LEARNING, ADAPTIVE CURVE FITTING AND STOCHASTIC TRACKING*

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Two basic and related problems in learning systems are studied with optimum filtering and control methods. The first problem is how to approximate a curve or surface accurately with a minimum number of parameters. A solution is given using generalized maximum principle.[1,2] The second problem is optimum tracking of a changing parameter or parameters from noisy data using random sampling theory. The well known stochastic approximation methods of Robbins and Monroe[3] and Kiefer and Wolfowitz[4] track stationary parameters only and are shown to agree with special cases of the present stochastic tracking theory.

Previous studies of learning and adaptive systems usually describe devices and systems which accomplish certain goals, and relatively fewer papers are addressed to the problem of the fastest or best way of accomplishing these goals.[5-12] These latter group presents many diversified methods with few links in between. In addition to developing useful methods towards the latter objective, the present paper also gives a unified theoretical link to the various diversified methods.

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ADAPTIVE CURVE FITTING & SUBOPTIMIZATION

In a given problem, the gain curves etc. are usually best approximated by a number of arcs such that each individual arc satisfies a certain differential equation though the gain curve does not.[13] One such example is piecewise linear approximation. Each individual arc is a straight line which satisfies the equation

$$\dot{x} = \alpha_i \qquad t_{i-1} < t < t_i \tag{1}$$

where α_i is a constant slope, but the curve to be approximated does not satisfy (1). The problem of approximation is to choose α_i and t_i. An attractive feature of adaptive curve fitting is that the duration covered by an arc is automatically shortened at places where the curve is more curly.

The problem of adaptive curve fitting can be formulated as follows:

$$R_N = \int_{t_o}^{t_N} ||y(t) - x_1(t)|| \, dt = \text{minimum} \tag{2}$$

$$\dot{x} = Ax + Bu(t) \tag{3}$$

$$u(t) = \alpha_i \qquad t_{i-1} \leq t < t_i \tag{4}$$

where x and u are n- and m- dimensional vectors, m<n; y(t) is a scalor function; $x_1(t)$ is the first component of x; and A and B are constant matrices; the constants t_o and t_N are given; α_1, $\alpha_2 \ldots \alpha_N$ and t_1, $t_2 \ldots t_{N-1}$

are to be selected so that R_N is minimum.

Sometimes the problem is not of finding the closest approximation to a given curve y(t) but to minimize some functional of the curve x(t). However the problem is different from a straight calculus of variation problem in that x(t) is to be made up by N arcs as specified by (3) and (4). Then (2) is replaced by

$$R_N = \int_{t_o}^{t_N} g(x,u,t)\, dt = \text{minimum} \tag{5}$$

The expression (5) includes (2) as a special case.

A generalized version of Pontryagin's maximum principle gives the following condition to be satisfied by α_i and t_i:

$$H(\psi,x,u) = \psi'Ax + \psi'Bu - g(x,u,t) \tag{5a}$$

$$\dot{\psi}' = -\psi'A + \frac{\partial}{\partial x} g(x,u,t) \tag{6}$$

where ψ' is an n-dimensional row vector, and

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{t_o}^{t_N} [H(\psi,x,u+\Delta u_\varepsilon) + H(\psi,x,u)]\, dt \leq 0 \tag{7}$$

for all variations $\Delta u_\varepsilon(t)$ satisfying

$$\int_{t_o}^{t_N} ||\Delta u_\varepsilon(t)||dt < \varepsilon K \tag{8}$$

and K is any finite constant.

One way of varying u is changing t_i in (4) by ε while keeping

α_i, and t_j, $j \neq i$ the same. As t_i can be made to vary in one direction and then the opposite, inequality (7) gives

$$H(\psi(t_i),x(t_i), \alpha_i) = H(\psi(t_i),x(t_i),\alpha_{i+1}) \qquad i=1,2,\cdots N-1 \quad (9)$$

Taking variations of α_i in both directions, inequality (7) gives

$$\int_{t_{i-1}}^{t_i} \frac{\partial}{\partial \alpha_i} H(\psi,x,\alpha_i)dt = 0 \qquad i = 1,2,,,N \qquad (10)$$

It can be readily verified that other ways of varying u do not introduce any new condition which cannot be derived from (9) and (10). If $x(t_o)$ and $x(t_N)$ are unspecified, the transversality conditions give:

$$\psi'(t_o) = 0 \qquad \psi'(t_N) = 0 \qquad (11)$$

There are N-1 + mN + 2n conditions in (9), (10), and (11). These allow the initial values of ψ', x, and the constants α_i, and t_i to be completely determined.

Example I. Curve Fitting by Straight Lines. We have n = 1, $x = x_1$, and $\dot{x} = u$, A = 0

$$H = \psi u - ||y - x||$$

Equations (6), (9), (10), and (11) give respectively

$$\dot{\psi} = \frac{\partial}{\partial x} ||y(t) - x(t)|| \qquad (12)$$

$$\psi(t_i) = 0 \qquad i = 0,1,2...\bar{N} \qquad (13)$$

$$\int_{t_i}^{t_{i+1}} \psi(t)\, dt = 0 \qquad i = 0,1,2...N-1 \qquad (14)$$

Substituting (12) into (13) and (14) gives

$$J_o \equiv \int_{t_i}^{t_{i+1}} \frac{\partial}{\partial x} ||y(t) - x(t)|| dt = 0 \tag{15}$$

$$J_i \equiv \int_{t_i}^{t_{i+1}} t \frac{\partial}{\partial x} ||y(t) - x(t)|| dt = 0 \tag{16}$$

Let

$$||y - x|| = \frac{1}{2} (y - x)^2$$

Then

$$\frac{\partial}{\partial x} ||y - x|| = x(t) - y(t) \tag{17}$$

Equations (15) and (16) have a simple geometric interpretation. Referring to Fig. 1, the value $x(t_i)$ and t_i are assumed known. We choose a slope for the straight line segment and integrate out to a point P so that the net area between the curve and the straight line is zero ($J_o = 0$). If the selected slope is too small or too large, $J_i \lessgtr 0$ as shown by the line segements RP' and RP". There is one line segment RP which gives simultaneously $J_o = J_1 = 0$. The values of t_{i+1} and $x(t_{i+1})$ are given by the position of the end point P. Thus given $x(t_o)$, the subsequent points t_1, $x(t_1)$, t_2, $x(t_2)$... etc. are completely determined in this step by step manner.

If the initial point $x(t_o)$ is selected closer to $y(t_o)$, the approximation is better but the required number of line segments is also increased. By choosing the value of $x(t_o)$, we can control the number of line segments or the accuracy for approximating a given curve.

Example II. Curve Fitting by Parabolic Segments with Continuous $x(t)$ and dx/dt at t_i, i-1...N-1.

We have n = 2, m = 1, and

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u$$

In the solution, Equations (13) and (14) are replaced by, i = 0, 1...N-1

$$\int_{t_i}^{t_{i+1}} \psi_1(t)\, dt = 0 \tag{18}$$

$$\int_{t_i}^{t_{i+1}} t\psi_1(t)\, dt = 0 \tag{19}$$

where

$$\psi_1(t) = \int_{t_o}^{t} \frac{\partial}{\partial x_1} \, ||y(t) - x_1(t)||\, dt \tag{20}$$

Once $x_1(t_o)$ and $x_2(t_o)$ are selected, the subsequent $t_i, x_1(t_i)$ and $x_2(t_i)$ are completely determined. The freedom in the choice of the initial slope $x_2(t_o)$ is necessary for meeting the condition

$$\psi_1(t_N) = 0$$

<u>Example III. Suboptimization</u> Consider the classical problem of minimizing

$$J(x) = \int_0^T g(x,\dot{x})dt \tag{21}$$

but only piecewise linear functions are allowed:

$$\dot{x}(t) = b_i, \qquad t_{i-1} \leq b_i \leq t_i \tag{22}$$

with $t_o = 0$, $t_N = T$, where b_i and t_i, $i = 1,2...N-1$, are to be chosen.

The Hamiltonian is

$$H(\psi,x,u) = \psi' u - g(x,u) \tag{23}$$

Then (6), (9), and (10) become

$$\dot{\psi}' = -\frac{\partial H}{\partial x} = \frac{\partial}{\partial x} g(x,u) \tag{24}$$

$$H(\psi,x,b_i)\Big|_{t_i} = H(\psi,x,b_{i+1})\Big|_{t_i} \tag{25}$$

$$\int_{t_{i-1}}^{t_i} [\psi' - \frac{\partial}{\partial b_i} g(x,b_i)]\, dt = 0 \tag{26}$$

Equation (25) gives

$$\psi'(t_i) = \frac{g(x(t_i),b_{i+1}) - g(x(t_i),b_i)}{b_{i+1} - b_i} \tag{27}$$

Equations (24), (26), and (27) taken together constitute the "Euler-Lagrange" equations in crude form. The relation

$$\psi' = \frac{\partial g(x,\dot{x})}{\partial \dot{x}}$$

is replaced by (26) and (27) for the suboptimized solution.

Stochastic Tracking

General Model

In stochastic approximation, the unknown parameters are assumed constant. If the parameters vary in some random fashion, it becomes important to keep track of the parameter variations. Figure 2 gives a model of an adaptive control system in which the "plant" in itself represents a closed loop system with varying condition ξ and adjustable parameters γ. The totality of all possible plant conditions is represented by the set Σ which may or may or may not be finite. At any one time the condition of the plant is $\xi \varepsilon \Sigma$. Measurements $\eta(t)$, $t = 0, 1 \ldots$ are made on the plant and the conditional probabilities $q(\xi,t)$ at time t for each $\xi \varepsilon \Sigma$ is give by

$$q(\xi(t),t) \equiv \frac{\sum\limits_{\xi(1)} \sum\limits_{\xi(2)} \cdots \sum\limits_{\xi(t-1)} p(\xi(1), \xi(2) \cdots \xi(t), \eta(1), \eta(2) \cdots \eta(t))}{\sum\limits_{\xi(1)} \sum\limits_{\xi(2)} \cdots \sum\limits_{\xi(t)} p(\xi(1), \xi(2) \cdots \xi(t), \eta(1), \eta(2) \cdots \eta(t))} \tag{28}$$

The best values of the adjustable parameters γ are calculated from the set of conditional probabilities $q(\xi,t)$.

However, (28) is very difficult to use as t becomes large. The following theorem gives a simpler but equivalent method of calculating $q(\xi,t)$.

Theorem Let the transition probabilities of the plant and measuring system be specified by $p(\xi(t)/\xi(t-1))$, and $p(\eta(t)/\xi(t))$.

Then

$$r(\xi(t+1), t+1) = \sum_{\xi(t)} q(\xi(t),t)\, p(\xi(t+1)/\xi(t)) \tag{29}$$

$$q(\xi(t+1),t+1) = \frac{r(\xi(t+1),t+1)\, p(\eta(t+1)/\xi(t+1))}{\sum_{\xi(t+1)} r(\xi(t+1),t+1)\, p(\eta(t+1)/\xi(t+1))} \tag{30}$$

In other words, every bit of information in $p(\xi(1))$, $\eta(1)$, $\eta(2) \cdots \eta(t)$ is represented in the distribution $q(\xi(t),t)$.

Proof To prove the theorem, it will be shown that (28) is equivalent to (29) and (30). Let the following short notations be defined:

$P(\xi,\eta;t)$ and $P(\eta;t)$ denote joint probabilities

$P(\xi,\eta;t) \equiv p(\xi(1), \xi(2), \cdots \xi(t), \eta(1), \eta(2) \cdots \eta(t))$

$P(\eta;t) \equiv p(\eta(1), \eta(2) \cdots \eta(t))$

and $\sum_{\xi(1,t)}$ denote the multiple sum $\sum_{\xi(1)} \sum_{\xi(2)} \cdots \sum_{\xi(t)}$. Then

$$P(\eta;t) = \sum_{\xi(1,t)} P(\xi,\eta;t) \tag{31}$$

$$q(\xi(t),t) = \sum_{\xi(1,t-1)} \frac{P(\xi,\eta;t)}{P(\eta;t)} \tag{32}$$

By definition of the transition probabilities

$$P(\xi,\eta;t+1) = P(\xi,\eta;t)\, p(\xi(t+1)/\xi(t))\, p(\eta(t+1)/\xi(t+1)) \tag{33}$$

$$q(\xi(t+1),t+1) = \frac{\sum_{\xi(t)} \left\{ \sum_{\xi(1,t-1)} P(\xi,\eta,t) \right\} p(\xi(t+1)/\xi(t))\, p(\eta(t+1)/\xi(t+1))}{\sum_{\xi(t+1)} \sum_{(t)} \left\{ \sum_{\xi(1,t-1)} P(\xi,\eta,t) \right\} p(\xi(t+1)/\xi(t))\, p(\eta(t+1)/\xi(t+1))} \tag{34}$$

Dividing both the denominator and numerator of (34) by $P(\eta,t)$ gives:

$$q(\xi(t+1),t+1) = \frac{\sum_{\xi(t)} q(\xi(t),t)\ p(\xi(t+1)/\xi(t))\ p(\eta(t+1)/\xi(t+1))}{\sum_{\xi(t+1)} \sum_{\xi(t)} q(\xi(t),t)\ p(\xi(t+1)/\xi(t))\ p(\eta(t+1)/\xi(t+1))} \tag{35}$$

Equation (35) is the same as (29) and (30) when (29) is regarded as a definition of $r(\xi(t+1),t+1)$.

Optimum Adjustment

The optimum setting of adjustment parameters $\hat{\gamma}$ is obtained by minimizing the expected penalty function:

$$\sum_{\xi(t)} F(\gamma,\xi)\ q(\xi(t),t) = \min,$$

where $F(\gamma,\xi)$ represents the penalty if the plant is in condition ξ, and the setting γ is used.

For each state ξ there is a set of best parameters $\gamma = (\gamma_1,\gamma_2\cdots\gamma_n)$. Let $\gamma(\xi)$ be the best γ for the state ξ. If the penalty function F is

$$F = ||\gamma(\xi) - \gamma||^2 Q$$

where Q is an nxn positive definite matrix, and $||\alpha||^2 Q$ represents the scalar $\alpha' Q\alpha$. Then the best γ which minimizes the probable penalty is

$$\hat{\gamma}(t) = \sum_{\xi(t)} \gamma(\xi(t))\ q(\xi(t),t) \tag{36}$$

In the above section, the number of states can be infinite, and ξ is a continuous vector or scalor. The summations are then replaced by integrals.

Relation to Kalman Filter[14]

In Kalman's optimum filtering problem, the state and observed variables are assumed to be generated by

$$\xi(t+1) = A(t)\ \xi(t) + w(t) \tag{37}$$

$$\eta(t) = H(t)\ \xi(t) + v(t) \tag{38}$$

where ξ and η are of dimensions n and m respectively and w and v are white Gaussian random processes. It is readily shown that (37) and (38) lead to a Markov process with

$$p(\xi(t+1)/\xi(t)) = \frac{1}{(2\pi)^{n/2}\,\Delta_w^{1/2}}\ \exp.\left\{-\tfrac{1}{2}\|\xi(t+1) - A(t)\ \xi(t)\|^2_{\phi_w^{-1}}\right\} \tag{39}$$

$$p(\eta(t)/\xi(t)) = \frac{1}{(2\pi)^{m/2}\,\Delta_v^{1/2}}\ \exp.\left\{-\tfrac{1}{2}\|\eta(t) - H(t)\ \xi(t)\|^2_{\phi_v^{-1}}\right\} \tag{40}$$

where $\phi_w = \overline{w(t)w'(t)}$, $\phi_v = \overline{v(t)\ v'(t)}$.

It will be shown that with the transition probabilities (39) and (40), (29) and (30) reduce to the two step updating process of the Kalman Filter.[15] The following algebraic identity is readily verified:

$$(x - By)'\ M^{-1}(x - By) + y'N^{-1}y \equiv x'K^{-1}x + (y - Cx)'\ L^{-1}\ (y - Cx) \tag{41}$$

where x and y are column vectors, and B,C, K, L, M, N are matrices:

$$L^{-1} = B'M^{-1}B + N^{-1} \tag{42}$$

$$K^{-1} = M^{-1} - M^{-1}B\ L\ B'M^{-1} \tag{43}$$

and $C = L\ B'M^{-1}$ (44)

From (42) and (43) the following expressions are derived:

$$K = B\,N\,B' + M \tag{45}$$

$$L = N - N\,B'\,[BNB' + M]^{-1}\,B\,N \tag{46}$$

To derive Kalman filter, Gaussian distribution is assumed for $q(\xi(t))$.

$$q(\xi(t),t) = \frac{1}{(2\pi)^{n/2}\,\Delta_{\xi}^{1/2}}\ \exp.\left\{\frac{1}{2}||\xi(t) - \bar{\xi}(t)||^{2}{}_{\phi(t)^{-1}}\right\} \tag{47}$$

where $\tilde{\phi}(t) = \overline{[\xi(t) - \bar{\xi}(t)][\xi(t) - \bar{\xi}(t)]'}$ and Δ_{ξ} is the determinant of $\tilde{\phi}(t)$.

Substituting (39) and (47) into (29), the resulting integral has an exponent $-\frac{1}{2}E$:

$$E = ||\xi(t) - \bar{\xi}(t)||^{2}{}_{\tilde{\phi}(t)^{-1}} + ||\xi(t+1) - A(t).\xi(t)||^{2}{}_{\phi_W^{-1}} \tag{48}$$

Let $y = \xi(t) - \bar{\xi}(t)$, and $x = \xi(t+1) - A(t)\,\bar{\xi}(t)$ identity (41) gives

$$E = ||\xi(t+1) - A(t)\,\bar{\xi}(t)||^{2}{}_{K^{-1}} + ||\xi(t) - \bar{\xi}(t) - Cx||^{2}{}_{L^{-1}}$$

The second term disappears by integrating with respect to $\xi(t)$, and the first term gives $A(t)\,\bar{\xi}(t)$ as the best estimate of $\xi(t+1)$ and $K = A\,\tilde{\phi}(t)\,A' + \phi_W$ as the error covariance $\tilde{\phi}(t+1,t)$ before measurement $\eta(t+1)$ is made:

$$r(\xi(t+1)) = \frac{1}{(2\pi)^{n/2}\,\Delta^{1/2}}\ \exp.\left\{\frac{1}{2}\,||\xi(t+1) - A(t)\,\bar{\xi}(t)||^{2}{}_{\tilde{\phi}(t+1,t)^{-1}}\right\} \tag{49}$$

Substituting (40) and (49) into (30) and making use of (41), (44) and (46) the second step in Kalman filter is obtained.

Quasi-Equivalent Generating Process

A quasi-equivalent generating process is a generating process in the form of (37) which gives the same covariance matrix

$$\phi(t,\tau) \equiv \overline{\xi(t)\ \xi(\tau)'}$$

though not the same transition probabilities $p(\xi(t+1)/\xi(t))$. Consider the following example in which ξ is continuous, $|\xi| \leq 1$:

$$p(\xi(1)) = \frac{1}{2} \tag{50}$$

$$p(\xi(t+1)/\xi(t)) = \nu\ \delta(\xi(t+1) - \xi(t)) + \frac{1-\nu}{2} \tag{51}$$

Then

$$\phi(t,\tau) = \frac{1}{3}\ \nu^{|t-\tau|} \tag{52}$$

The same $\phi(t,\tau)$ can be obtained by the following process

$$\xi(t+1) = \nu\ \xi(t) + w(t) \tag{53}$$

where $w(t)$ is white. But (50) and (51) cannot be obtained from (53). The first step in Kalman filter is still valid:

$$r(\xi(t+1),t+1) = \int_{-1}^{1} q(\xi(t),t) \left[\nu\,\delta(\xi(t+1) - \xi(t)) + \frac{1-\nu}{2}\right] d\xi(t)$$

$$= \nu\ q(\xi(t+1),t) + \frac{1-\nu}{2}$$

$$\bar{\xi}(t+1,t) = \int_{-1}^{1} \xi(t+1)\ r(\xi(t+1),t+1)d\ \xi(t+1)$$

$$= \nu \int_{-1}^{1} \xi\ q(\xi,t)d\xi = \nu\ \bar{\xi}\ (t) \tag{54}$$

The second step in Kalman filter gives a different expected value of $\bar{\xi}(t+1)$ from that obtained from (30) but the result is very close.

If a Markov process has a quasi equivalent generating process, Kalman filter for the latter gives an approximately optimal estimate for the former.

Kalman Filter, Stochastic Tracking, Stochastic Approximation

Kalman filter establishes a link between stochastic tracking of a stationary process and Robbins and Monroe stochastic approximation. If $\xi(t)$ varies randomly and the process is stationary, Kalman filter gives the same steady state solution as the author's previous stochastic tracking solution.[8] If $\xi(t)$ is a constant $(A(t) = 1,\ w(t) = 0$ in (37), Kalman filter gives

$$\bar{\xi}(t+1) = \bar{\xi}(t) + K(\eta - H\,\bar{\xi}(t))$$

where Kt approaches a constant as t approaches infinity. The result is in agreement with Robbins and Monroe stochastic approximation.[3]

Stochastic peak tracking

A stochastic peak tracking problem is given as follows:

$$m(t) = m_o(t) - \frac{a(t)}{2}(\xi(t) - x(t))^2 \qquad (55)$$

where m is a merit parameter, x is an adjustable parameter, and $\xi(t)$ is the best setting for x at t. It varies with t randomly as follows:

$$\xi(t+1) = A(t)\ \xi(t) + w(t) \qquad (56)$$

The measured merit is y

$$y(t) = m(t) + v(t) \qquad (57)$$

where w(t) and v(t) are white with covariance functions ϕ_w and ϕ_v respectively. In order to determine $\xi - x$, x is set at $\bar{\xi}(t) + C$ and $\bar{\xi}(t) - C$ for each t, where $\bar{\xi}(t)$ is the best estimate of ξ at t, and the resulting measurements yield y(t) and y′(t):

$$y(t) - y'(t) = -2ac\,[\bar{\xi}(t)-\xi(t)] + v - v' \tag{58}$$

Kalman filter solution to the above problem yields:

$$\bar{\xi}(t+1) = A\,\bar{\xi}(t) + K(t)\,[y(t) - y'(t)] \tag{59}$$

$$K(t) = \frac{a\,c\,A\,\tilde{\phi}(t)}{2a^2c^2\,(t) + \tilde{\phi}_v} \tag{60}$$

where $\tilde{\phi}(t)$ is the expected value of $\overline{[\bar{\xi}(t) - \xi(t)]^2}$, and

$$\tilde{\phi}(t+1) = \frac{A\,\phi_v\,\tilde{\phi}(t)}{2a^2c^2\tilde{\phi}(t) + \phi_v} + \phi_w \tag{61}$$

<u>Determination of c(t)</u>

One basis for determining c(t) is to maximize the expected value of m(t) :

$$m(t) = m_o(t) - \frac{a}{4}(\xi - \bar{\xi} - c)^2 - \frac{a}{4}(\xi - \bar{\xi} + c)^2$$

$$= m_o(t) - \frac{a}{2}\left[(\xi - \bar{\xi})^2 + c^2\right]$$

The criteria is

$$J = \sum_{t=0}^{t=T} a(t)\left[\tilde{\phi}(t) + c(t)^2\right] = \text{min.} \tag{62}$$

A variable J(t′) is defined as the above sum from t=0 to t = t′ - 1. Then (62) is equivalent to:

$$J(0) = 0$$

$$J(t+1) = J(t) + a(t) \;\; [\tilde{\phi}(t) + c(t)^2] \qquad (63)$$

$$J(T+1) = \min \qquad (64)$$

Equations (61), (63), and (64) can be treated as an optimal discrete control problem with $\begin{pmatrix} \tilde{\phi} \\ J \end{pmatrix}$ as the state vector and $u = ac^2$ as the control variable. The Hamiltonian function is

$$H(\psi,\phi,u,t) = -(J + a\tilde{\phi} + u) + \psi \left[\frac{A\,\phi_V\,\tilde{\phi}}{2au\tilde{\phi} + \phi_V} + \phi_W \right]$$

$$\psi(t-1) = \frac{\partial H(\psi,\tilde{\phi},t)}{\partial\tilde{\phi}}$$

$$\psi(T) = 0$$

and u is determined by maximizing the Hamiltonian. In the limit of large T, and constant a and A, the solution agrees with previous result on stochastic peak-seeking.[8]

Stochastic Approximation

If $\xi(t)$ and $a(t)$, are constants, the problem reduces to that of stochastic approximation. Let $A(t) = 1$, and $\phi_W(t) = 0$. Let a new state variable $z(t)$ be defined as $1/\tilde{\phi}(t)$. Equations (61) and (63) become:

$$z(t+1) = z(t) + \left(\frac{2a}{\phi_V}\right) u(t) \qquad (65)$$

$$J(t+1) = J(t) + \frac{a}{z(t)} + u(t) \qquad (66)$$

The Hamiltonian function is

$$H(\psi,z,u,t) = -\left(J + \frac{a}{z} + u\right) + \psi\left(z + \frac{2au}{\phi_V}\right) \qquad (67)$$

$$\psi(T) = 0 \qquad (68)$$

$$\psi(t-1) = \frac{\partial H}{\partial z} = \frac{a}{z^2} + \psi(t) \tag{69}$$

From (67) and (68), $\psi(t)$ is seen to be a positive but monotonously decreasing function which decreases to zero at $t = T$. Equation (69) gives the following solution:

(1) $t < t_1$, and $\psi > \frac{\phi_v}{2a}$

$u = u_m$

(2) $t > t_1$ and $\psi < \frac{\phi_v}{2a}$

$u = 0$

(3) $t = t_1$

$$\psi(t_1) = \frac{a}{z(t_1+1)^2} (T - t_1) = \frac{\phi_v}{2a}$$

and $u(t_1)$ is equal to the value required to reduce $z(t_1+1)$ to

$$z(t_1+1) = \sqrt{\frac{2a^2(T-t_1)}{\phi_v}} \tag{70}$$

The solution is a bang-bang solution which requires reduction of $\phi(t)$ to $1/z(t_1+1)$ and then let $c(t)$ equal to zero afterwards. The criterion (62) does not lead to Kiefer-Wolfowitz stochastic approximation.

An alternative criterion is to require $\bar{\phi}(t)$ and $c(t)^2$ to approach 0 at the same logrithmic rate as $t\to\infty$:

$$\tilde{\phi}(t) = C_1 t^{-\alpha} \qquad c^2 = C_2 t^{-\alpha}$$

$$\frac{2a^2c^2}{\phi_v} = \frac{1}{\tilde{\phi}(t+1)} - \frac{1}{\tilde{\phi}(t)} = C_1 \alpha t^{\alpha-1} + \ldots$$

Therefore $\alpha - 1 = -\alpha$, $\alpha = \frac{1}{2}$. It is readily verified that the result agrees with Kiefer-Wolfowitz stochastic approximation.[4]

One essential significance, of the above result is that the total error, $\tilde{\phi}(t) + c(t)^2$, reduces atmost at the rate $t^{-1/2}$ as $t \to \infty$, and cannot be any faster.

Conclusion

Two problems in learning and adaptive systems are studied in this paper:

1. How to approximate a curve most accurately with a given number of arcs? In case the curve is known, a step by step method is derived from the generalized maximum principle. The method is readily reducible to computer algarithm. In case the curve is not known, but a certain functional of the curve is to be minimized, the method gives a necessary condition which can be called a discrete version of Euler Lagrange equation.

In many applications, the approximated curve is unknown, or time varying or both. Then α_i and t_i are determined for a nominal curve initially, and the subsequent variations of α_i or improvements on the estimation of α_i, can be determined by a stochastic tracking method.

2. What is the optimal way of tracking an unknown or varying parameter from noisy data? A general stochastic tracking problem is

formulated in terms of a Markov process with noisy measurements. The parameter estimation problem is simplified by proving that all the apriori information and information obtained from past measurements are contained within a set of conditional probabilities. The calculation of the conditional probabilities can be done by a two-step updating process which reduces to Kalman filter if the distributions are Gaussian.

In many applications the random variations of the parameters of a system cannot be described by state variable equations. The general procedure reduces to a Kalman filter if there is a quasi-equivalent process (same covariance but different distributions) whose parameters can be described by state variable equations, and reduces further to Robbins and Munro stochastic approximation if the parameters are fixed but unknown.

The peak tracking problem can be reduced to the stochastic tracking problem with the additional variable of the excursion amplitude δ, which is then obtained by maximizing the expected peak. In the special case that the parameters do not change, the method yields two tracking procedures depending on the criterion used:

(a) Tracking for a limited time and then settle for the parameter value so determined. It is shown that the expected error is proportional to t^{-1}, where t is the tracking time.[16]

(b) A procedure which agrees with the Kiefer-Wolfowitz stochastic approximation method. It is shown further that the expected total reduction is peak value (due to error and hunting loss) is proportional to $t^{-1/2}$.

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List of Figures

Fig. 1. Choice of slope and length of the line segments approximating y(t).

Fig. 2. Model of a Stochastic Adaptive System.

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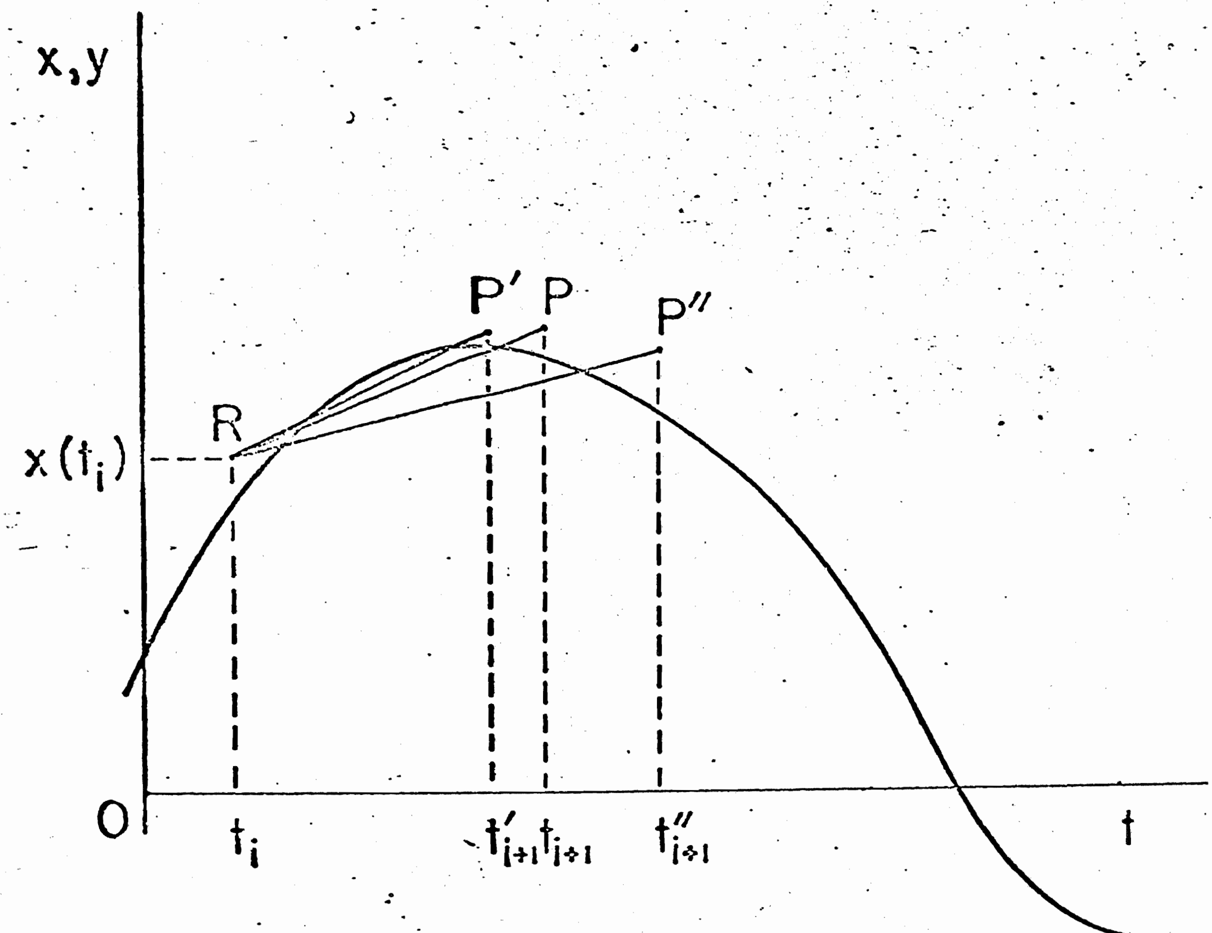

x,y
P′
P
P″
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x(t_i)
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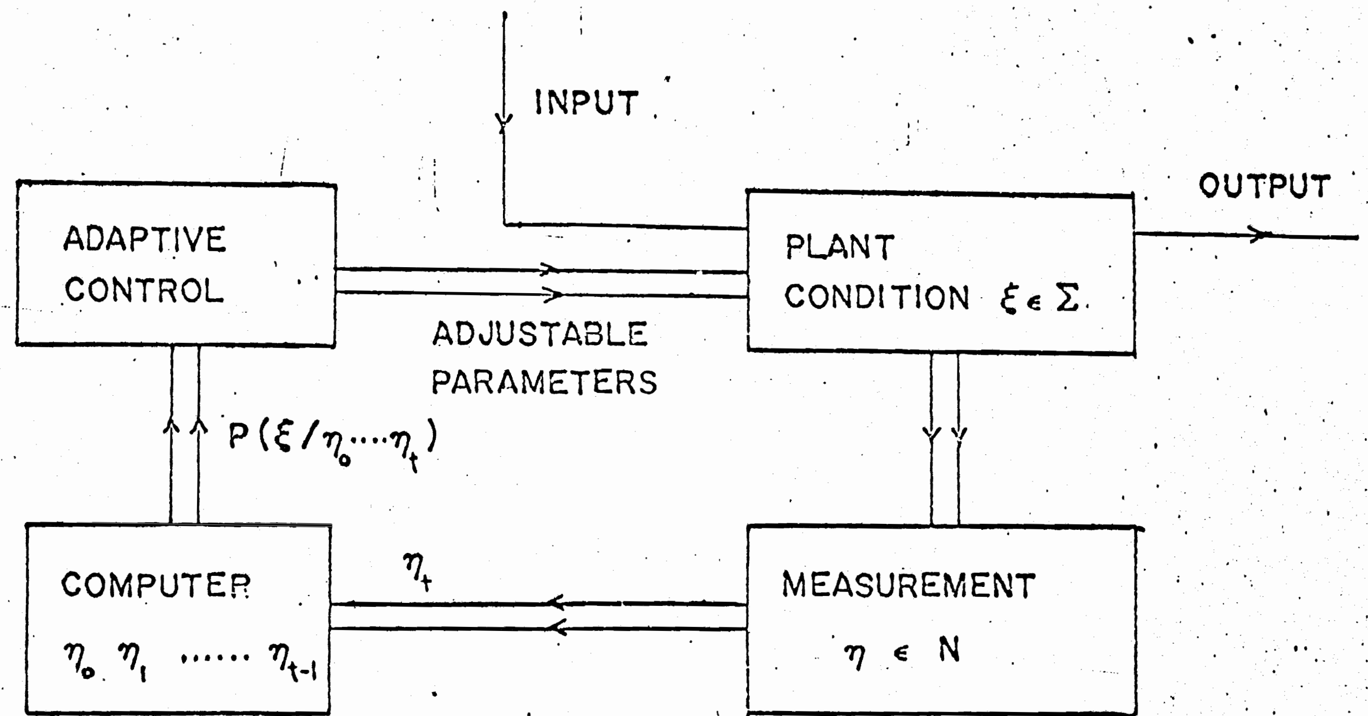
t_i
t′_{i+1}
t_{i+1}
t″_{i+1}
t



MODEL OF A STOCHASTIC ADAPTIVE SYSTEM