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INCOMPRESSIBLE VISCOUS FLOWS RESULTING FROM THE STEADY ROTATION OF A FINITE DISK

by

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Abstract

A solution for the flow in the half plane bounded by a solid plane is obtained. A circular part of this plane rotates steadily while the rest of the plane is stationary. It is assumed that this rotation is the sole source of disturbance in the fluid. The mathematical problem and the method of solution are therefore basically different from Karman's and Cochrane's treatments of an apparently similar case. Comparison of these works with the solution obtained brings up the rarely discussed question of determinacy of differential systems governing viscous incompressible flows.

Introduction

This is an analysis of the incompressible viscous flow in the half space bounded by the solid plane $z' = 0$, where (r', θ', z') are cylindrical co-ordinates. The central part of this plane $0 < r' < a$ $z' = 0$ rotates with a constant angular velocity Ω. The rest of the plane $a < r'$ $z' = 0$ is assumed to be stationary. This analysis shows that due to the viscosity angular momentum diffuses through the otherwise stagnant medium. This quantity is therefore small at infinity but large at the vicinity of the source of motion. Due to the centrifugal acceleration, the fluid near the disk is pushed radially. In the same time, because of continuity, fluid is drawn towards the disk along the axis of symmetry. A typical spiral particle-path is sketched in Figure 1.

It is stressed that this qualitative explanation of the mechanism under discussion, and the corresponding sequence of steps in the analysis are based on the assumption that the rotation is the sole source of disturbance. As will be shown this is not true for the flow considered by Karman[1] and Cochrane [2]. Hence, despite certain similarities between the geometries of the boundaries and the resulting flow patterns (see Figure 1) the case considered here is basically different. In the Karman-Cochrane solution the sum of the tangential stresses on any central part of the disk (which they assume to be infinitely wide) result in a moment M which is proportional to $\Omega^{3/2}$[3]. Thus, though M represents a generalized force and Ω is a measure of the resulting motion these two vectorial quantities do not change signs simultaneously. This feature of the $M - \Omega$ relationship reflects the fact that the moment M is not the only generalized force which disturbs the fluid. Conversely, in the solution proposed M is indeed an odd function of Ω so that the two change signs simultaneously.

The features of the $M - \Omega$ relationship cannot form the basis for the proof that the rotation of the disk is not the only cause of the Karman-Cochrane flow. We therefore propose to examine that solution more carefully. As pointed out[4] it belongs to the class of solutions[1, 2, 5, 6] in which the radial and circumferential components of velocity are proportional to r' while the z' component is independent of this variable. Since the r' dependence is assumed to be known, no conditions are prescribed at $r' = \text{const.} = \infty$. At $z' = \infty$ Karman and Cochrane impose two conditions while at $z' = 0$, all three velocity components are prescribed. This underdeterminate formulation gives the impression that the boundary conditions which hold at $z' = 0$, or rather the fluid-motion that these represent, is of particular importance. It probably led to the common belief that the Karman-Cochrane flow is "due to a rotating disk"[3] and no other cause. However, it is possible to treat the Karman-Cochrane solution as one which is governed by a well-posed problem in which the three velocity components are prescribed everywhere on the bounding surface. In such formulation the purely rotational motion is prescribed at $z' = 0$ as before. However, the velocity fields that were <u>obtained</u> in Refs. 1 and 2 for $z' = \infty$ and $r' = \infty$, appear in the properly posed formulation as the boundary conditions which should be <u>imposed</u> at these parts of the bounding surface. Therefore, overlooking the question of uniqueness, one concludes that the Karman-Cochrane flow results no more from the rotation of the disk than from the axial inflow and radial outflow which are found to prevail at $z' = \infty$ and $r' = \infty$, respectively.

Following Collins[7] and others[4] we expand dependent variables in powers of Reynolds number Re. The coefficients in these series satisfy differential equations of either the second or the fourth order. Corresponding number of boundary conditions are obtained from the requirements that the fluid should not move with respect to the solid surfaces and that its velocity at infinity should be finite. The last requirement is a consequence of the facts that the medium is dissipative and that finite amount of power is supplied in order to maintain the only disturbance. The Green's functions for the fourth and second order differential system are obtained. In terms of these any finite number of coefficients can be evaluated. Attention is focused on the physical features of the solution in which only the first two are retained and which is, therefore, valid only when Reynolds number is small.

Analytical Development

In terms of the non dimensional independent variables defined by

$$(r, \theta, z) = ((r'/a), \theta', (z'/a)) ,$$

the governing equations are

$$-R_e \frac{1}{r} \frac{\partial(\psi, rv)}{\partial(r, z)} = D^2(rv) , \tag{1}$$

$$-R_e \left[2v\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial(\psi, D^2\psi)}{\partial(r, z)} + \frac{2}{r^2}\frac{\partial\psi}{\partial z} D^2\psi \right] = D^4\psi , \tag{2}$$

where

$$D^2 \equiv r\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \qquad\qquad R_e \equiv \frac{\Omega a}{\nu}$$

Here ν is the kinematic viscosity, ψ is the Lagrange's stream function so that the velocity components in the direction of (r, θ, z) are $(r^{-1}\partial\psi/\partial z,\ v,\ -r^{-1}\partial\psi/\partial r)$. The dependent variables v and ψ are nondimensionalized so that the boundary conditions are

$$\left.\begin{array}{llll} v = r & \psi = \partial\psi/\partial z = 0 & & z = 0,\ 0 < r < 1 , \\ v = \psi = \partial\psi/\partial z = 0 & & & z = 0,\ 1 < r < \infty , \\ v ,\ r^{-1}\partial\psi/\partial r ,\ r^{-1}\partial\psi/\partial z & & \text{are finite} & r, z \to \infty \end{array}\right\} \tag{3}$$

The following expansions are assumed to hold

$$v = \sum_{i=0}^{\infty} V_{2i}(r,z)\, Re^{2i} \qquad\qquad \Psi = \sum_{i=0} \Psi_{2i+1}(r,z)\, Re^{2i+1} \ . \tag{4}$$

Equations (1) and (2) thus yield[4, 7]

$$-\sum_{\alpha=0}^{i-1} \frac{1}{r} \frac{\partial(\Psi_{2(i-\alpha)-1},\, r V_{2\alpha})}{\partial(r,z)} = D^2(r V_{2i}) \ , \tag{1_{2i}}$$

$$\begin{aligned} -\sum_{\alpha=0}^{i} \Big[2 V_{2(i-\alpha)} \frac{\partial V_{2\alpha}}{\partial z} &+ \frac{1}{r} \frac{\partial(\Psi_{2(i-\alpha)-1},\, D^2 \Psi_{2\alpha+1})}{\partial(r,z)} \\ &+ \frac{2}{r} \frac{\partial \Psi_{2(i-\alpha)-1}}{\partial z} D^2 \Psi_{2\alpha+1} \Big] = D^4 \Psi_{2i+1} \end{aligned} \tag{2_{2i+1}}$$

in which coefficients with a negative index are taken to vanish identically. The function V_o satisfies the conditions imposed on v while the coefficients V_{2i} for $i > 0$ vanish everywhere on z = 0 and are finite at infinity. We let functions Ψ_{2i+1} for any index satisfy the conditions imposed on Ψ .

The coefficient V_o is governed by

$$\left. \begin{aligned} 0 &= D^2(r V_0) \\ V_0(r,0) &\begin{cases} = r & r < 1 \ , \\ = 0 & 1 < r \ , \end{cases} \qquad V_0 \ \text{finite} \quad r, z \to \infty \end{aligned} \right\} \tag{5}$$

The governing equation suggests a solution of the type $J_1(\mu r)\, e^{\pm \mu z}$ or

$Y_1(\mu r)e^{\pm\mu z}$ where J_1 and Y_1 are Bessel Function of the first order. In view of the finiteness of the velocity field at infinity the argument of the exponents has to be negative. Inclusion of the axis of symmetry in the domain under discussion makes $Y_1 e^{\pm\mu z}$ inadmissable, so that V_0 is assumed to have the form

$$V_0 = \int_0^\infty g(\mu) J_1(\mu r) e^{-\mu z} \mu \, d\mu \quad . \tag{6}$$

The function $g(\mu)$ is obtained by combining Henkel's inversion formula with the boundary conditions at z = 0. When use is made of the recurrence relationships of Bessel Functions the solution can be expressed thus:

$$V_0 = \int_0^\infty J_2(\mu) J_1(\mu r) e^{-\mu z} d\mu \quad . \tag{6'}$$

In order to solve for V_{2i} , $i > 0$, use is made of the auxiliary functions W_{2i} defined thus:

$$V_{2i} = \partial W_{2i} / \partial r \quad .$$

These functions satisfy the following conditions and equation

$$\left.\begin{aligned} -\int_r^\infty \hat{r}^{-1} q_{2i}(\hat{r}, z)\, d\hat{r} &= \nabla^2 W_{2i} \; , \\ W_{2i}(r,0) = 0 \; , \quad W_{2i} \to 0 \quad &\text{as} \quad r, z \longrightarrow \infty \; , \end{aligned}\right\} \tag{7}$$

where $q_{2i}(r, z)$ is the left hand side of equation (1_{2i}). Here ∇^2 is the Laplace operator in cylindrical co-ordinates which is related to D^2 thus:

$$D^2 \, r(\partial/\partial r) = r(\partial/\partial r)\nabla^2$$

The resulting potential problem (7) has a solution of the form

$$W_{2i} = \int_0^\infty \int_0^\infty \left[- \int_{\bar{r}}^\infty \hat{r}^{-1} q_{2i}(\hat{r}, z)\, d\hat{r} \right] G(r, z, \bar{r}, \bar{z})\, \bar{r}\, d\bar{r}\, d\bar{z} \,, \qquad (8_{2i})$$

where G is the appropriate Green's function. It is $(4\pi)^{-1}$ times the potential due to two uniform ring shaped "charges" or "masses" located at $r = \bar{r}$, $z = \bar{z}$ and at $r = \bar{r}$ $z = -\bar{z}$.

This function is given by

$$G = -\pi^{-1}\left\{ \sqrt{(r+\bar{r})^2 + (z-\bar{z})^2}\; K\left(\sqrt{\frac{4r\bar{r}}{(r+\bar{r})^2 + (z-\bar{z})^2}}\right) \right.$$
$$\left. - \sqrt{(r+\bar{r})^2 + (z+\bar{z})^2}\; K\left(\sqrt{\frac{4r\bar{r}}{(r+\bar{r})^2 + (z+\bar{z})^2}}\right) \right\} \qquad (9)$$

where K is the Complete Elliptic Integral of the first kind.

It is noted that V_o could have been evaluated by solving first for W_o which is harmonic throughout, has assigned values at z = 0 and vanishes at infinity. By differentiating W_o the following is obtained

$$V_0 = \frac{3}{4\pi}\int_0^1\int_{-\pi}^{\pi}(1-\bar{r}^2)\frac{z(r+\bar{r}\cos\bar{\theta})}{(r^2+\bar{r}^2+2r\bar{r}\cos\bar{\theta}+z^2)^{5/2}}\,d\bar{\theta}\,\bar{r}\,d\bar{r} \quad . \tag{10}$$

This form is more complicated than that of equation (6'), but its more recognizable order properties are useful.

Just as D^2 of equation (1_{2i}) is related to ∇^2 so is D^4 which appears in equation (2_{2i+1}) related to the biharmonic operator ∇^4 . Therefore, again use is made of auxilliary function Γ_{2i+1} which is defined in this case by

$$r\,\partial\Gamma_{2i+1}/\partial r = \Psi_{2i+1} \, .$$

Evidently Γ_{2i+1} is governed by

$$\left.\begin{aligned} -\int_r^\infty \hat{r}^{-1}\,p_{2i+1}(\hat{r},z)\,d\hat{r} &= \nabla^4\Gamma_{2i+1} \\ \Gamma_{2i+1}(r,0) = \frac{\partial\Gamma_{2i+1}}{\partial z}(r,0) = 0 \qquad & \frac{1}{r}\frac{\partial\Gamma_{2i+1}}{\partial z},\ \frac{1}{r}\frac{\partial\Gamma_{2i+1}}{\partial z}\ \text{finite}\quad r,z\to\infty \end{aligned}\right\} \tag{11}$$

where p_{2i+1} stands for the left hand side of equation (2_{2i+1}). In the solution of the type

$$\Gamma_{2i+1} = \int_0^\infty\int_0^\infty\left[-\int_{\bar{r}}^\infty \hat{r}^{-1}\,p_{2i+1}(\hat{r},\bar{z})\,d\hat{r}\right]H(r,z,\bar{r},\bar{z})\,\bar{r}\,d\bar{r}\,d\bar{z} \tag{12_{2i+1}}$$

we let the Green's function H have the form

$$H = -(8\pi)^{-1}\left\{ \int_{-\pi}^{\pi} \left[(r^2 + 2r\bar{r}\cos\bar{\theta} + \bar{r}^2 + (z-\bar{z})^2)^{1/2} \right.\right.$$

$$\left.\left. - (r^2 + 2r\bar{r}\cos\bar{\theta} + \bar{r}^2 + (z+\bar{z})^2)^{1/2} \right] d\bar{\theta} + \Theta(r, z, \bar{r}, \bar{z}) \right\} . \tag{13}$$

In this expression the square roots are the distances between $(r, 0, z)$ and the points $(\bar{r}, \bar{\theta}, \bar{z})$ and $(\bar{r}, \bar{\theta}, -\bar{z})$. Since with R as the radial spherical co-ordinate the following holds

$$\nabla^2 (R/8\pi) = (4\pi)^{-1} R^{-1}$$

these two terms are the biharmonic potential due to point "masses" or "charges" situated at ($\bar{r}$, $\bar{\theta}$, $\bar{z}$) and ($\bar{r}$, $\bar{\theta}$, $-\bar{z}$). Therefore, the integral with respect to $\bar{\theta}$ is biharmonic everywhere except along $r = \bar{r}$, $z = \bar{z}$, $-\pi < \theta < \pi$ where it has the appropriate type of singularity. Because of the mirror image at $r = \bar{r}$ $z = -\bar{z}$ this integral vanishes on z = 0 and is finite at infinity. The function Θ is a singularity-free biharmonic function which, like the integral with respect to $\bar{\theta}$, vanishes at z = 0 and is fin[illegible] at infinity. It is added to the expression for H so as to make its derivative [illegible] the plane z = 0 vanish. This requirement yields the relationship

$$2\bar{z}\int_{-\pi}^{\pi} \left[r^2 + 2r\bar{r}\cos\bar{\theta} + \bar{r}^2 + \bar{z}^2 \right]^{-1/2} d\bar{\theta} + (\partial\Theta/\partial z)_{z=0}$$

which suggests the following solution for Θ

$$\Theta = \int_{-\pi}^{\pi} [\bar{r}^2 + 2r\bar{r}\cos\bar{\theta} + r^2 + (z+\bar{z})^2]^{-1/2} \, 2z\bar{z} \, d\bar{\theta}$$

Hence H is given by

$$H = -(8\pi)^{-1}\int_{-\pi}^{\pi} \Big\{ [\bar{r}^2 + 2r\bar{r}\cos\bar{\theta} + r^2 + (z-\bar{z})^2]^{1/2} - [\bar{r}^2 + 2r\bar{r}\cos\bar{\theta} + r^2 + (z+\bar{z})^2]^{1/2} + 2z\bar{z}[\bar{r}^2 + 2r\bar{r}\cos\bar{\theta} + r^2 + (z+\bar{z})^2]^{-1/2} \Big\} d\bar{\theta}. \qquad (13')$$

Results and Discussion

In order to show that at least,under some circumstances, the flow obtained has the predicted pattern we examine the solution in which only the first term in each of the two expansions (4) are retained. Though the trunkation of the terms of $0(Re^2)$ limits the applicability of the solution to cases in which Re is small, previous works suggests that such results are nevertheless meaningful[4, 7, 8]. With $v = V_0$ equations (6') and (10) imply that the circumferential velocity is small far from the disk. More precisely, it follows from the latter that v is very nearly equal to $LzrR^{-5}$ where L is a constant, and is therefore of $0(R^{-3})$ for large R. Therefore p_1, which is equal to $(-\partial V_0^2/\partial z)$, decreases like R^{-7}. Consequently provided ρ

and ζ are large but finite not too big an error is committed in setting $\bar{r} = \rho$ and $\bar{z} = \zeta$ as the upper limits of the integral in equation (12_1). Thus, since expansion in powers of (1/z) and (1/r) yields

$$H = (8\pi)^{-1} \int_{-\pi}^{\pi} \Big\{ (\bar{z}/z) - (\bar{z}/z)^2$$

$$\frac{\bar{z}^3 - (3/2)\bar{z}(r^2 + \bar{r}^2 + 2r\bar{r}\cos\bar{\theta})}{z^3} + O(z^{-4}) \Big\} \, d\bar{\theta} ,$$

$$H = (1/2)\, \bar{z}^2 z^2 r^{-3} + O(r^{-4}) ,$$

the following approximate relationship holds

$$\psi \approx Re\, r(\partial\Gamma_1/\partial r) \simeq \begin{cases} Re\, Q\, r^2 z^{-3} & \text{for large z} \\ Re\, Q\, z^2 r^{-3} & \text{for large r} \end{cases}$$

The constant Q, which is given by

$$Q = -\frac{3}{2} \int_0^{\zeta}\!\!\int_0^{\rho} \Big[\int_{\bar{r}}^{\infty} \hat{r}^{-1} 2 V_0(\hat{r}, \bar{z})\, \partial V_0/\partial\bar{z}\, d\hat{r} \Big] \bar{r}\, \bar{z}^2\, d\bar{r}\, d\bar{z} ,$$

can be shown to be positive. The flow far from the disk can therefore be approximated by

$$u = -3\, Re\, Q\, r z^{-4} \qquad v = L\, r z^{-4} \qquad w = -2\, Re\, Q\, z^{-3} , \tag{14}$$

$$u = 2\, Re\, Q\, z\, r^{-4} \qquad v = L\, z\, r^{-4} \qquad w = 3\, R\, Q\, z^2 r^{-5} , \tag{15}$$

for large and small (z/r), respectively. These results imply that far from the disk near the horizontal plane fluid particle paths spiral outward. Far out near the axis the flow is directed towards the disk (see Figure 1.). At the vicinity of the disk the swirl is predominant over the motion in the meridian planes.

It is noted that the tangential stress at the disk $s_{\theta z}$ like the (dimensional) circumferential velocity is proportional to Ω times a series of even powers of Re. Therefore M is an odd function of Ω and the $M-\Omega$ relationship satisfies the desired sign criterion.

As mentioned, the Karman-Cochrane solution is commonly believed to represent the flow that can be produced by rotating a disk which is contact with otherwise stagnant liquid. However, in any laboratory experiment that one can visualize, the disk is not infinite as assumed in the analysis. It is therefore necessary to assume that in so far as the vicinity of the origin, $R \ll 1$ is concerned the situation prevailing at the rim, $r \sim 1$, is of secondary importance. However, if 'edge effects' (as Goldstein[3] calls them) are negligible then for small R the solution obtained here should be identical with that of Karman and Cochrane. Evidently this is not the case. In view of equation (6') the following holds

$$v = \sum_{i=0}^{\infty} \left\{ \frac{(-1)^i}{i!\,(i+1)!} \int_0^{\infty} \left(\frac{\mu}{2}\right)^{2i+1} J_2(\mu)\, e^{-\mu z}\, d\mu \right\} r^{2i+1} + O(Re^2).$$

For small z the coefficients of r^{2i+1}, $i > 0$, do not vanish while in the Karman-Cochrane solution v is proportional to r (for any Re). The in-admissability of the assumption about 'edge effects' is in line with our basic premise that situations prevailing at different parts of the bounding surface should be treated on equal footings. This attitude is evidently not, by any means, universal in the contemporary literature on viscous incompressible flows.

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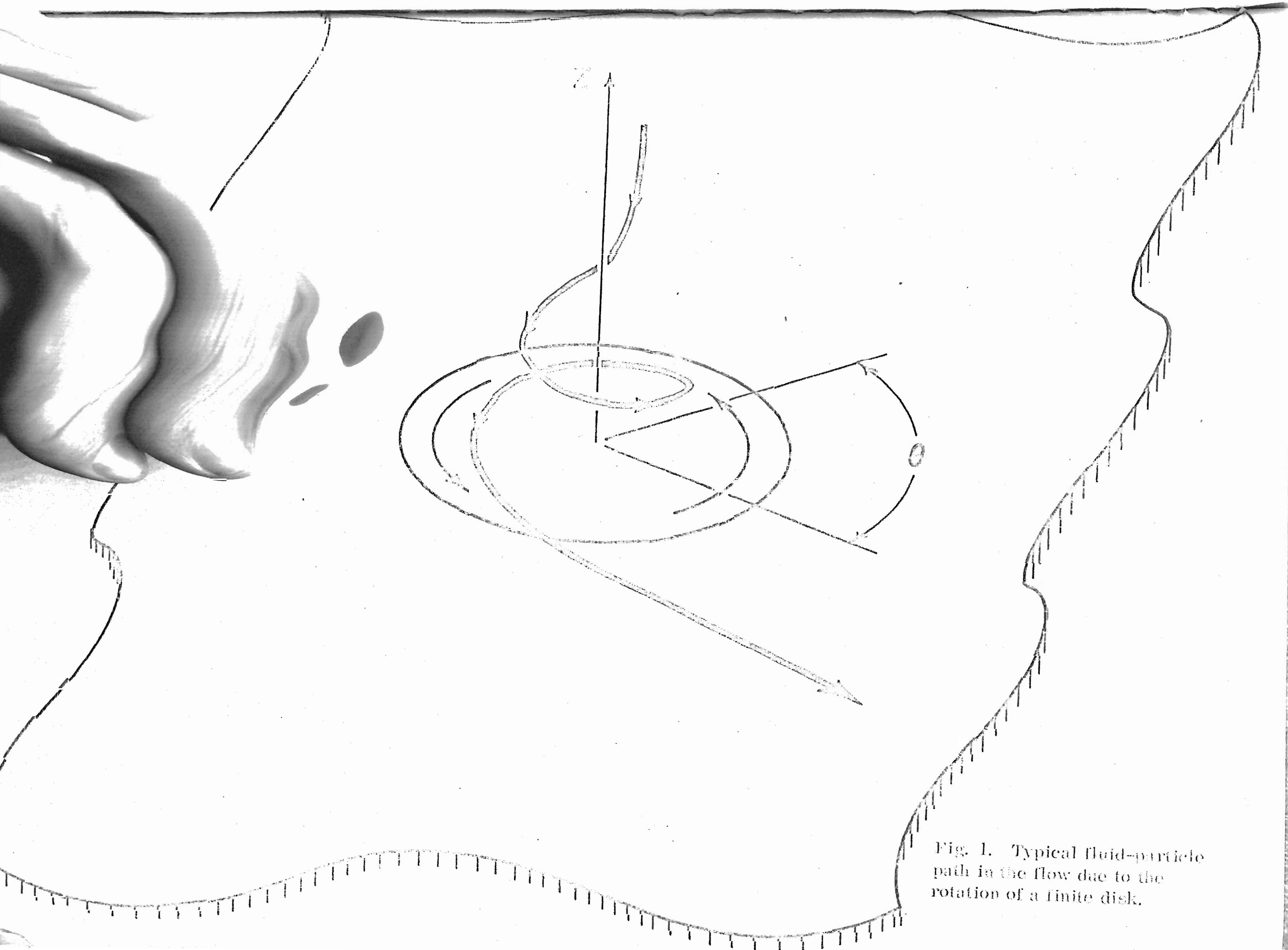

Fig. 1. Typical fluid-particle path in the flow due to the rotation of a finite disk.