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A PERTURBATION SOLUTION FOR THE NONLINEAR RADIATION HEAT TRANSFER PROBLEM

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Abstract

A perturbation technique is developed for determining the transient temperature in a slab, insulated on one face and subject to nonlinear thermal radiation at the other face. The slab is initially at a uniform temperature and is assumed to be homogeneous and isotropic; the physical properties are assumed to be independent of temperature. Temperature distributions and heat flux at the radiating boundary are presented in a dimensionless, graphical form for a wide range of parameters, and the former are compared with previously obtained analog computer results.

Das Abstrakt

Eine Perturbationsmethode für die Ermittlung des Zeitverlaufs von Temperatur in einer Platte, die an einer Oberfläche isoliert und an der anderen einer nichtlinearen Wärmestrahlung unterworfen ist, wird entwickelt. Es wird vorausgesetzt, dass die Platte ursprünglich eine gleichmässige Temperatur hat, und dass sie homogen und isotropisch ist; ausserdem wird vorausgesetzt, dass die physikalischen Eigenschaften von der Temperatur unabhängig sind. Die Temperaturverbreitungen und die Wärmestromung an der strahlenden Oberfläche sind für eine umfangreiche Parameterreihe in einer graphischen dimensionsfreien Form dargestellt, und jene sind mit den früher an Analogrechenmaschinen erhaltenen Resultaten verglichen.

Краткий обзор

В этом докладе разрабатывается пертурбационный метод для установления зависимости температуры в слое, который изолирован на одной поверхности и подвергнут нелинейной термической радиации на другой. Слой начально имеет равномерную температуру и предполагается, что он однородный и изотермический; кроме того, его физические свойства предполагаются независимыми от температуры. Распределения температуры и тепло-потока через радиационную поверхность заданы в безразмерной графической форме для широкого диапазона параметров, и приведено сравнение с результатами полученными заранее при помощи вычислительных машин.

Introduction In problems of heat transfer involving convection, radiation, or evaporation at the surface of a body, the flux of heat at the surface temperature is, in general, a nonlinear function of the surface temperature. Furthermore, the thermal properties of the body may also vary with temperature. One commonly employed approximation to the real phenomenon assumes constant properties in the medium but with heat transfer at the surface a given nonlinear function of surface temperature. Mathematically such problems occupy an interesting position between the classical linear theory and the general case in which both the differential equation and the boundary conditions are nonlinear.

In heat transfer between solids and gases it is commonly assumed that the rate of heat exchange across a gas-solid interface is proportional to the difference between the temperature of the solid surface and the surrounding medium which gives rise to the boundary condition of the form[1]

$$-k\left(\frac{\partial T}{\partial x}\right) = H\Delta T \qquad (1)$$

where $\partial T/\partial x$ is the thermal gradient at the surface; and H is a factor of proportionality, frequently called the film transfer factor; and k is the thermal conductivity. If H is independent of temperature the above boundary condition is linear. For small temperature differences, ΔT,

[1] This is, of course, "Newton's law of cooling."

and where most of the heat transfer is due to conduction-convection, H varies but slightly with temperature and may be approximated by some constant number. For large temperature differences radiation plays a dominant role so that the film transfer factor is strongly temperature dependent. When the conduction and convection may be neglected, the film transfer factor is given by Stefan-Boltzmann Radiation Law,

$$H_r = A\epsilon \frac{T_s^4 - T_g^4}{T_s - T_g} \qquad \text{(ii)}$$

where T_s is the absolute temperature of the solid at the surface, T_g the absolute temperature of the ambient gas, ϵ the emissivity, and A is a constant depending upon the units of measurement. It is clear that when the film coefficient for the radiating boundary (ii) is substituted in (i) the resulting boundary condition is nonlinear. Since a nonlinearity in either the differential equation or the boundary condition renders the entire boundary value problem nonlinear, the solution to the unsteady heat transfer problem with a radiating boundary condition is difficult.

The present problem was studied by Mann and Wolf (1); their investigation was, however, primarily concerned with the existence and uniqueness of the solution to the integral equation associated with the problem and less concerned with a practical method for its solution. Jaeger (2) solved the problem in terms of a power series expansion which is adequately convergent only for early times. Chambre (3) obtained a solution in terms of an approximating polynomial evaluated by means of the "heat balance" integral. Abarbanel (4) presents approximate solutions for the surface temperature of slabs, spheres, cylinders, and semi-infinite solids, for very small and large values of time. Richardson (5) utilizes the Biot variational method and a polynomial approximation to obtain an approximate solution for a narrow range of exponents in the nonlinear boundary condition. Fairall, et al, (6) present some computer results for surface temperatures. A great quantity of data for this problem has been obtained by Zerkle and Sunderland (7) who have obtained the numerical results by the use of an analog computer and have plotted graphs for a wide range of parameters.

Despite the extensive amount of work previously done on this problem there is yet no general analytical solution available. In the present study an asymptotic solution is obtained, by means of a perturbation procedure, which appears to be satisfactory for most engineering purposes. The large number of graphs provided by Zerkle and Sunderland provide a convenient standard for demonstrating the relative accuracy of the present solution. Because the latter is analytical, important functions of the temperature distribution such as flux at any point and total heat are readily obtained.

<u>Statement of the Problem</u> A slab initially at a uniform temperature is suddenly exposed to radiant heat transfer on one or both faces. It is assumed, following ref. (7) that the boundary conditions are uniform over each boundary surface (this implies that the heat flow is one-dimensional); the environment temperature (T_e) is constant; the slab is a homogeneous, isotropic, and opaque, and the physical properties are independent of temperature; the radiation interchange factor (F_{se}) is independent of slab surface temperature; the slab is exposed to continual heating. From the first assumption, the general heat conduction equation reduces to

$$\frac{\partial T}{\partial \tau} = \frac{k}{\rho c}\frac{\partial^2 T}{\partial x^2} \qquad (0 \le x \le L;\ \tau > 0) \qquad (1)$$

A solution to equation (1) subject to the following initial and boundary conditions is sought.

$$T(x,0) = T_i \qquad (2)$$

$$\frac{\partial T}{\partial x}(0,\tau) = \frac{\sigma F_{se}}{k}[T^4(0,\tau) - T_E^4] \qquad (3)$$

$$\frac{\partial T}{\partial x}(L,\tau) = \frac{\sigma F_{se}}{k}[T_e^4 - T^4(L,\tau)] \qquad (4)$$

<u>Outline of Perturbation Technique</u> It is easily shown that there exists a steady state solution, corresponding to $\tau = \infty$, for the above problem, as follows,

As $\partial T/\partial \tau = 0$, equation (1) becomes

$$T = Ax + B. \qquad (5)$$

Since T must satisfy the boundary conditions, substituting (5) in (3) and (4) gives

$$\frac{\partial T}{\partial x}(0,\tau) = A = \frac{\sigma F_{se}}{k}[B^4 - T_E^4] \qquad (6)$$

$$\frac{\partial T}{\partial x}(L,\tau) = A = \frac{\sigma F_{se}}{k}[T_e^4 - (AL + B)^4] \qquad (7)$$

The constants A and B are obtained by solving the algebraic equations (6) and (7) so that (1) is determined. This solution will be designated by $T^{(0)}$.

The zeroth approximation in the perturbation technique is taken to be latter solution, $T^{(0)}$. Further, let the first approximation be written

$$T^{(1)} = T^{(0)} + \varphi_1 \qquad (8)$$

where it is assumed that

$$|\varphi_1(x,\tau)| < T^{(0)}(x) \qquad (9)$$

Substituting (8) into (1), (2), (3) and (4), yields a boundary-value problem on φ_1. After solving for φ_1 the second approximation may be found, in principle, by perturbing $T^{(1)}$;

$$T^{(2)} = T^{(1)} + \varphi_2 = T^{(0)} + \varphi_1 + \varphi_2 \qquad (10)$$

In general, however, the solution for φ_1 will be an infinite series, where the second and subsequent terms decay at a faster rate than the first. The perturbation, therefore, is taken about $T^{(o)} + \varphi_1'$, which results in a second approximation

$$T^{(2)} = T^{(o)} + \varphi_1' + \varphi_2 \qquad (11)$$

where φ_1' is the first term of φ_1.[1]

Assuming further that $|\varphi_2| < |\varphi_1'|$ and $|\varphi_1| < T_e$ as previously,[2] one obtains a boundary-value problem on φ_2. Proceeding to n^{th} approximation, and in a like manner retaining the first term of each of the (n-1) solutions which is assumed to satisfy

$$|\varphi_n| < |\varphi_{n-1}'| < \ldots\ldots < |\varphi_1'| < T_e \qquad (12)$$

yields

$$T^{(n)} = T_e + \varphi_1' + \varphi_2' + \ldots\ldots + \varphi_{n-1}' + \varphi_n \qquad (13)$$

to obtain φ_n once $\varphi_1', \varphi_2' \ldots\ldots \varphi_{n-1}'$ are known, one proceeds as follows. Let

$$\varphi_n = T^{(n)} - (T_e + \varphi_1' + \varphi_2' + \ldots + \varphi_{n-1}') \qquad (14)$$

Substituting the derivatives of φ_n in (1) and (2), and realizing that each φ_k' and T_e satisfies the heat equation, the equation on φ_n is

$$\frac{\partial \varphi_n}{\partial \tau} = \alpha \frac{\partial^2 \varphi_n}{\partial x^2} \qquad (15)$$

with the initial condition

$$\varphi_n(x,o) = (T_i - T_e) - \varphi_1'(x,o) - \varphi_2'(x,o) \ldots\ldots - \varphi_{n-1}'(x,o) \qquad (16)$$

Substituting the derivatives of φ_n into (3) and (4) and expanding the right hand by means of the binomial expansion, taking (12) into account and retaining only those terms of order φ_n yields,

$$\frac{\partial \varphi_n}{\partial x}(o,\tau) = F_1(T_e, \varphi_1', \ldots\ldots \varphi_{n-1}') - k_s \varphi_n \qquad (17)$$

$$\frac{\partial \varphi_n}{\partial x}(L,\tau) = F_2(T_e, \varphi_1', \ldots\ldots \varphi_{n-1}') - k_s \varphi_n \qquad (18)$$

The equations (15), (16), (17), and (18) represent a linear boundary-value problem on φ_n with the φ_k' $(k < n)$ appearing as nonhomogeneous terms. This problem is readily solved in closed form for each approximation provided the functions φ_k' are known, i.e., if one wanted the third approximation, it would be necessary to have first solved for the second and first approximation. The solutions for the various approximations will now be developed. Inasmuch as the specific example to be evaluated in the following section will be for the problem solved numerically by Zerkle and Sunderland (7), the development will be for the boundary value problem considered by them, namely: (1), (2), (4), and at $x = 0$,

$$\frac{\partial T}{\partial x} = 0 \qquad (19)$$

SOLUTION FOR VARIOUS APPROXIMATIONS

Zeroth Approximation It is easily verified that

$$T^{(o)} = T_e. \qquad (20)$$

Clearly, this is the exact solution.[1] It is this solution which will be perturbed to obtain the various approximations.

First Approximation Write

$$T(x,\tau) = T_e + \varphi_1(x,\tau) \qquad (21)$$

where it is assumed that $|\varphi_1(x,\tau)| < T_e$. Substituting for T, and its derivatives from (21) in the original equation (1) and boundary conditions (2), (3) and (4) gives

$$\frac{\partial \varphi_1}{\partial \tau} = \alpha \frac{\partial^2 \varphi_1}{\partial x^2}(x,\tau) \qquad (0 \le x \le L,\ \tau > 0) \qquad (22)$$

$$\varphi_1(x,0) = (T_i - T_e) \qquad (23)$$

$$\frac{\partial \varphi_1}{\partial x}(0,\tau) = 0 \qquad (24)$$

$$\begin{aligned}\frac{\partial \varphi_1}{\partial x}(L,\tau) &= \frac{\sigma F_{se}}{k}[T_e^4 - (T_e + \varphi_1(L,\tau))^4] \\ &= \frac{\sigma F_{se}}{k}[T_e^4 - (T_e^4 + 4T_e^3\varphi_1 + 6T_e^2\varphi_1^2 + 4T_e\varphi_1^3 + \varphi_1^4)] \\ &= -\frac{\sigma F_{se}}{k} T_e^4 [4\frac{\varphi_1}{T_e} + 6(\frac{\varphi_1}{T_e})^2 + 4(\frac{\varphi_1}{T_e})^3 + (\frac{\varphi_1}{T_e})^4]\end{aligned} \qquad (25)$$

Under the a priori assumption $|\varphi_1| < T_e$, the following is valid:

$$(\frac{\varphi_1}{T_e})^3 < (\frac{\varphi_1}{T_e})^2 < (\frac{\varphi_1}{T_e}) < 1$$

Hence it is clear that if $|\frac{\varphi_1}{T_e}| < \frac{2}{3}$ that the first

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[1] The remainder of the series reappears in the evaluation of φ_2. For the numerical example considered later on, the terms of the series are proportional to $e^{-\delta_1^2 N_{Fo}}$, $e^{-\delta_2^2 N_{Fo}}$, etc., where $\delta_1 < \delta_2 < \ldots\ldots$.

[2] The assumptions regarding the relative magnitudes are verified a posteriori.

[1] There are actually four possible solutions for $T^{(o)}$ since the latter is obtained from $[T^{(o)}]^4 = T_e^4$. The other three roots, however, are not physically meaningful.

term on the right hand side of (25) will be the dominant one, and for small values of $|\varphi_1/T_e|$ all but the first term may be neglected. Thus (25) becomes

$$\frac{\partial\varphi_1}{\partial x}(L,\tau) = -4\left(\frac{\sigma F_{se}}{k}T_e^3\right)\varphi_1$$

$$= -\frac{4}{LN_{rh}}\varphi_1$$

$$= -k_5\,\varphi_1 \quad \text{where } k_5 \equiv \frac{4}{LN_{rh}} \qquad (26)$$

The problem, as it stands, is equivalent to the case of a heated slab of thickness L which is initially at temperature T_i and which has one face ($x = 0$) insulated, while the surface at $x = L$ of the slab is heated according to Newton's Law of heating (linear radiation). The general solution of equation (22) is easily obtained, Chapman (8), and is

$$\varphi_1 = e^{-\beta^2\alpha\tau}[B_1\sin(\beta x) + B_2\cos(\beta x)] \qquad (27)$$

where $(\beta L)\tan(\beta L) = \frac{4}{N_{rh}}$ defines the value of β i.e. $\delta_n \tan\delta_n = \frac{4}{N_{rh}}$ where $\delta_n = (\beta L)$ (28)

Using the (23), (24), (26), (27), the solution for $T^{(1)}$ is[1]

$$\frac{T^{(1)} - T_i}{T_e - T_i} = 1 - 2\sum_{n=1}^{\infty} e^{-\delta_n^2\alpha\tau}\frac{\sin\delta_n\cos(\delta_n\xi)}{\delta_n + \sin\delta_n\cos\delta_n} \qquad (29)$$

where δ_n is the n^{th} root of the equation (28), values of which have been evaluated and are tabulated in Carslaw and Jaeger (9).

__Second Approximation__ Since (29) involved an approximation (i.e. $|\varphi_1/T_e| << 1$) it does not represent, of course, a general solution for the transient problem.[2] To obtain a closer approximation, for larger values of $|\varphi_1/T_e|$, (29) is perturbed, i.e., $T = T_e + \varphi_1 + \varphi_2$. (30)

As noted earlier, because of the decay factor, all terms but the first are negligible for large times. Moreover, since one is quite free to choose the value about which to perturb, rather than include all of φ_1 in the second approximation only its first term is retained. Denoting the first term of φ_1 series by φ_1', (30) becomes

$$T = T_e + \varphi_1' + \varphi_2 \qquad (31)$$

where it is assumed that $|\varphi_2/T_e| < |\varphi_1'/T_e|$ and where

$$\varphi_1' = 2e^{-\delta_1^2\alpha\tau/L^2}\frac{\sin\delta_1\cos(\delta_1\xi)}{\delta_1 + \sin\delta_1\cos\delta_1}(T_i - T_e) \qquad (32)$$

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[1] The variable $\frac{T^{(1)} - T_i}{T_e - T_i}$ has been selected to conform to (7).

[2] The solution obviously fails at $\tau = 0$.

Substituting (31) and its derivatives into (1), (2), (3), and (4) yields, after symplifying,

$$\frac{\partial\varphi_2}{\partial\tau} = \alpha\frac{\partial^2\varphi_2}{\partial x^2} \qquad (0 \le x \le L,\ \tau > 0) \qquad (33)$$

$$\varphi_2(x,0) = (T_i - T_e) - \varphi_1'(x,0)$$

$$= (T_i - T_e) - 2\frac{\sin\delta_1\cos(\delta_1\xi)}{\delta_1 + \sin\delta_1\cos\delta_1}(T_i - T_e)$$

$$= (T_i - T_e) - k_1\cos(\delta_1\xi)(T_i - T_e) \qquad (34)$$

$$\frac{\partial\varphi_2}{\partial x}(0,\tau) = 0 \qquad (35)$$

$$\frac{\partial T}{\partial x}(L,\tau) = \sigma F_{se}[T_e^4 - T^4(L,\tau)]$$

i.e. $$\frac{\partial\varphi_1'}{\partial x}(L,\tau) + \frac{\partial\varphi_2}{\partial x}(L,\tau) = \frac{\sigma F_{se}}{k}[T_e^4 - T^4(L,\tau)]$$

i.e. $$-\frac{\sigma F_{se}}{k}(4T_e^3)\varphi_1' + \frac{\partial\varphi_2}{\partial x}(L,\tau)$$

$$= \frac{\sigma F_{se}}{k}[T_e^4 - \{T_e + (\varphi_1' + \varphi_2)\}^4] \qquad (36)$$

As with the first approximation only the terms of order φ_2 are retained, which reduces (36) to the linear equation

$$\frac{\partial\varphi_2}{\partial x}(L,\tau) = -6\frac{\sigma F_{se}}{k}\varphi_1'^2 T_e^2 - 4\frac{\sigma F_{se}}{k}T_e^3\varphi_2$$

$$= -6\left(\frac{\sin 2\delta_1}{\delta_1 + \sin\delta_1\cos\delta_1}\right)^2\frac{(T_i - T_e)^2}{T_e L N_{rh}}e^{-k_2\tau} - \frac{4}{LN_{rh}}\varphi_2$$

$$= -k^{(1)}\cdot e^{-k_2\tau} - k_5\varphi_2 \qquad (37)$$

where

$$k^{(1)} = 6\left(\frac{\sin 2\delta_1}{\delta_1 + \sin\delta_1\cos\delta_1}\right)^2\frac{(T_i - T_e)^2}{T_e L N_{rh}},\quad k_2 = 2\delta_1^2\frac{\alpha}{L^2}$$

While the restrictions on φ_2 are similar to that formerly imposed on φ_1 (in the discussion of the first approximation), at this point it is no longer required that $|\varphi_1/T_e|$ (or $|\varphi_1'/T_e|$) be vanishingly small.

__Solution for Second Approximation__ What appears the simplest way to effect a solution of (33), (34), (35), and (37) is to let

$$\varphi_2(x,\tau) = \zeta(x,\tau) + f_1(x)g_1(\tau) \qquad (38)$$

which after substitution gives

$$\alpha\zeta_{xx} = \zeta_\tau(x,\tau) - \{\alpha f_1'' g_1 - f_1\dot{g}_1\} \qquad (39)$$

$$\zeta_x(0,\tau) + f_1'(0)g_1(\tau) = 0 \qquad (40)$$

$$\zeta_x(L,\tau) + f_1'(L)g_1(\tau) + k_5 f_1(L)g_1(\tau) = -k^{(1)}e^{-k_2\tau} - k_5\zeta(L,\tau) \qquad (41)$$

Since $f_1(x)$ and $g_1(\tau)$ are arbitrary functions they are chosen such that

$$g_1(\tau)[f_1'(L) + k_5 f_1(L)] = -k^{(1)}e^{-k_2\tau} \qquad (42)$$

from which it follows from (41) that

$$\zeta_x(L,\tau) = -k_5\zeta(L,\tau)$$

Further, it is convenient to let

$$f_1'(L) + k_5 f_1(L) = 1 \qquad (43)$$

so that $g_1(\tau) = -k^{(1)}e^{-k_2\tau}$ (44)

Furthermore choose $f'(0) = 0$ (45)

then (40) becomes $\zeta_x(0,\tau) = 0$ (46)

Finally let $\alpha f_1'' g_1 - f_1\dot{g}_1 = 0$ (47)

so that (39) becomes $\alpha\zeta_{xx} = \zeta_\tau$ (48)

Substituting (44) into (47) gives

$\alpha f_1''(-k^{(1)}e^{-k_2\tau}) - f_1(k_2k^{(1)}e^{-k_2\tau}) = 0$ and upon dividing by $k_4e^{-k_2\tau}$ yields $\alpha f_1'' + k_2 f_1 = 0$ (49)

From the boundary conditions on f_1, (43), (45), and (47), the solution for f_1 becomes

$$f_1(x) = \frac{\mathrm{Cos}\,qx}{k_5\,\mathrm{Cos}\,ql - q\,\mathrm{Sin}\,ql} \quad \text{where } q = \sqrt{\frac{k_2}{\alpha}} \qquad (50)$$

Substituting (38) for φ_2 in the initial condition (34) gives for the initial condition on ζ

$$\zeta(x,0) = (T_i - T_e) - k_1\mathrm{Cos}(\delta_1\xi) + k^{(1)}\frac{\mathrm{Cos}\,qx}{k_5\,\mathrm{Cos}\,ql - q\,\mathrm{Sin}\,ql} \qquad (51)$$

Hence the completion of the solution for φ_2 requires the solution of

$$\alpha\zeta_{xx}(x,\tau) = \zeta_\tau(x,\tau) \qquad (52)$$

$$\zeta(x,0) = (T_i - T_e) - k_1\mathrm{Cos}(\delta_1\xi) + k_4\frac{\mathrm{Cos}\,qx}{(k_5\,\mathrm{Cos}\,ql - q\,\mathrm{Sin}\,ql)} \qquad (53)$$

$$\zeta_x(0,\tau) = 0 \qquad (54)$$

$$\zeta_x(L,\tau) = -k_5\zeta(L,\tau) \qquad (55)$$

(52)-(55) is similar to the boundary-value problem on φ_1 and may be solved in an identical fashion, ref. (8), to yield

$$\zeta(x,\tau) = 2\sum_{n=1}^{\infty} e^{-\delta_n^2 N_{Fo}} \frac{\mathrm{Sin}\,\delta_n \mathrm{Cos}(\delta_n\xi)}{\delta_n + \mathrm{Sin}\,\delta_n\mathrm{Cos}\,\delta_n}(T_i - T_e) - 2\sum_{n=1}^{\infty}\delta_n e^{-\delta_n^2 N_{Fo}}\frac{\mathrm{Cos}(\delta_n\xi)}{\delta_n + \mathrm{Sin}\,\delta_n\mathrm{Cos}\,\delta_n} \cdot \frac{k_1}{2}\left[\frac{\mathrm{Sin}(\delta_1+\delta_n)}{(\delta_1+\delta_n)} + \frac{\mathrm{Sin}(\delta_1-\delta_n)}{(\delta_1-\delta_n)}\right] + 2\sum_{n=1}^{\infty}\delta_n \cdot e^{-\delta_n^2 N_{Fo}}\frac{\mathrm{Cos}(\delta_n\xi)}{\delta_n + \mathrm{Sin}\,\delta_n\mathrm{Cos}\,\delta_n}\frac{k_6}{2} \cdot \left[\frac{\mathrm{Sin}(q+\delta_n)}{(q+\delta_n)} + \frac{\mathrm{Sin}(q-\delta_n)}{(q-\delta_n)}\right] \qquad (56)$$

where

$$k_6 = \frac{6}{T_e L N_{rh}}\left(\frac{\mathrm{Sin}\,2\delta_1}{\delta_1 + \mathrm{Sin}\,\delta_1\mathrm{Cos}\,\delta_1}(T_i - T_e)\right)^2 / (k_5\mathrm{Cos}\,ql - q\,\mathrm{Sin}\,ql)$$

The solution for φ_2 is given by (56), (50), (44) and (38), the temperature by (31), and the non-dimensional temperature by

$$\frac{T^{(2)} - T_i}{T_e - T_i} = 1 - \Big[2e^{-\delta_1^2 N_{Fo}}\frac{\mathrm{Sin}\,\delta_1\mathrm{Cos}(\delta_1\xi)}{\delta_1 + \mathrm{Sin}\,\delta_1\mathrm{Cos}\,\delta_1} + 2\sum_{n=1}^{\infty} e^{-\delta_n^2 N_{Fo}}\frac{\mathrm{Sin}\,\delta_n\mathrm{Cos}(\delta_n\xi)}{\delta_n + \mathrm{Sin}\,\delta_n\mathrm{Cos}\,\delta_n}$$

$$- 2\sum_{n=1}^{\infty}\delta_n e^{-\delta_n^2 N_{Fo}}\frac{\mathrm{Sin}\,\delta_1\mathrm{Cos}(\delta_n\xi)}{(\delta_1 + \mathrm{Sin}\,\delta_1\mathrm{Cos}\,\delta_1)} \cdot \frac{1}{(\delta_n + \mathrm{Sin}\,\delta_n\mathrm{Cos}\,\delta_n)}\left[\frac{\mathrm{Sin}(\delta_1+\delta_n)}{(\delta_1+\delta_n)} + \frac{\mathrm{Sin}(\delta_1-\delta_n)}{(\delta_1-\delta_n)}\right]$$

$$+ \sum_{n=1}^{\infty}\delta_n e^{-\delta_n^2 N_{Fo}}\frac{\mathrm{Cos}(\delta_n\xi)}{(\delta_n + \mathrm{Sin}\,\delta_n\mathrm{Cos}\,\delta_n)} \cdot \frac{D_i}{L(k_5\mathrm{Cos}\,q_1 l - q_1\mathrm{Sin}\,q_1 l)}\left[\frac{\mathrm{Sin}(\sqrt{2}\,\delta_1+\delta_n)}{(\sqrt{2}\,\delta_1+\delta_n)} + \frac{\mathrm{Sin}(\sqrt{2}\,\delta_1-\delta_n)}{(\sqrt{2}\,\delta_1-\delta_n)}\right] - k^{(1)}\frac{\mathrm{Cos}\,q_1(x)\,e^{-k_2\tau}}{(k_5\mathrm{Cos}\,q_1 l - q_1\mathrm{Sin}\,q_1 l)}\Big] \qquad (57)$$

where

$$D_i = \frac{6}{N_{rh}}\left\{\frac{\mathrm{Sin}\,2\delta_1}{\delta_1 + \mathrm{Sin}\,\delta_1\,\mathrm{Cos}\,\delta_1}\right\}^2\left(\frac{T_i}{T_e} - 1\right) \qquad (58)$$

It was empirically determined, based upon numerical computation, that the deletion of the $n > 1$ terms in the several infinite series of (58) caused no observable change in the temperature function.[1] Hence, an entirely satisfactory en-

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[1] That is, in the range where the second approximation may be considered inadequate, the inclusion of additional terms in each infinite series does not improve the accuracy. Improvement in accuracy can be obtained only by going to a higher approximation of the perturbation solution.

gineering solution is provided when only the $n = 1$ terms are retained, i.e.

$$\begin{aligned}\frac{T^{(2)}-T_i}{T_e-T_i} = 1 - \Big[& 2\, e^{-\delta_1^2 N_{Fo}} \frac{\sin\delta_1 \cos(\delta_1\xi)}{(\delta_1+\sin\delta_1\cos\delta_1)} \\ & + (D_i)\frac{\cos\delta_1}{(\delta_1+\sin\delta_1\cos\delta_1)} \\ & \cdot \frac{e^{-\delta_1^2 N_{Fo}}}{\left(\frac{4}{N_{rh}}\cos\sqrt{2}\,\delta_1-\sqrt{2}\,\delta_1\sin\sqrt{2}\,\delta_1\right)} \\ & \cdot \left\{\frac{\sin(\sqrt{2}+1)\delta_1}{(\sqrt{2}+1)}+\frac{\sin(\sqrt{2}-1)\delta_1}{(\sqrt{2}-1)}\right\} \\ & - \frac{(D_i)\cos(\sqrt{2}\,\delta_1\xi)\,e^{-2\delta_1^2 N_{Fo}}}{\left(\frac{4}{N_{rh}}\cos\sqrt{2}\,\delta_1-\sqrt{2}\,\delta_1\sin\sqrt{2}\,\delta_1\right)}\Big]\end{aligned} \qquad (59)$$

This completes the solution for the second approximation.

Third Approximation Proceeding as in the second approximation let

$$\varphi_3 = \zeta(x,\tau) + f_1 g_1 + f_2 g_2 \qquad (60)$$

After substitution and identifying f_2 and g_2 with suitable nonhomogeneous terms, it is found that f_2 is the solution of the same boundary value problem governing f_1 but where the coefficient k_2 is replaced by $3/2\ k_2$. The non-dimensional solution turns out to be

$$\begin{aligned}\frac{T^{(3)}-T_i}{T_e-T_i} = 1 - \Big[& 2\, C_1 \sin\delta_1\cos(\delta_1\xi)e^{-\delta_1^2N_{Fo}} \\ & + C_1C_2C_3(D_i)\cos(\delta_1\xi)e^{-\delta_1^2N_{Fo}} \\ & - C_2(D_i)\cos(\sqrt{2}\,\delta_1\xi)e^{-\delta_1^2N_{Fo}} + 2\sum_{n=1}^{\infty} e^{-\delta_n^2N_{Fo}} \\ & \cdot \frac{\cos(\delta_n\xi)\sin\delta_n}{(\delta_n+\sin\delta_n\cos\delta_n)} - 2\, C_1\sum_{n=1}^{\infty}\delta_n\, e^{-\delta_n^2N_{Fo}} \\ & \cdot \frac{\cos(\delta_n\xi)\sin\delta_n}{(\delta_n+\sin\delta_n\cos\delta_n)}\left\{\frac{\sin(\delta_1+\delta_n)}{(\delta_1+\delta_n)}+\frac{\sin(\delta_1-\delta_n)}{(\delta_1-\delta_n)}\right\} \\ & - C_1C_2C_3(D_i)\sum_{n=1}^{\infty}\delta_n\, e^{-\delta_n^2N_{Fo}} \\ & \cdot \frac{\cos(\delta_n\xi)}{(\delta_n+\sin\delta_n\cos\delta_n)}\left\{\frac{\sin(\delta_1+\delta_n)}{(\delta_1+\delta_n)}+\frac{\sin(\delta_1-\delta_n)}{(\delta_1-\delta_n)}\right\} \\ & + C_2C_3\sum_{n=1}^{\infty} e^{-\delta_n^2N_{Fo}}\frac{\cos(\delta_n\xi)}{(\delta_n+\sin\delta_n\cos\delta_n)} \\ & + 2\, C_1C_2C_3C_4\sum_{n=1}^{\infty}\delta_n\, e^{-\delta_n^2N_{Fo}}\frac{\cos(\delta_n\xi)}{\delta_n+\sin\delta_n\cos\delta_n} \\ & \cdot \frac{6}{N_{rh}}(D_i)\cos\delta_1\left\{\frac{\sin(\sqrt{2}\,\delta_1+\delta_n)}{(\sqrt{2}\,\delta_1+\delta_n)}\right. \\ & \left.+\frac{\sin(\sqrt{2}\,\delta_1-\delta_n)}{(\sqrt{2}\,\delta_1-\delta_n)}\right\} - \frac{2}{3}C_2C_4\sum_{n=1}^{\infty}\delta_n\, e^{-\delta_n^2N_{Fo}} \\ & \cdot \frac{\cos(\delta_n\xi)}{(\delta_n+\sin\delta_n\cos\delta_n)}(D_i)\left\{\frac{\sin(\sqrt{3}\,\delta_1+\delta_n)}{(\sqrt{3}\,\delta_1+\delta_n)}\right. \\ & \left.+\frac{\sin(\sqrt{3}\,\delta_1-\delta_n)}{(\sqrt{3}\,\delta_1-\delta_n)}\right\}\left\{\frac{18}{N_{rh}}C_2\cos(\sqrt{2}\,\delta_1\xi)-1\right\} \\ & - \frac{12}{N_{rh}}C_1C_2C_3C_4(D_i)\cos\delta_1\cos(\sqrt{2}\,\delta_1\xi)\,e^{-2\delta_1^2N_{Fo}} \\ & + \frac{2}{3}C_4(D_i)\left\{\frac{18}{N_{rh}}C_2\cos(\sqrt{2}\,\delta_1\xi)-1\right\} \\ & \cdot\left\{e^{-3\delta_1^2N_{Fo}}\cos(\sqrt{3}\,\delta_1\xi)C_5\right\}\Big]\end{aligned} \qquad (61)$$

where

$$C_1 = \frac{1}{(\delta_1+\sin\delta_1\cos\delta_1)}$$

$$C_2 = \frac{1}{(4/N_{rh}\cos\sqrt{2}\,\delta_1-\sqrt{2}\,\delta_1\sin\sqrt{2}\,\delta_1)}$$

$$C_3 = \frac{\sin(\sqrt{2}+1)\delta_1}{(\sqrt{2}+1)}+\frac{\sin(\sqrt{2}-1)\delta_1}{(\sqrt{2}-1)}$$

$$C_4 = \frac{\sin 2\delta_1}{(\delta_1+\sin\delta_1\cos_1\delta)}$$

$$C_5 = \frac{1}{4/N_{rh}\cos\sqrt{3}\,\delta_1-\sqrt{3}\,\delta_1\sin\sqrt{3}\,\delta_1}$$

$$C_6 = 18/N_{rh}C_2\cos(\sqrt{2}\,\delta_1\xi) - 1$$

Truncating each series after $n = 1$, gives the engineering formula

$$\begin{aligned}\frac{T^{(3)}-T_i}{T_e-T_i} = 1 - \Big[& 2\, C_1\sin\delta_1\cos(\delta_1\xi)\, e^{-\delta_1^2N_{Fo}} \\ & - C_2(D_i)\cos(\sqrt{2}\,\delta_2\xi)\, e^{-\delta_1^2N_{Fo}} \\ & + C_1C_2C_3\ \cos\delta_1\xi\ e^{-\delta_1^2N_{Fo}} \\ & + \frac{12}{N_{rh}}C_1^2C_2C_3^2C_4(D_i)\cos\delta_1\cos(\delta_1\xi) \\ & \cdot e^{-\delta_1^2N_{Fo}} - \frac{2}{3}C_1C_2C_4C_6(D_i)\cos(\delta_1\xi) \\ & \cdot\left\{\frac{\sin(\sqrt{3}+1)\delta_1}{(\sqrt{3}+1)}+\frac{\sin(\sqrt{3}-1)\delta_1}{(\sqrt{3}-1)}\right\}e^{-\delta_1^2N_{Fo}} \\ & - \frac{12}{N_{rh}}C_1C_2C_3C_4(D_i)\cos\delta_1\cos(\sqrt{2}\,\delta_1\xi) \\ & \cdot e^{-2\delta_1^2N_{Fo}} + \frac{2}{3}\ C_4C_5C_6(D_i)\cos(\sqrt{3}\,\delta_1\xi) \\ & \cdot e^{-3\delta_1^2N_{Fo}}\Big]\end{aligned} \qquad (62)$$

This completes the solution for the third approximation.

n^{th} Approximation The n^{th} approximation is obtained in an analogous manner by writing

$$\varphi_n = \zeta(x,\tau) + f_1g_1 + f_2g_2 + \dots + f_{n-1}g_{n-1} \qquad (63)$$

which results in a set of equations for the "f" functions as follows:

$$\alpha f_1''(x)\, g_1(\tau) - f_2(x)\, \dot{g}_1(\tau) = 0$$

$$f_1'(0) = 0$$

$$f_1'(L) + k_5 f_1(L) = 1, \text{ where } g_1(\tau) = -k^{(1)} e^{-k_2\tau} \tag{64}$$

$$\alpha f_2''(x)\, g_2(\tau) - f_2(x)\, \dot{g}_2(\tau) = 0$$

$$f_2'(0) = 0$$

$$f_2'(L) + k_5 f_2(L) = 1, \text{ where } g_2(\tau) = -k^{(2)} e^{-3/2 k_2\tau}$$

....(65)

$$\alpha f_{n-1}'' g_{n-1} - f_{n-1} \dot{g}_{n-1} = 0$$

$$f_{n-1}'(0) = 0$$

$$f_{n-1}'(L) + k_5 f_{n-1}(L) = 1, \text{ where}$$

$$g_{n-1}(\tau) = -k^{(n-1)} e^{-(n/2)k_2\tau} \tag{66}$$

These are easily solved to yield

$$f_1(x) = \frac{\operatorname{Cos}(q_1 x)}{(k_5 \operatorname{Cos} q_1 l - q_1 \operatorname{Sin} q_1 l)} \text{ where } q_1 = \sqrt{\left(\frac{k_2}{\alpha}\right)} \tag{67}$$

$$f_2(x) = \frac{\operatorname{Cos} q_2(x)}{(k_5 \operatorname{Cos} q_2 l - q_2 \operatorname{Sin} q_2 l)} \text{ where } q_2 = \sqrt{\left(\frac{3k_2}{2\alpha}\right)}$$

....(68)

$$f_{n-1}(x) = \frac{\operatorname{Cos}(q_{n-1} x_1)}{(k_5 \operatorname{Cos} q_{n-1} l_1 - q_{n-1} \operatorname{Sin} q_{n-1} l_1)}$$

$$\text{where } q_{n-1} = \sqrt{\left(\frac{n}{2}k_2\right)} \tag{69}$$

Hence $\varphi_n(62)$ is given by

$$\varphi_n = \zeta + \frac{\operatorname{Cos}(q_1 x)}{(k_5 \operatorname{Cos} q_1 l - q_1 \operatorname{Sin} q_1 l)} (-k^{(1)} e^{-k_2\tau}) + \frac{\operatorname{Cos}(q_2 x)}{(k_5 \operatorname{Cos} q_2 l - q_2 \operatorname{Sin} q_2 l)} (-k^{(2)} e^{-\frac{3}{2}k_2\tau}) + \cdots + \frac{\operatorname{Cos}(q_{n-1} x_1)}{(k_5 \operatorname{Cos} q_{n-1} l_1 - q_{n-1} \operatorname{Sin} q_{n-1} l_1)} (-k^{(n-1)} e^{-\frac{n}{2}k_2\tau}) \tag{70}$$

and the corresponding initial condition is

$$\varphi_n(x,0) = \zeta(x,0) + f_1(x) g_1(0) + \cdots + f_{n-1}(x) g_{n-1}(0) \tag{71}$$

The function $\zeta(x,\tau)$ is obtained in the usual way for this new initial condition, thus completing the n^{th} approximation.

Comparison with Analog Computer Results: By use of the formulas developed it is possible to compute numerical values simply by slide rule, and, in fact, data for several curves were computed in this manner. However, because it was felt desirable to compare the perturbation solutions with the curves obtained by analog computer in ref. (7) (which are quite numerous) most of the numerical calculations were performed on an IBM 7040 and the results plotted with the help of Calcomp. 545. There are several key parameters and various arrangements are possible for plotting purposes. The scheme chosen follows ref. (7). The basic parameter is T_i/T_e; values are plotted for 0.75, 0.5, 0.25. For each T_i/T_e, three stations were chosen, $x/L = 0, 0.5, 1.0$. Finally for a given value of T_i/T_e and x/L, curves of non-dimensional temperature $(T-T_i)/(T_e-T_i)$ versus non-dimensional time $\alpha\ \tau/L^2$ are plotted for three values of the parameter N_{rh}. These are plotted in Figures (1), (2), (3). It is found that for $T_i T_e = 0.75$ the first approximation is quite accurate and the second approximation gives the values very close to that obtained by ref. (7) so that this approximation may be satisfactory for engineering applications. As the ratio T_i/T_e is reduced to 0.5, it is seen that third approximation is quite close to the analog computer results. As expected the results for $T_i/T_e = 0.25$ do not compare quite as favorably to ref. (7) as do the higher ratios but, nevertheless, should be of sufficient accuracy for most applications.

Conclusion It has been demonstrated foregoing that a perturbation technique yields a satisfactory approximation to the nonlinear radiation heating problem. Numerical results computed by the formulas developed compare quite favorably to previously obtained analog results for a T_i/T_e as small as 0.25.

One major advantage of an analytical solution over an analog (or digital) computer solution is that values of interest may be obtained without the need for interpolating between plotted values or chart values. To obtain the flux, for example, by means of the analog computer solution in general, requires obtaining differences between two curves which are a small distance apart. With the data given in ref. (7) however, one would have to interpolate between values given at $x/L = 1$, $x/L = 0.5$, and $x/L = 0$. On the other hand to obtain the flux by means of the solution developed herein requires only the differentiation of (61). The third approximation for the flux is indicated below.

$$\frac{L}{T_e-T_i}\left.\frac{\partial T}{\partial x}\right|_{x=L} = \delta_1 e^{-\delta_1^2 N_{Fo}}\Big[2 C_1 \operatorname{Sin}\delta_1 \operatorname{Cos}\delta_1 - \sqrt{2}\, C_2(D_i)\operatorname{Sin}(\sqrt{2}\,\delta_1) + C_1 C_2 C_3 \operatorname{Sin}(\delta_1) + \frac{12}{N_{rh}} C_1^2 C_2 C_3^2 C_4 (D_i) \operatorname{Sin}\delta_1 \operatorname{Cos}\delta_1 - \frac{2}{3} C_1 C_2 C_4 C_6'(D_i)\operatorname{Sin}(\delta_1)\left\{\frac{\operatorname{Sin}(\sqrt{3}+1)\delta_1}{(\sqrt{3}+1)} + \frac{\operatorname{Sin}(\sqrt{3}-1)\delta_1}{(\sqrt{3}-1)}\right\} + \frac{12/2}{N_{rh}} C_1 C_2^2 C_4 (D_i) \operatorname{Sin}(\sqrt{2}\,\delta_1)\operatorname{Cos}(\delta_1)\left\{\frac{\operatorname{Sin}(\sqrt{3}+1)\delta_1}{(\sqrt{3}+1)} + \frac{\operatorname{Sin}(\sqrt{3}-1)\delta_1}{(\sqrt{3}-1)}\right\} - \frac{12/2}{N_{rh}} C_1 C_2 C_3 C_4 (D_i)$$

$$\cdot \sin(\sqrt{2}\ \delta_1)\cos(\delta_1)\ e^{-\delta_1^2 N_{Fo}} + \frac{2}{\sqrt{3}} C_4 C_5 C_6{}'(D_i)$$

$$\cdot \sin(\sqrt{3}\ \delta_1)\ e^{-2\delta_1^2 N_{Fo}} - \frac{12/2}{N_{rh}} C_2 C_4 C_5 (D_i)$$

$$\cdot \sin(\sqrt{2}\ \delta_1)\cos(\sqrt{3}\ \delta_1)\ e^{-2\delta_1^2 N_{Fo}}]$$

$$\text{where } C_6{}' = \{18/N_{rh}\ C_2 \cos(\sqrt{3}\ \delta_1) - 1\} \quad (72)$$

References

1. W. R. Mann and F. Wolf, "Heat Transfer Between Solids and Gases Under Non-Linear Boundary Condition," Quart. Appl. Math., Vol. 9, no. 2, 1951, p. 163.

2. J. C. Jaeger, "Conduction of Heat in a Solid With a Power Law of Heat Transfer at Its Surface," Proc. Camb. Phil. Soc., Vol. 46, 1950, p. 634.

3. P. L. Chambre, "Non-Linear Heat Transfer Problem," Journal Appl. Phys., Vol. 30, no. 11, 1959, p. 1683.

4. S. S. Abarbanel, "Time Dependent Temperature in Radiative Solids, Journal Maths. Phys., Vol. 39, no. 4, 1960, p. 246.

5. P. D. Richardson, "Unsteady One-Dimensional Heat Conduction with a Non-Linear Boundary Condition," Trans. of ASME, Jour. Heat Transfer Series C, Vol. 86, May 1964, p. 298.

6. R. S. Fairall, R. A. Wells, and R. L. Belcher, "Unsteady State Heat Transfer in Solids With Radiation at One Boundary," Trans. of ASME, Jour. Heat Transfer, Series C, Vol. 84, Aug. 1962, p. 266.

7. R. D. Zerkle and J. Edward Sunderland, "The Transient Temperature Distribution in a Slab Subject to Thermal Radiation," Proc. of ASME, Jour. Heat Transfer, Series C, Vol. 87, Feb. 1965, p. 117.

8. A. J. Chapman, "Heat Transfer, Macmillan Company, New York, 1960, p. 113-116.

9. H. S. Carslaw and J. C. Jaeger, "Conduction of Heat in Solids," Clarendon Press at Oxford, 1959, p. 491.

Nomenclature

c = specific heat, Btu/lb deg R
F_{se} = radiation interchange factor between slab and environment
L = slab width, Ft.
N_{Fo} = Fourier number, $\alpha\tau/L^2$, dimensionless
N_{rh} = radiation number for heating, $k/\sigma\ F_{se} Te^3\ L$, dimensionless
T = absolute temperature, deg R
T_e = environment temperature, deg R
T_i = initial slab temperature, deg R
x = space coordinate normal to slab faces, ft.
α = thermal diffusivity, $k/\rho c$, ft. ^{2}/hr.
ΔT = temperature difference, deg R
Δx = distance increment in x-direction, ft.
$\Delta\tau$ = increment in time, hr.
ρ = density, lb/ft^3
σ = Stefan-Boltzman constant, Btu/hr. ft^2 deg R^4

subscripts:
e = refers to environment
i = refers to initial conditions

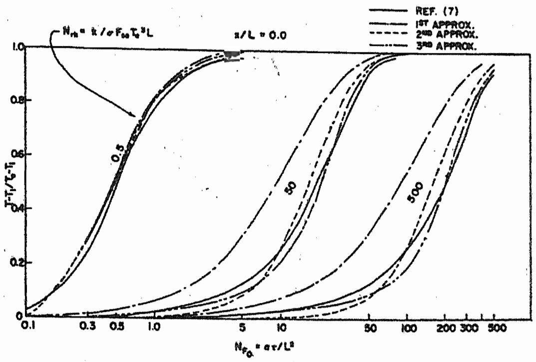

FIGURE 1-a
$T_i/T_e = 0.25$
TRANSIENT TEMPERATURE AT INSULATED SURFACE

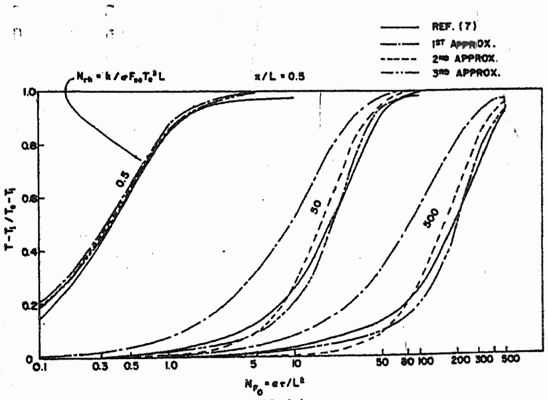

FIGURE 1-b
$T_i/T_e = 0.25$
TRANSIENT TEMPERATURE AT THE CENTER PLANE

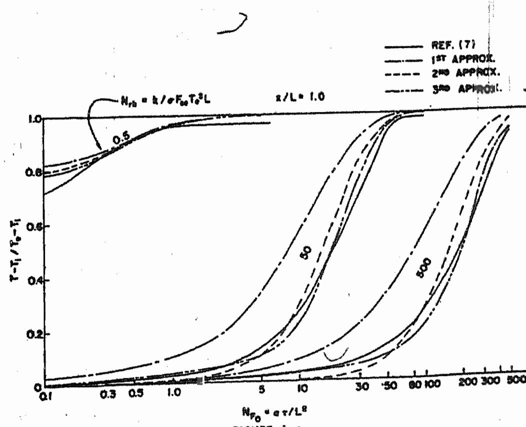

FIGURE 1-c
$T_i/T_e = 0.25$
TRANSIENT TEMPERATURE AT THE HEATED FACE

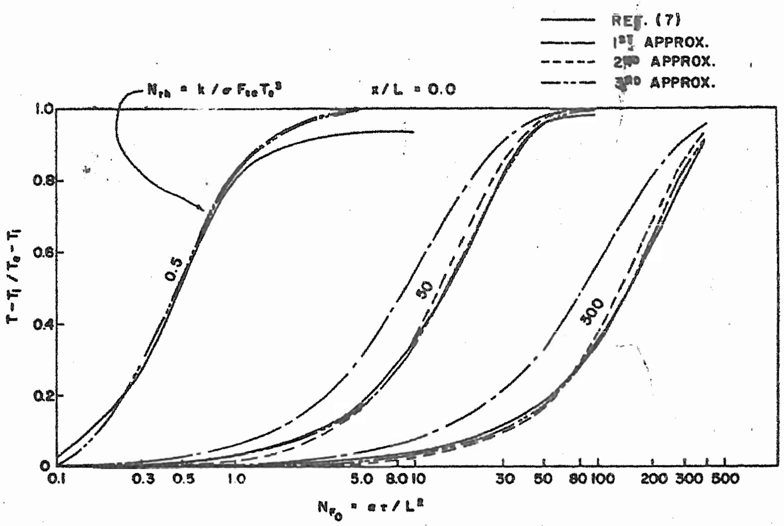

FIGURE 2-a

$T_i / T_e = 0.5$

TRANSIENT TEMPERATURE AT INSULATED SURFACE

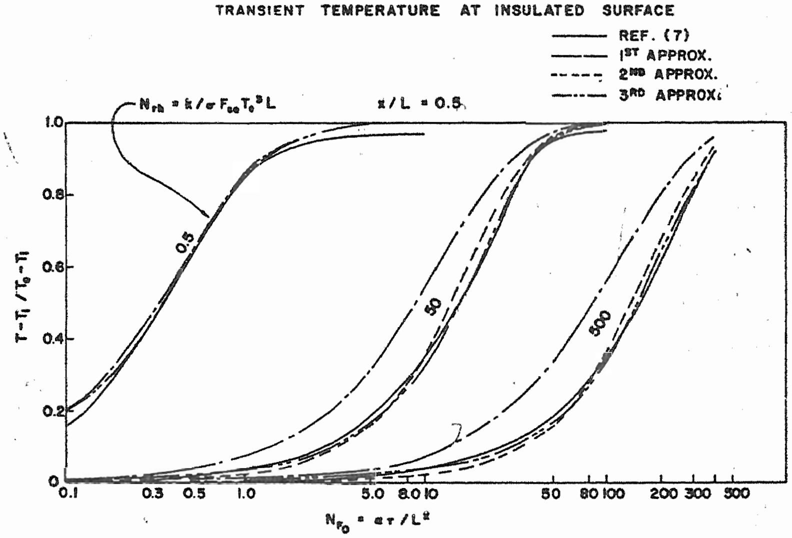

FIGURE 2-b

$T_i / T_e = 0.5$

TRANSIENT TEMPERATURE AT THE CENTER PLANE

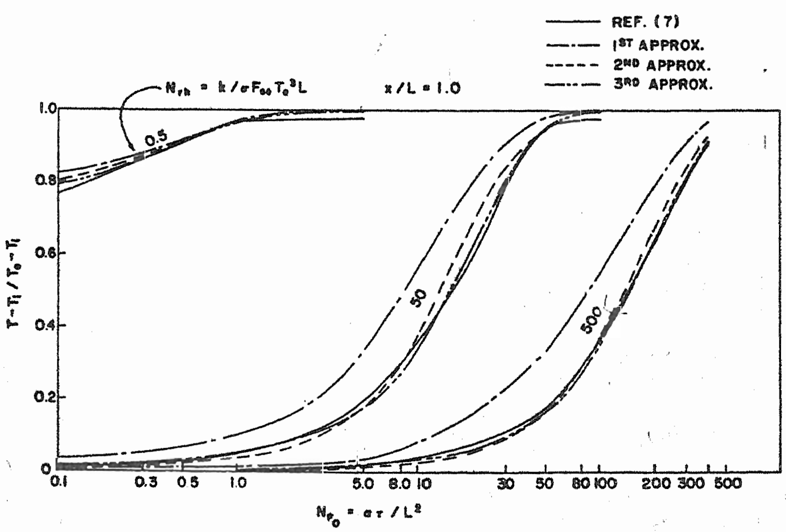

FIGURE 2-c

$T_i / T_e = 0.5$

TRANSIENT TEMPERATURE AT THE HEATED FACE

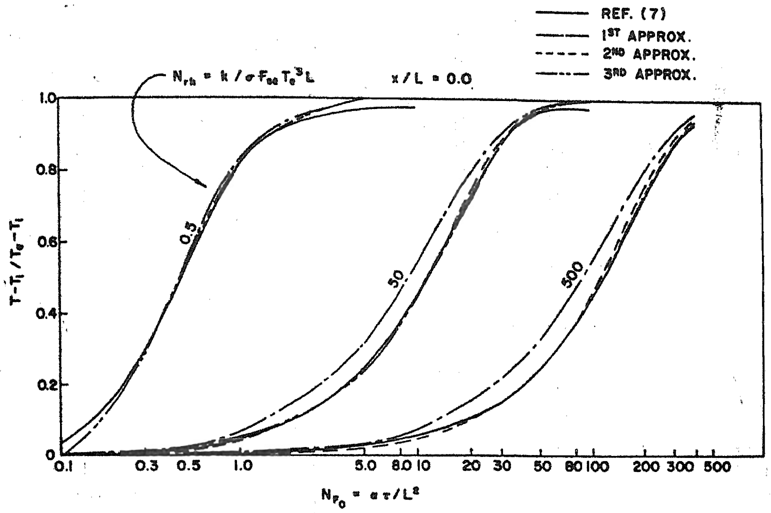

FIGURE 3-a

$T_i / T_e = 0.75$

TRANSIENT TEMPERATURE AT INSULATED SURFACE

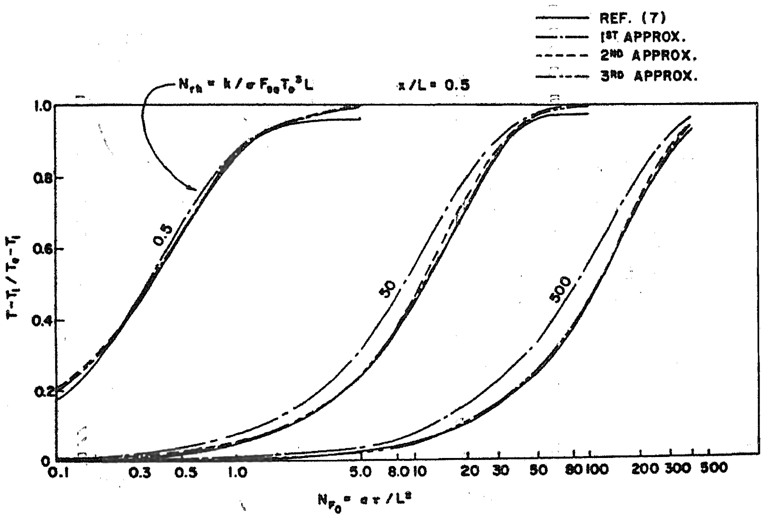

FIGURE 3-b

$T_i / T_e = 0.75$

TRANSIENT TEMPERATURE AT THE CENTER PLANE

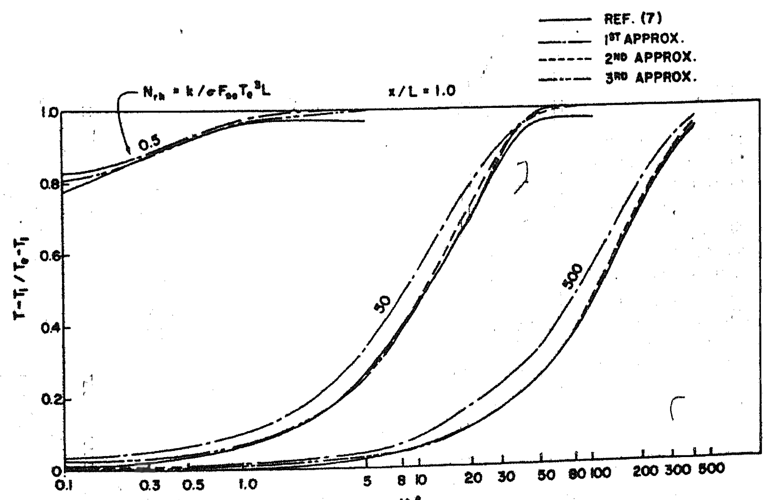

FIGURE 3-c

$T_i / T_e = 0.75$

TRANSIENT TEMPERATURE AT THE HEATED FACE