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A REPRESENTATION OF POSITIVE-REAL OPERATORS ON A HILBERT SPACE

by

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A function $f(\lambda)$ of a complex variable λ is half plane positive or simply positive whenever it is analytic and satisfies $\operatorname{Re} f(\lambda) > o$ in the half plane $\operatorname{Re} \lambda > o$. If, in addition, $f(\bar{\lambda}) = \overline{f(\lambda)}$ we say the function is positive real. By virtue of Schwarz's reflection principle the last condition is equivalent to asserting that $f(\lambda)$ is real for real λ in the half plane. A theorem of Nevanlinna [1] which is based on an earlier result of Herglotz [2] tells us that a necessary and sufficient condition for $f(\lambda)$ to be a positive function is that it can be represented by the Stieltjes integral

$$f(\lambda) = a\lambda + iq + \int_{-\infty}^{\infty} \frac{i\zeta\lambda - 1}{i\zeta - \lambda}\, d\mu(\zeta) \tag{1}$$

where $a \geq o$ and q are real scalars and μ is a non-decreasing and bounded function which we normalize by $\mu(-\infty) = o$. Moreover, as shown by Cauer [3], $f(\lambda)$ is positive real if and only if $q = o$ and $\mu(-\zeta) = -\mu(\zeta)$.

We wish to extend (1) to a class of operator valued functions R_λ on a Hilbert space which are, in a sense to be made precise below, positive real. Such functions occur in the study of passive Hilbert systems and the notation R_λ refers to the fact that the operators often arise as the resolvents of certain unbounded operators (c.f. [4], [10]).

H will denote an arbitrary complex Hilbert space and H_0 is to consist of all elements in H which satisfy the symmetry condition $(u,v) = (v,u)$, where parenthesis denotes inner product in H. It is clear that H_0 forms a Hilbert space over the real numbers. We now define a one parameter family of operators R_λ on H to be positive if $f(\lambda) = (R_\lambda u,u)$ is positive in λ for all u in the domain of R_λ. In order to define positive real operators we take our clue from the fact that on E^n the matrix valued operator $R(\lambda)$ satisfies $(R(\bar{\lambda})u,u) = \overline{(R(\lambda)u,u)}$ for all real u whenever

$R(\bar{\lambda}) = \overline{R(\lambda)}$. Hence we say that a positive family R_λ is positive real whenever $f(\bar{\lambda}) = \overline{f(\lambda)}$ for all $u \in H_0$.

The next theorem was first established on E^n in a different way by Youla [5]. Before proving it we need a lemma.

Lemma 1 Let $f_\lambda(u,v) = a(u,v)\,\lambda + q(u,v) + \int_{-\infty}^{\infty} \frac{i\zeta\lambda - 1}{i\zeta - \lambda}\, d\mu_\zeta(u,v)$ for $\operatorname{Re}\lambda > 0$ where a, q, μ are complex valued function on $H \oplus H$ and where μ is of bounded variation in ζ on the real axis, normalized so that $\mu_{-\infty} = 0$. The quantities a, q, μ are uniquely determined by f_λ and if f_λ is a bilinear functional then so are a, q, μ.

Proof: Let $F_\lambda(u_1, u_2, v) = f_\lambda(u_1 + u_2, v) - f_\lambda(u_1, v) - f_\lambda(u_2, v)$ for any $u_1, u_2, v \in H$. By assumption $F_\lambda = 0$ for $\operatorname{Re}\lambda > 0$ and it suffices to show the linearity of a, q, μ in this, the first argument, u. Now $F_\lambda = \alpha(u_1, u_2, v)\,\lambda + \beta(u_1, u_2, v) + \int \frac{i\zeta\lambda - 1}{i\zeta - \lambda}\, d\eta_\zeta(u_1, u_2, v)$ where $\alpha(u_1, u_2, v) = a(u_1 + u_2, v) - a(u_1, v) - a(u_2, v)$, etc. Since f_λ is defined for $\operatorname{Re}\lambda > 0$ then $F_{\bar{\lambda}} = 0$ also so that

$$F_\lambda - F_{\bar{\lambda}} = \alpha + \int \frac{1-\zeta^2}{\lambda\bar{\lambda} - i\zeta(\lambda - \bar{\lambda}) - \zeta^2}\, d\eta = 0.$$ If $\lambda = 1$ we obtain $\alpha + \int d\eta = 0$.

Hence $F_1 = \beta = 0$ or q is bilinear. Now from $\alpha + \int d\eta = 0$ we obtain $F_\lambda = (1-\lambda^2)\cdot \int \frac{d\eta}{i\zeta - \lambda} = 0$ or $\int \frac{d\eta}{i\zeta - \lambda} = 0$ for $\operatorname{Re}\lambda > 0$. Hence $L_\lambda(u,v) = \int \frac{d\mu_\zeta(u,v)}{i\zeta - \lambda}$ is also bilinear. By the Stieltjes inversion formula (see, for example, [6], pg 357),

$\mu_\zeta(u,u) - \mu_{\zeta_0}(u,u) = \lim_{\sigma \to 0} \int_{\zeta_0}^{\zeta} \operatorname{Re} L_\lambda(u,u)\, d\omega$ where $\lambda = \sigma + i\omega$. But $\mu_{-\infty} = 0$ so that

$\mu_\zeta(u,u) = \lim_{\sigma \to \infty} \int_{-\infty}^{\zeta} \operatorname{Re} L_\lambda(u,u)\, d\omega$. Now polarize μ to obtain

$$4\,\mu_\zeta(u, v) = \mu_\zeta(u+v, u+v) - \mu_\zeta(u-v, u-v) + i\,[\mu_\zeta(u+iv, u+iv) - \mu_\zeta(u-iv, u-iv)]$$

$= \lim_{\sigma \to 0} \int_{-\infty}^{\zeta} \operatorname{Re} L_\lambda(u, v)\, d\zeta$ since L_λ, being bilinear, is uniquely determined by its quadratic form. This formula exhibits the fact that $\mu_\zeta(u, \nu)$ is also bilinear. Hence $\zeta(u_1, u_2, \nu) = o$ and so $F_\lambda = \alpha(u_1, u_2, \nu) = o$ which shows that a is bilinear.*

If f_λ can be represented by different a, q, μ then by taking differences essentially the same argument as above shows that the difference in q's are zero and that the differences in a and μ's satisfy $\alpha + \int d\eta = o$ or $\int \frac{d\eta}{i\zeta - \lambda} = o$. The Stieltjes formula applied to this last expression shows that the difference in μ values is also zero. This shows that f_λ uniquely determines a, q, μ. An alternate argument can be based on the fact that a is determined by $\lim_{\sigma \to \infty} f(\sigma)/\sigma$ where $\sigma = \text{Re}\,\lambda$ (see [7], pg. 24). Since $f(\sigma) = o$ for $\sigma > o$ then so is a.

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*It is worth remarking here on a similar result which is also quite useful. If we know $L_t(u,\nu) = \int_{-\infty}^{\infty} e^{it\zeta}\, d\eta_\zeta(u, \nu)$ is bilinear then so is η whenever η is of bounded variation in ζ. The proof can be found, for example, in [9], pp. 35-6 and is equivalent to showing that if

$\int e^{it\zeta}\, d\eta_\zeta = o$ then $\eta_\zeta = o$.

Theorem: Let R_λ be a one-parameter family of bounded linear operators on the Hilbert Space H. Then a necessary and sufficient condition in order that R_λ be positive is that it admit the spectral representation

$$R_\lambda = A\lambda + Q + \int_{-\infty}^{\infty} \frac{i\zeta\lambda - 1}{i\zeta - \lambda}\, d\psi_\zeta . \tag{2}$$

for all Re $\lambda > 0$, where A is a bounded self adjoint and positive operator on H and Q is skew self adjoint (i.e., $Q = Q^*$) and where ψ_ζ is a one-parameter family of bounded self adjoint operators on H which satisfy $\psi_\zeta \geq \psi_{\zeta'}$ for $\zeta > \zeta'$, i.e., $(\psi_\zeta u, u) \geq (\psi_{\zeta'} u,u)$ for all u, $\psi_{-\infty} = 0$ and $(\psi_\zeta u,u)$ bounded in ζ. In addition R_λ is positive real if and only if $(Qu,u) = 0$ for $u \epsilon H_0$ and $\psi_{-\zeta} = -\psi_\zeta$.

Proof: If (2) holds then $f(\lambda) = (R_\lambda u,u) = (Au,\nu)\lambda + (Qu,u) + \int \frac{i\zeta\lambda-1}{i\zeta-\lambda} d(\psi_\zeta u,u)$ is clearly positive in Re $\lambda > 0$ since $(Au,u) \geq 0$, Re $(Qu,u) = 0$, and $(\psi_\zeta u,u)$ is non-decreasing, bounded in ζ and non-negative. If $f(\lambda)$ is positive real then $(Qu,u) = 0$ for $u \epsilon H_0$ and $\overline{\int \frac{i\zeta\lambda-1}{i\zeta-\lambda} d(\psi_\zeta u,u)} = \int \frac{i\zeta\bar{\lambda}-1}{i\zeta-\bar{\lambda}} d(\psi_\zeta u,u)$ so that $f(\bar{\lambda}) = \overline{f(\lambda)}$ for $u \epsilon H_0$.

To establish the converse let R_λ be positive and polarize $(R_\lambda u,\nu)$ by writing $4(R_\lambda\, u+\nu,\ u+\nu) - (R_\lambda\, u-\nu,\ u-\nu) + i\,(R_\lambda\, u+i\nu,\ u+i\nu) - i\,(R_\lambda\, u-i\nu,\ u-i\nu)$. Each term on the right is a scalar positive function in Re $\lambda > 0$ and so (1) holds for each of these terms. If we combine the terms in a, q, η for each expression we obtain another representation like (1) except that now q no longer will be real, nor a positive, nor η monotone increasing. In fact one obtains $4(R_\lambda u,\nu) = a(u,\nu)\ \lambda + q(u,\nu) + \int_{-\infty}^{\infty} \frac{i\zeta\lambda-1}{i\zeta-\lambda} d\eta_\zeta(u,\nu)$ (3)
where a, q, η are complex valued and η is simply of bounded variation in ζ. Since $(R_\lambda u,\nu)$ is bilinear in u,ν so are a, q, η by lemma 1. Also, by lemma 1, the representation (1) determines a, q, η uniquely so that if we let $\nu = u$ (3) then, since $(R_\lambda u,u)$ is positive in λ, we see from (1) that $a(u,u) \geq 0$, $q(u,u)$ is imaginary, and $\eta_\zeta(u,u) \geq 0$. Thus the quadratic forms associated with $a(u,\nu)$,

$q(u,\nu)$, $\eta_\zeta(u,\nu)$ are non-negative and zero if and only if $u = o$. Hence, by Schwarz's inequality $|(a(u,\nu)| \le \|a(u,u)\| \; \|a(\nu,\nu)\|$ and $|\eta_\zeta(u,\nu)| \le \|\eta_\zeta(u,u)\| \; \|\eta_\zeta(\nu,\nu)\|$. Since R_λ is bounded for each Re $\lambda > 0$ then $|R_\lambda(u,u)| \le$ constant (u,u) where the constant depends on λ. For $\lambda = 1$ we obtain from (3) that $a(u,u) + \int d\eta_\zeta(u,u) \le$ constant (u,u). But $o \le a$, $o \le \eta_\zeta \le \eta_\infty \le \int_{-\infty}^{\infty} d\eta_\zeta$ (since $\eta_{-\infty} = o$) and so $a(u,u) \le$ constant (u,u), $\eta_\zeta(u,u) \le$ constant (u,u) for all $u \epsilon H$. By virtue of (4) this shows that $a(u,\nu)$ and $\eta_\zeta(u,\nu)$ are <u>bounded</u> bilinear forms on H, for each ζ. By a theorem of Riesz (see, for example, Riesz and Nagy [8], pg. 202) there exist bounded operators A, ψ_ζ such that $a(u,\nu) = (Au,\nu)$ and $\eta_\zeta(u,\nu) = (\psi_\zeta u,\nu)$. Since a, η are both non-negative then A, ψ_ζ are self adjoint positive operators for each ζ (Reisz-Nagy [8], pg. 229). Also, since η_ζ is bounded and non-decreasing so is the family ψ_ζ. From equation (3) we now see that $q(u,\nu)$ is also bounded and bilinear (the sum of bounded functionals is bounded) and so there exists, as above, a bounded operator Q on H for which $q(u,\nu) = (Qu,\nu)$. Since $q(u,u)$ is imaginary we have $(Qu,u) = (Q^*u,u) = -(u,Qu)$ or Q is skew self adjoint. Finally, if R_λ is positive real then $q(u,u) = 0$ for all $u \epsilon H_o$ since $q = 0$ in (1) for positive real functions. For the same reason $(\psi_{-\zeta}u,u) = -(\psi_\zeta u,u)$ when $u \epsilon H_o$. Thus

$$R_\lambda(u,\nu = (Au,\nu)\lambda + (Qu,\nu) + \int_{-\infty}^{\infty} \frac{i\zeta\lambda - 1}{i\zeta - \lambda} d(\psi_\zeta u,\nu)$$

where A, Q, ψ_ζ have the desired properties from which the theorem follows.

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The following is a corrected version of Lemma 1 on page 2 of Report. No. 108 by E. J. Beltrami.

Lemma 1 Let $f_\lambda(u,\nu) = a(u,\nu)\lambda + q(u,\nu) + \int_{-\infty}^{\infty} \frac{i\zeta\lambda - 1}{i\zeta - \lambda} d\mu_\zeta(u,\nu)$ for Re $\lambda > o$ where a, q, μ are complex valued functions on $H \oplus H$ and where μ is of bounded variation in ζ on the real axis, normalized so that $\mu_{-\infty} = o$. Moreover, suppose that $a(u,u) \geq o$, $q(u,u)$ is imaginary, and $\mu_\zeta(u,u)$ is non-decreasing in ζ. Then the quantities a, q, μ are uniquely determined by f_λ and if f_λ is a bilinear functional then so are a, q, μ.

Proof: Let $F_\lambda(u_1, u_2, \nu) = f_\lambda(u_1 + u_2, \nu) - f_\lambda(u_1, \nu) - f_\lambda(u_2, \nu)$ for any $u_1, u_2, \nu \in H$. By assumption $F_\lambda = o$ for Re $\lambda > o$ and it suffices to show the linearity of a, q, μ in this, the first argument, u. Now $F_\lambda = \alpha(u_1, u_2, \nu)\lambda + \beta(u_1, u_2, \nu) + \int \frac{i\zeta\lambda - 1}{i\zeta - \lambda} d\eta_\zeta(u_1, u_2, \nu)$ where $\alpha(u_1, u_2, \nu) = a(u_1 + u_2, \nu) - a(u_1, \nu) - a(u_2, \nu)$ etc.

By the Stieltjes inversion formula (see, for example, [6], pg. 357),

$$\gamma_\zeta(u,u) - \gamma_{\zeta o}(u,u) = \lim_{\sigma \to o} \frac{1}{\pi} \int_{\zeta o}^{\zeta} \text{Re } f_\lambda(u, u)\, d\omega,$$

where $\lambda = \sigma + i\omega$ and $\gamma_\zeta(u,u) = \int_o^\zeta (1+\omega^2)\, d\mu_\omega(u,u)$. But $\gamma_o = o$ and so $\gamma_\zeta(u,u) = \lim_{\sigma \to o} \frac{1}{\pi} \int_o^\zeta \text{Re } f_\lambda(u,u)\, d\omega$. Now polarize γ to obtain $4\gamma_\zeta(u,\nu) = \gamma_\zeta(u+\nu, u+\nu) - \mu_\zeta(u-\nu, u-\nu) + i[\mu_\zeta(u+i\nu, u+i\nu) - \mu_\zeta(u-i\nu, u-i\nu)] = \lim_{\sigma \to o} \frac{1}{\pi} \int_o^\zeta \text{Re } f_\lambda(u, \nu)\, d\omega$ since f_λ, being bilinear, is uniquely determined by its quadratic form. This formula exhibits the fact that $\gamma_\zeta(u, \nu)$ is bilinear. Moreover, since $\mu_{-\infty} = o$ then $\mu_\zeta(u,u) = \int_{-\infty}^{\zeta} \frac{d\gamma_\omega(u,u)}{1+\omega^2}$ and the same argument shows that $\mu_\zeta(u, \nu)$ is also bilinear. Hence $F_\lambda = \alpha(u_1, u_2, \nu)\lambda$ $\beta(u_1, u_2, \nu)$. Since f_λ is defined for Re $\lambda > o$ then $F_{\bar\lambda} = o$ also and $F_\lambda - F_{\bar\lambda} =$

$(\lambda-\bar{\lambda})\ \alpha = o$ which shows that a is bilinear. But then $F_\lambda = \beta = o$ and so q is also bilinear.*

If f_λ can be represented by a different a, q, μ then by taking differences essentially the same argument as above shows that f_λ uniquely determines a, q, μ. An alternate argument can be based on the fact that a is determined by $\lim_{\sigma\to\infty} f(\sigma)/\sigma$ where $\sigma = \text{Re}\ \lambda$ (see [7], pg. 24). Since $f(\sigma) = o$ for $\sigma > o$ then so is a.

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