conjecture a possible change of flow direction across $r = r^*$ is allowed by retaining A_0^1, B_1^1, A_0^2 and B_1^2, rather than only the first two constants. An algebraic equation quadratic in is then obtained. For $c < 5$ the lower of the two solutions is indeed quite close to the expression in equation (30).

The Rayleigh number which corresponds to the more 'complicated' circulation pattern is higher. Of course for large c or 'coin shaped' cylindrical containers the 'simple' modes are more stable. These can take place only if the work done by the buoyancy forces along the upward fluid particle-paths is ballanced by the high viscous dissipation due to the comparatively long horizontal paths. Thus the more 'complicated' patterns, in which the horizontal paths are not too long, are likely to be marginally stable for lower values of Ra . The critical values of Ra for such modes are governed by equations ($28_{m,j}$) and ($29_{m,j}$). Those values of Ra can therefore be approximated by allowing a change of flow direction across one or more cylindrical surfaces, in the manner shown. Again, convection patterns in which W is antisymmetric with respect to z = 0 are more 'complicated' than symmetric ones in the sense that u_3 changes sign across an internal surface. Since for any value of c the former are bound to be more stable they are not treated within the framework of this analysis.

Though more accurate, the explicit expression for the solution obtained by retaining four constants is lengthy and involved. On the other hand equation (30) readily demonstrates the effect of various circumstances on the stability of the completely confined fluid. Thus when the cylindrical surface is insulated, and ε is unity, Ra is smaller than when this surface is highly conductive. Indeed highly conductive walls tend to inhibit instability by reducing the deviation of the temperature from its linear distribution. When the area of the cylindrical surface is small compared with the horizontal ones its conductivity plays a less important role. Thus when c is large, yet $(\alpha_1^n c)$ is kept constant, the multiplyer of is very small. Similarly Rayleigh number for very slender column can be obtained by letting c become very small. Equation (30) then degenerates into

$$\bar{R}_a = R_a c^4 \doteq (\alpha_1^n c)^4 (1 + 2\varepsilon/n)^{-1} \qquad (30^*)$$

where $\bar{R}_a$ is the Rayleigh number defined with the radius of the container as the characteristic length. In Table I the values of $\bar{R}_a$ for very slender insulated containers which were obtained by Yih are compared with those of Equation (30*). The closeness of the results is particularly reassuring when it is born in mind that in Ref. 2 an altogether different mathematical procedure is used.

Table I

n	1	2
from equation (30)	71.9	347.8
from Ref. 2	67.9	329.1

It is noted that not very meaningful results are obtained by carrying this process in the other direction, namely, by finding the limit of equation (30) for very large c. As mentioned, the mode represented by non-vanishing A_0' and B_1' is not the least stable unless c is moderate or small.

When W vanishes on $z = \pm 1/2$ the series expression of equation (24) can be differentiated with respect to z(11). It can be similarly shown that if $\nabla_1^2 \Phi$ is expressed as

$$\nabla_1^2 \Phi = \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} b_j^m J_n(\alpha_m^n r) \sin(2j\pi z)$$

where b_j^m are constants, then the following holds

$$b_j^m \int_0^c r J_n^2(\alpha_m^n r)\, dr \int_{-1/2}^{1/2} \sin^2(2j\pi z)\, dz = \int_0^c \int_{-1/2}^{1/2} r \nabla_1^2 \Phi J_n(\alpha_m^n r) \sin(2j\pi z)\, dz\, dr$$

$$= -(\alpha_m^n)^2 \int_0^c \int_{-1/2}^{1/2} r \Phi J_n(\alpha_m^n r) \sin(2j\pi z)\, dr\, dz = -(\alpha_m^n)^2 B_j^m \int_0^c r J_n^2(\alpha_m^n r)\, dr \int_{-1/2}^{1/2} \sin^2(2j\pi z)\, dz.$$

In the integration by parts the boundary terms vanish because $\Phi(c, z)$ (like $W(r, \pm 1/2)$ in the previous instant) is zero. Since W vanishes on all the boundaries one can also justify the differentiation which was carried out in order to derive $\nabla^2 W$. The termwise differentiations involved in expressing the other components of the integrated vorticity equations are similarly permissable. In many other works on stability the series expressions which are to be differentiated have to contain terms which depend on the boundary values of the variable expressed or its derivatives. This complicates the analysis considerably when a certain expression has to be differentiated as many as four or six times. Here four rather than one or two dependent variables are solved for. Each is therefore differentiated fewer times.

It is consequently possible to express these variables by comparatively simpler series forms.

When the top surface is exposed one can no longer expect modes of instability to be symmetric with respect to the central horizontal plane. Therefore, in these cases the origin is taken to be on the bottom of the container. The following expression for the vertical component of velocity is assumed

$$W = \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \tilde{A}_k^m J_n(\alpha_m^n r) \sin(k\pi z) , \tag{31}$$

where $\tilde{A}_k^m$ are unknown constants. Since it is intended to account for surface tension variations the following rather general condition

$$N\Theta + \partial\Theta/\partial z = 0 \tag{6'}$$

is assumed to hold at z = 1. The nondimensional parameter N is defined by

$$N = hd/k$$

Its physical significance is amply explained by Pearson[6]. On the other hand $\Theta(r, 0)$ and $\Theta(c, z)$ are assumed, for simplicity, to vanish. The solution for the temperature distribution is therefore

$$\Theta = \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \tilde{A}_k^m J_n(\alpha_m^n r) \left[(\alpha_m^n)^2 + (k\pi)^2\right]^{-1} \sin(k\pi z)$$

$$- \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \frac{\tilde{A}_k^m (-1)^k}{(\alpha_m^n)^2 + (k\pi)^2} (k\pi) \frac{J_n(\alpha_m^n r)}{N \sinh(\alpha_m^n) + \alpha_m^n \cosh(\alpha_m^n)} \sinh(\alpha_m^n z) \tag{32}$$

When non-dimensionalized, equation (20) reduces to

$$B\Theta = \partial \Phi / \partial z , \qquad \partial \Psi / \partial z = 0 , \qquad (20')$$

where the Marangoni number B is defined by

$$B \equiv \left(-\frac{\partial S}{\partial T}\right)\left(\frac{\beta d}{\rho \nu K}\right)$$

It is assumed that Φ can be expressed in terms of the unknowns $\tilde{B}_i^m$ in the form

$$\Phi = \sum_{m=1}^{\infty}\sum_{i=0}^{\infty} \tilde{B}_i^m J_n(\alpha_m^n r) \sin\left[(2i+1)\pi z/2\right]$$

$$- B \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \frac{\tilde{A}_k^m}{(\alpha_m^n)^2 + (k\pi)^2} \frac{\sinh(\alpha_m^n)}{N \sinh(\alpha_m^n) + \alpha_m^n \cosh(\alpha_m^n)} J_n(\alpha_m^n r) \sin(k\pi z), \qquad (33)$$

which satisfies (19) and (20'). A third set of constants, $\tilde{C}_i^m$, is introduced in the assumed expansion for Ψ

$$\Psi = \sum_{m=1}^{\infty}\sum_{i=0}^{\infty} \tilde{C}_i^m \alpha_m^n J_n'(\alpha_m^n c) \frac{I_n'((2i+1)\pi c/2)}{I_{n+1}((2i+1)\pi c/2)} \left[\frac{r^n}{n c^{n-1}} - \frac{I_n((2i+1)\pi r/2)}{((2i+1)\pi/2) I_n'((2i+1)\pi c/2)}\right] \sin\left(\frac{2i+1}{2}\pi z\right) \qquad (34)$$

This form satisfies condition (19), (20) and the governing equation (16'). From condition (21) one gets the relationship

$$\sum_{i=0}^{\infty} (\tilde{B}_i^m - \tilde{C}_i^m) \sin\left(\frac{2i+1}{2}\pi z\right) = B \sum_{k=1}^{\infty} \frac{\tilde{A}_k^m}{(\alpha_m^n)^2 + (k\pi)^2} \cdot \frac{\sinh(\alpha_m^n)}{N \sinh(\alpha_m^n) + \alpha_m^n \cosh(\alpha_m^n)} \sin(k\pi z)$$

which yields an infinite set of algebraic equations linear in $\tilde{A}_k^m$, $\tilde{B}_i^m$ and $\tilde{C}_i^m$. Two other such sets are obtained from the equations of continuity and vorticity. Trunkation of these in the manner which was discussed previously yields

$$Ra\left\{\frac{(\alpha_i^n)^2}{(\alpha_i^n)^2+\pi^2}\left[1 \quad + \frac{3}{4}\,\frac{\pi^2}{N\sinh(\alpha_i^n)+(\alpha_i^n)\cosh(\alpha_i^n)}\cdot\frac{(\pi/2)\sinh(\alpha_i^n)-(\alpha_i^n)\cosh(\alpha_i^n)}{(\alpha_i^n)^2+(\pi/2)^2}\right]\right\}$$

$$B\,\frac{\pi}{2}\cdot\frac{3}{4}\cdot\frac{\sinh(\alpha_i^n)}{N\sinh(\alpha_i^n)+(\alpha_i^n)\cosh(\alpha_i^n)}\cdot\frac{(\alpha_i^n)^2\,\pi^2}{(\alpha_i^n)^2+\pi^2} \quad =$$

$$= (\alpha_i^n)^2\left[(\alpha_i^n)^2+\pi^2\right]+\left[(\alpha_i^n)^2+(\pi/2)^2\right]\left(\frac{\pi}{2}\right)^2 \quad - \frac{2}{n}\cdot\frac{J_{n+1}(\alpha_i^n c)\; I_n'(\pi c/2)}{I_{n+1}(\pi c/2)\; J_n'(\alpha_i^n c)}\;. \tag{35}$$

Though inaccurate this relationship is in agreement with available results and is physically plausible. It implies that the fluid is unstable when a linear combination of Ra and B exceed a critical value. Thus as found by Nield[3] instability can occur when the fluid is cooled rather than heated below, provided the Marangoni number is sufficiently large. Equation (35) also indicates that an increase of N or a decrease in c retard the Marangoni type of instability. Indeed, when N is large the temperature of the exposed surface tends to be uniform and there are only small variations in the surface tension. Again, such variations across a comparatively small area have little influence on the fluid below. It is then noted that in the extreme case of $N \longrightarrow \infty$, $\tilde{C}_i^m = \tilde{B}_i^m$ and equation (35) degenerates into a form which is similar to that of equation (30) when $\varepsilon = 0$. The numerator in the latter is larger than the right hand side of equation (35). Fluid in an open container is thus found to be less stable than that which is completely confined. In the other limiting case of very slender column equation (35) reduces to equation (30^*) in which ε vanishes. This is to be expected because of the aforementioned reasons.

4. Approximate Solutions for Rotationally Symmetric Modes

It was pointed out in the last section that the assumed expansions are quite general and differentiable. This trouble was taken in order to show that it is possible to solve for more than one mode or obtain more accurate values of R_a. However, the properties of the assumed expansions hardly affect the quality of the solutions which were actually obtained by drastic trunkations. The results turn out to be acceptable mainly because the trunkation gives rise to local rather than overall violations of the physical principles. Thus two of the governing equations and either all or all but one of the boundary conditions are satisfied exactly, i.e. pointwise. The relationships that are not satisfied exactly are reduced to identities between integrals over either the entire volume or the cylindrical surface. The quality of the results is also affected by the deviation of the assumed trunkated expressions for W and ϕ from the exact solution for those variables. As pointed out the trunkated expressions represent physically plausible flow patterns so that this deviation is quantitative rather than qualitative. The errors in the resulting values of R_a and B are consequently not too serious.

Conversely, the trunkation carried out in Section 3 gives rise to serious errors when $n = 0$. If only A'_0 and B'_1 are non-zero the relationship (which was first introduced in Ref. 7).

$$\int_0^c \int_{-\pi}^{\pi} u_3 \, d\theta \, r \, dr = 0 \qquad \text{on} \qquad z = \text{const} \tag{36}$$

does not hold unless $n \neq 0$. In other words, for trunkated rotationally symmetric solution continuity is not satisfied even in an overall manner and the arguments about the quality of the solution hold no longer. The general analysis of the last section is nevertheless valid for $n = 0$ so that meaningful results can be obtained by adopting less drastic trunkation. However, as mentioned, the solutions of two many homogeneous algebraic equations cannot be presented in the explicit form of equations (30) and (35) - toward which we strive. A solution of such form is therefore obtained here by assuming rather simple expressions for ϕ and W and carrying out the analysis as before. These expressions need not be the first terms in a complete differentiable series. Instead they must satisfy equation (36) and other physical requirements which are believed to have bearing on the quality of the solution. It is thus required that the flow pattern should be compatible with the destabilizing forces. For example, the pattern inside a closed container should bear resemblance to that of a 'convection cell'. However the assumed flow should be stagnant at the solid boundaries.

In view of the above mentioned requirements and the rotational symmetry of the flow pattern the fluid is constrained to move in the vertical planes $\theta = \text{const}$. It therefore circulates forming horizontal vortex rings. Unless c is large, with the least stable mode it is expected that only one such ring will be formed. It is therefore assumed that for a closed container the vertical velocity is given by

$$W = a\left[J_0(\omega_1 c)\, I_0(\omega_1 r) - J_0(\omega_1 r)\, I_0(\omega_1 c)\right] \cos(\pi z) \tag{37}$$

where a is a constant. The origin of the co-ordinate system is chosen as in the corresponding non-symmetric cases. The number ω_1 is the first nontrivial root of

$$J_0(\omega_1 c)\, I_0'(\omega_1 c) + J_0'(\omega_1 c)\, I_0(\omega_1 c) = 0$$

Hence the expression for W satisfies (36) and (9).
The operator ∇^2 appears frequently in the governing equations so that the assumed dependence on r involving Bessel Functions is convenient. Thus, the solution for Θ can be readily shown to be given by

$$\Theta = -a\left(\frac{J_0(\omega_1 c)}{\omega_1^2 - \pi^2} I_0(\omega_1 r) + \frac{I_0(\omega_1 c)}{\omega_1^2 + \pi^2} J_0(\omega_1 r) \right) \cos(\pi z)$$

$$+ \begin{cases} a \dfrac{\omega_1^2}{\omega_1^4 - \pi^4}\, 2\, J_0(\omega_1 c)\, I_0(\omega_1 c)\, \dfrac{I_0(\pi r)}{I_0(\pi c)} \cos(\pi z), & \text{if } \Theta(c,z) = 0 \\[2ex] a \dfrac{\pi^2}{\omega_1^4 - \pi^4}\, 2(\omega_1 c)\, J_0(\omega_1 c)\, I_0'(\omega_1 c)\, \dfrac{I_0(\pi r)}{(\pi c)\, I_0'(\pi c)} \cos(\pi z), & \text{if } \dfrac{\partial \Theta}{\partial r}(c,z) = 0. \end{cases} \tag{38}$$

It is assumed here too that both horizontal solid surfaces are kept at constant temperature.

In view of the axial symmetry of the flow pattern one of the momentum equations is satisfied identically. This is reflected in the present formulation of the problem by the possibility of eliminating one of the dependent variables. When equation (22) rather than (21) is assumed to hold on the cylindrical surface, the variable ψ satisfies homogeneous governing equation and boundary conditions. It therefore vanishes identically. The radial component of velocity is therefore $(-\partial\varphi/\partial r)$. The function ϕ is assumed to be given by

$$\phi = b\, J_0(\lambda_1 r) \sin(2\pi z) \qquad (39)$$

so that $(-\partial\varphi/\partial r)$ has the direction which is compatible with the assumed circulation pattern of a single ring-vortex. This expression satisfies the boundary conditions (17) and (20) when λ_1 is the first roots of

$$J_0'(\lambda_m c) = 0$$

Use is then made of the equations of continuity and vorticity. Procedure similar to that of Section 3 yields the following results

$$R_a = \frac{\lambda_1^2 + \pi^2}{\lambda_1^2} \cdot \frac{(\pi^2\lambda_1^2 + \omega_1^4) + 2(\lambda_1^2 + 4\pi^2)\pi^2}{\left(1 + \varepsilon(1-q)\dfrac{(\omega_1^4 - \lambda_1^4)\,\pi^2}{(\omega_1^4 - \pi^4)\,\lambda_1^2}\right)} \qquad (40)$$

Here q, which is defined by

$$q \equiv \frac{\sigma I_0'(\sigma c)\, I_0(\omega_1 c)}{\omega_1 I_0'(\omega_1 c)\, I_0(\sigma c)} ,$$

is unity when $\omega_1 = \sigma$. Hence as in equation (30) the multiplier of ε is always positive and when the cylindrical surface is well insulated the confined fluid is less stable. Again when $c \to 0$ $\bar{R}_a$ approaches the value obtained in Reference 2. However, since there equation (36) is not stated explicitly, agreement in the rotationally symmetric case appears at first surprising rather than reassuring. Nevertheless, closer examination shows that Yih's solution (accidently?) does satisfy that requirement. When c is very small z variations are negligible and by virtue of equation (17') the condition of vanishing $(\partial\Theta/\partial r)_{r=c}$, which is imposed there, reduces to

$$\frac{\partial}{\partial r} \nabla^2 W = 0 \qquad \text{on} \quad r = c . \tag{41}$$

Since in Reference 2 the solution has the form

$$W = E\, J_0(\delta r) \quad + \quad F\, I_0(\delta r)$$

equation (41) is mathematically equivalent to (36). The two conditions nevertheless are derived from two different physical principles.

Discussion

In Fig. 1 Ra is plotted as a function of c for three values of n and two values of ε. Completely confined fluid is unstable when the Rayleigh number exceeds the least of (infinitely many discrete) values of Ra which correspond to the slenderness of the container (see Fig. 1). Thus, convection never commences by forming a pattern in which there are strong θ variations. 'Simpler' modes, $n = 0, 1$, where there is only one surface across which u_3 changes sign are evidently less stable. Of the two possible the rotationally symmetric pattern is more stable for slender containers and less so for coin-shaped ones. The transition between the two is when c is about 0·4. It occurs probably because comparatively wide horizontal solid surfaces tend to inhibit long horizontal particle paths. When $n = 1$ fluid particles traverse shorter horizontal distances than when $n = 0$. As mentioned, these results are restricted to moderate values of c.

These stability curves also show that fluid is least stable when c is about 1.3. The corresponding critical Rayleigh numbers for highly conductive and insulated cylindrical surfaces are 2500 and 2150, respectively. Because of the increased viscous dissipation caused by the cylindrical wall both of these values are slightly higher than 1708 which was obtained in Reference 5 and 1.

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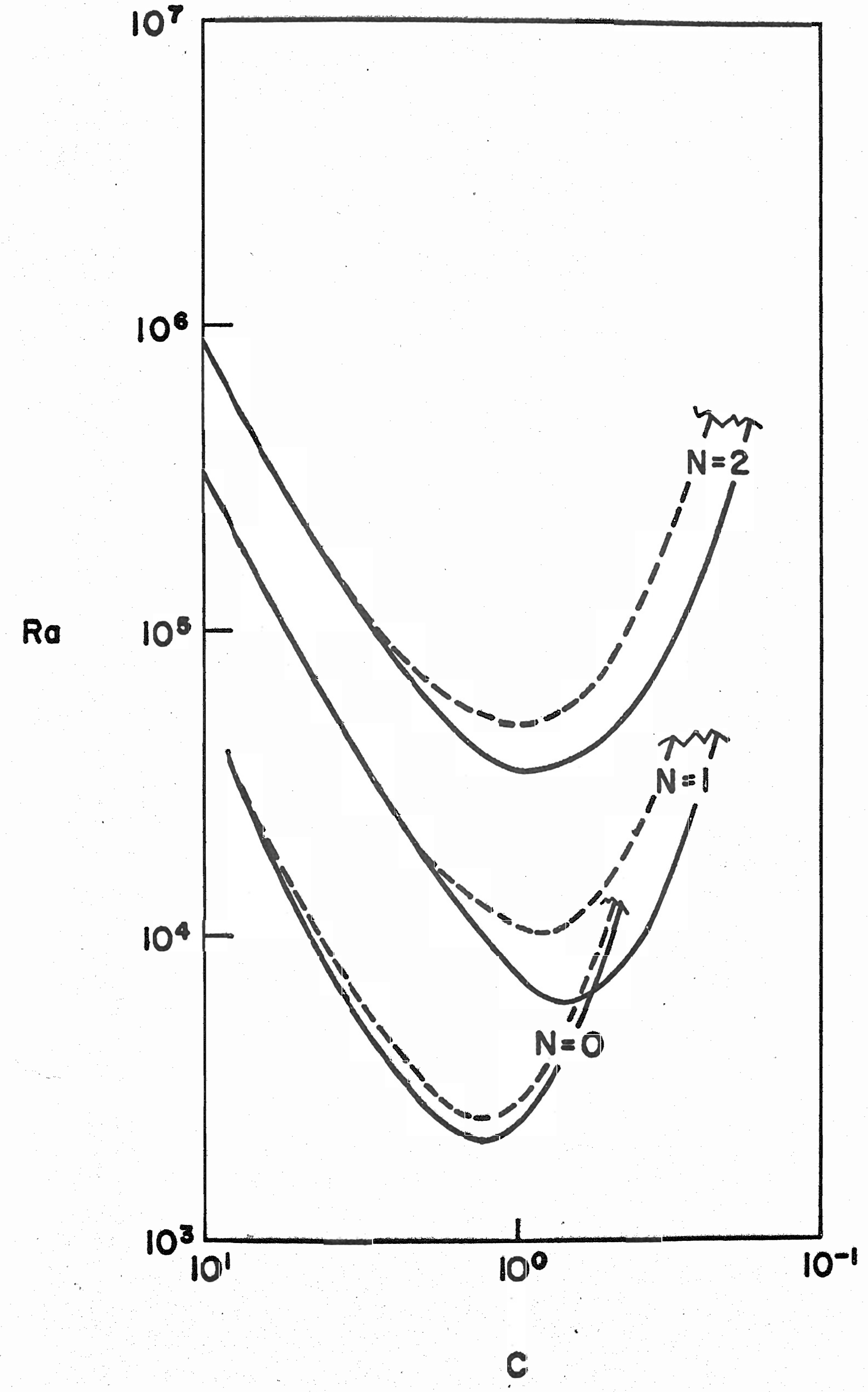

10^7
10^6
10^5
10^4
10^3
Ra
N=2
N=1
N=0
10^1
10^0
10^{-1}
C